A theorem about coprime action ⋆

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Abstract

It is well known that if an elementary abelian p-group P acts on a p'-group Q and $Q = [Q, P]$, then $Q = \langle [C_Q(A), P] \mid A \leq P \text{ of index } p \rangle$. Does a similar statement hold for $C_Q(P)$? Under further assumptions, the answer is yes. Goldschmidt proves theorems of this flavour in [1] and [2] and uses them to construct signalizer functors. For the same reason we prove a result of this type, under the assumption that Q is soluble.

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1 Preliminaries

We collect a few results about coprime action. These are well known and can be found in group theory books, for example in [3], Chapter 8. Throughout this paper, all groups are supposed to be finite and we follow standard notation (e.g. [3]).

Coprime Action

Let π be a set of primes and let P be a π-group which acts on a π'-group G. Let p be a prime. For any elementary abelian p-group P, we denote by $Hyp(P)$ and $Hyp^2(P)$ the set of all the subgroups of P of index p and p^2, respectively. We refer to the elements of $Hyp(P)$ as hyperplanes of P.

(i) If N is a P-invariant normal subgroup of G, then $C_{G/N}(P) = C_G(P)N/N$.

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(ii) We have $G = [G, P]C_G(P)$ and $[G, P] = [G, P, P]$. If G is abelian, then $G = [G, P] \times C_G(P)$.

(iii) Suppose that G is the product of two P-invariant subgroups G_1 and G_2. Then $C_G(P) = C_{G_1}(P)C_{G_2}(P)$.

(iv) If P is an elementary abelian p-group, then $G = \langle C_G(A) \mid A \in Hyp(P)\rangle$ and $[G, P] = \langle [C_G(A), P] \mid A \in Hyp(P)\rangle$.

2 A theorem about coprime action

Theorem

Let p be a prime. Suppose that the central product AA_0 acts coprimely on the soluble group G with $G = [G, A_0]$, where A is an elementary abelian p-group of rank at least 3. Furthermore, let $B \leq A$ and $H := C_G(A_0B)$. Then

$$H = \langle [C_G(X), A_0] \cap H \mid X \in Hyp^2(A)\rangle \ .$$

Proof. Let G be a minimal counterexample and set

$$H_0 := \langle [C_G(X), A_0] \cap H \mid X \in Hyp^2(A)\rangle.$$

We note that G is not abelian because otherwise $G = [G, A_0] \times C_G(A_0)$ by Coprime Action (ii). This implies that $C_G(A_0) \leq G'$ since the factor group G/G' is abelian.

Now let R be a maximal AA_0-invariant subgroup of G containing G', so that $R \trianglelefteq G$, and let $R_0 := [R, A_0]$. We note that $C_G(A_0) \leq G' \leq R$ and by Coprime Action (ii), therefore, $R = R_0C_G(A_0)$. Coprime Action (iv) implies that we can find a hyperplane Y of A such that $G = RC_G(Y)$. As $G = [G, A_0]$, the subgroup $U := [C_G(Y), A_0]$ is not contained in R. Now U is AA_0-invariant and so $G = RU$. Let N be a minimal AA_0-invariant normal subgroup of G.

We proceed towards a contradiction in small steps.

(1) $G = \langle R_0, U\rangle$.

Proof. We have $G = RU = C_G(A_0)\langle R_0, U\rangle$. As $\langle R_0, U\rangle$ is A_0-invariant, this gives $G = [G, A_0] = [C_G(A_0)\langle R_0, U\rangle, A_0] = [\langle R_0, U\rangle, A_0] \leq \langle R_0, U\rangle$. □

(2) $H = H_0(H \cap N)$.

Proof. The minimality of G implies that the theorem holds in the factor group G/N. Hence $HN/N = \langle [C_{G/N}(X), A_0] \cap HN/N \mid X \in Hyp^2(A)\rangle$. Using Coprime Action (i) and (iii) and the fact that the theorem holds in

G/N, we obtain $HN/N = H_0N/N$. Now $HN = H_0N$ and the statement holds by Dedekind's Law. □

(3) *Suppose that $V = [V, A_0]$ is a proper A-invariant subgroup of G. Then $N \not\leq V$. If $V \trianglelefteq G$, then $V \cap N = 1$.*

Proof. The theorem holds in V and therefore $H \cap V \leq H_0$. If N is contained in V then $H \cap N \leq H \cap V \leq H_0$ contradicting (2) together with the fact that G is a counterexample. If V is normal in G, it follows $V \cap N = 1$ by the minimal choice of N. □

(4) *Suppose that D is an AA_0-invariant normal subgroup of G and that $D \not\leq Z(G)$. Then*
(i