Transformation semigroups & minimal ideals

Wilf Wilson

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Computational semigroup theory

- We want to store semigroups and calculate facts about them without needing a list of all the elements to hand.
- Specifying a semigroup by a generating set of transformations is a good way.

Transformations (of a finite set)

What are transformations, and what are they like?

A $\mathbf{transformation}$ of a finite set Ω is a function $f : \Omega \to \Omega$.

We may as well insist that $\Omega = \{1, 2, \ldots, n\}$.
We call a transformation of $\{1, 2, \ldots, n\}$ a *transformation on n points*.

Semigroups

In group theory we have permutation groups.

- e.g. S_n.

In semigroup theory we have transformation semigroups.

- e.g. T_n.

Permutations vs. transformations

We can write permutations in two-line notation:

- e.g. $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$

And we can do the same for transformations:

- e.g. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 3 & 1 \end{pmatrix}$

Permutations vs. transformations

For permutations we have the *order* of a permutation.

- i.e. the least positive integer n such that $g^n = \text{id}$.
- $\langle g \rangle = \{g, g^2, \dots, g^{n-1}, g^n\}$.

For transformations we have the *index* and the *period* of a transformation.

- i.e. the least positive integers m, r such that $f^m = f^{m+r}$.
- $\langle f \rangle = \{f, f^2, \dots, \underbrace{f^m, f^{m+1}, \dots, f^{m+r-1}}_{\text{A cyclic group of order } r}\}$.

The kernel and the image of a transformation

Let f be a transformation on n points.

The **image** of f, $\mathrm{im}(f)$, is the set $\{(i)f : i \in \{1, 2, \ldots, n\}\}$.

The **kernel** of f, $\ker(f)$, is the equivalence relation on $\{1, 2, \ldots, n\}$ which relates i and j whenever $(i)f = (j)f$.

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For a permutation g, $\mathrm{im}(g) = \{1, 2, \ldots, n\}$ and $\ker(g)$ is equality.

Composition of transformations

The operation in a transformation semigroup is *composition of functions*. The composition $f \circ g$ of two transformations f and g is defined by:

$$(x)f \circ g = ((x)f)g \text{ for all } x \in \{1, 2, \ldots, n\}.$$

$$\begin{aligned} \text{Note that } \mathsf{im}(fg) &= \{((x)f)g : x \in \{1, 2, \ldots, n\}\} \\ &= \{(i)g : i \in \mathsf{im}(f)\} \\ &= \mathsf{im}(f) \cdot g \end{aligned}$$

The kernel and image of a transformation: an example

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 2 & 1 & 6 & 1 & 7 & 6 & 2 \end{pmatrix}$

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- Then $\mathrm{im}(f) = \{1, 2, 5, 6, 7\}$.
- $\ker(f)$ has equivalence classes:

$$\begin{aligned} \{4, 6\} &= (1)f^{-1}, \\ \{1, 3, 9\} &= (2)f^{-1}, \\ \{2\} &= (5)f^{-1} \\ \{5, 8\} &= (6)f^{-1}, \\ \text{and } \{7\} &= (7)f^{-1}. \end{aligned}$$

The rank of a transformation

Let f be a transformation.

The **rank** of f, rank(f), is equal to $|\text{im}(f)|$.

Equivalently:

The **rank** of f is the number of equivalence classes of ker(f).

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$$\text{i.e. } \mathrm{rank}(f) = \left| \frac{\{1, 2, \ldots, n\}}{\ker(f)} \right| = |\mathrm{im}(f)|$$

The rank of a transformation

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Proof.

- $\mathsf{rank}(xy) = |\mathsf{im}(xy)| = |\mathsf{im}(x) \cdot y| \leq |\mathsf{im}(x)| = \mathsf{rank}(x)$
- $\mathsf{rank}(xy) = |\mathsf{im}(xy)| = |\mathsf{im}(x) \cdot y| \leq |\{1, 2, \ldots, n\} \cdot y| = \mathsf{rank}(y)$

□

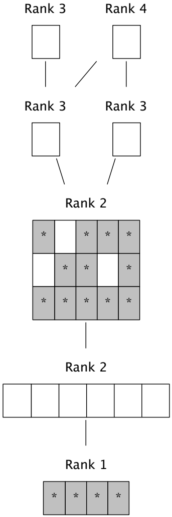
Composition of transformations

Our old example: let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 2 & 1 & 6 & 1 & 7 & 6 & 2 \end{pmatrix}$

Let g be another transformation.

- Consider two distinct points $i, j \in \mathrm{im}(f)$.
- If $(i)g = (j)g$ then we say that g *collapses the pair* $\{i, j\}$.
  - g collapses some pair in $\mathrm{im}(f)$ if and only if $\mathrm{rank}(fg) < \mathrm{rank}(f)$.

Semigroup diagram: rank



If you have been to a talk about semigroups before, you might recognise these semigroup diagrams.

Notice that rank decreases as you move further down the diagram.

Ideals

Let $S = (S, \cdot)$ be a semigroup and let $I \subseteq S$ be a non-empty subset of S.

Then I is an *ideal* if:

$$I \cdot S \subseteq I \text{ and } S \cdot I \subseteq I.$$

Equivalently:

$$is \in I \text{ and } si \in I \text{ for all } i \in I, s \in S.$$

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I like to think of ideals as **black holes**.

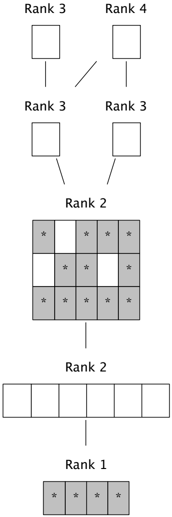
Principal ideals

For a semigroup S and an element $x \in S$ the set:

$$S^1 x S^1$$

is an ideal of S and is called *the principal ideal generated by* x.

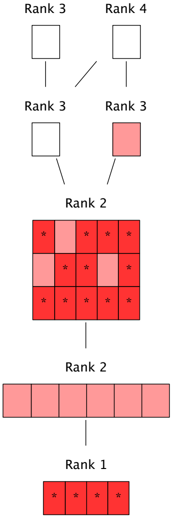
Semigroup diagram: principal ideal



Two elements lie in the same grid if they generate the same principal ideal.

(We call these grids $\mathscr{J}$-classes - this is how they are defined).

Semigroup diagram: principal ideal



Two elements lie in the same grid if they generate the same ideal.

(We call these $\mathscr{D}$- or $\mathscr{J}$-classes - this is how they are defined).

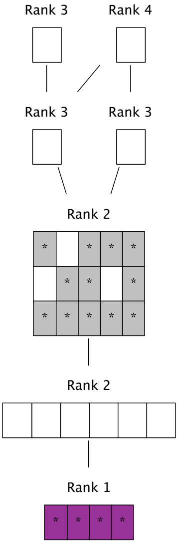
An element in the upper pink $\mathscr{D}$-class generates this ideal.

The minimal ideal

- A finite semigroup has a finite number of ideals.
- The intersection of two ideals is an ideal.

Therefore a finite semigroup has a *minimal ideal* (w.r.t. containment).

Semigroup diagram: minimal ideal



Minimal ideal $\leftrightarrow$ minimal rank

Let S be a finite transformation semigroup, and let $x \in S$.

$$x \text{ is in the minimal ideal of } S \Leftrightarrow \text{rank}(x) \text{ is smallest possible in } S.$$

That is, the minimal ideal is precisely the set of elements of minimal rank.

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Aim

For various reasons, we wish to be able to quickly get hold of an element of the minimal ideal of a transformation semigroup.

For example:

- To see if the semigroup is synchronising.
- To calculate the zero of a semigroup (or prove it doesn't exist).
- To calculate all the *other* elements of the minimal ideal.

Ways of calculating

- **BAD**: getting all of the elements and looking at them in turn.
  - Exponential complexity in n.
- **BETTER**: using the ideas I'm about to share.
  - Quadratic complexity in n.
  - Joint work with James Mitchell.

An example

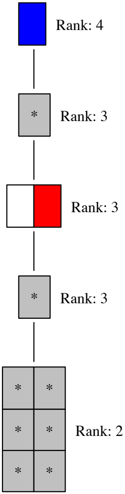
Let $S = \langle \sigma, \tau \rangle$ where:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 4 \end{pmatrix}$$

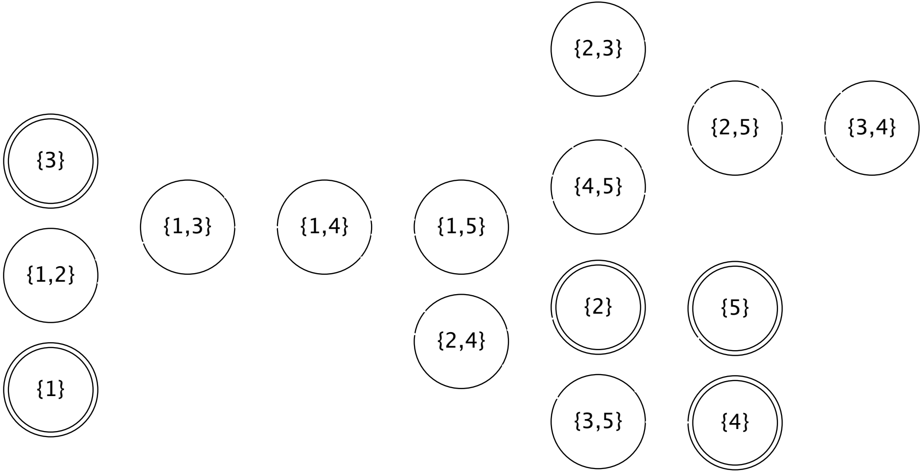
and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 2 & 5 \end{pmatrix}$$

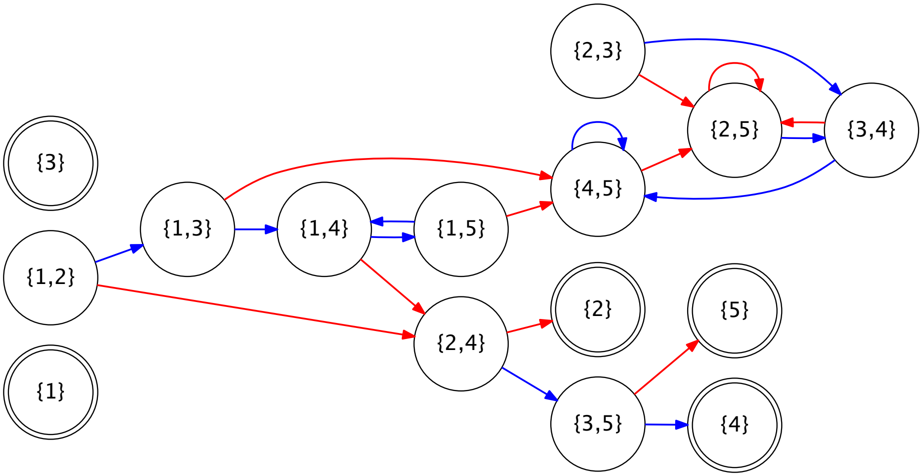
are transformations on 5 points.



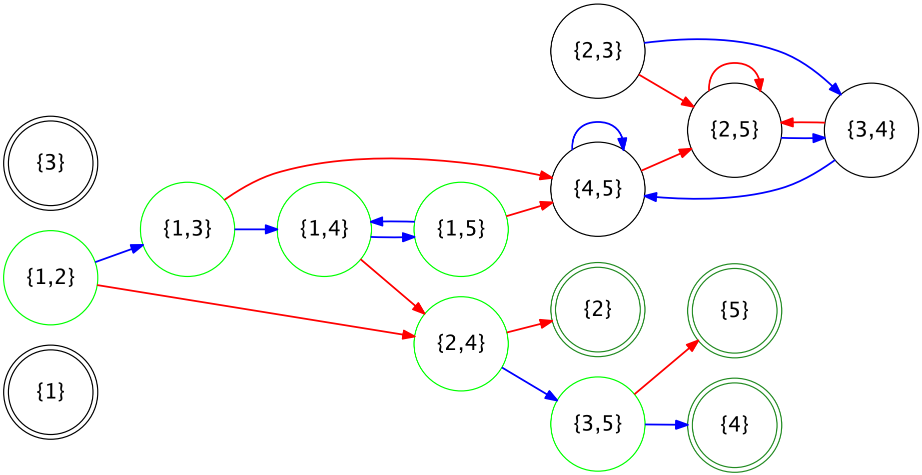
The graph: the $\binom{5}{2} + 5$ vertices



The graph: the $\binom{5}{2} \cdot 2$ edges



The graph: the 6 collapsible pairs



The two types of pairs

The $\binom{5}{2} = 10$ pairs fall into two types: those which can be collapsed, and those which can not.

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Our collapsible pairs:

$$\begin{aligned} \{3,5\}\sigma &= \{4\} \\ \{2,4\}\tau &= \{2\} \\ \{1,2\}\tau^2 &= \{2\} \\ \{1,4\}\tau^2 &= \{2\} \\ \{1,3\}\sigma\tau^2 &= \{2\} \\ \{1,5\}\sigma\tau^2 &= \{2\} \end{aligned}$$

The other pairs:

$$\begin{gathered} \{2,3\} \\ \{2,5\} \\ \{3,4\} \\ \{4,5\} \end{gathered}$$

Every element in S must have different images for i and j if $\{i,j\}$ is not collapsible.

The idea

S must have an element x with minimal rank r.

- Take an element $f \in S$.
- Then $f \cdot x$ also has minimal rank.

$\Rightarrow$ We can right-multiply any f to obtain an element of minimal rank.
$\Rightarrow$ For any non-minimal f, there is a collapsible pair of points in im(f).

- Start with any $f \in S$, and collapse pairs until you can't.
- You now have an element of the minimal ideal.

Let's do it

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Collapse pair $\{3,5\}$ in im(r_0):

$$r_1 := r_0\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 4 & 5 \end{pmatrix}$$

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Collapse pair $\{1,4\}$ in im(r_1):

$$r_2 := r_1\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 5 & 2 & 5 \end{pmatrix}$$

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Collapse pair $\{3,5\}$ in $\text{im}(r_0)$:

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$$r_2 := r_1\tau^2 = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 2 & 5 & 2 & 5\end{pmatrix}$$

No collapsible pairs $\Rightarrow r_2$ minimal.

End.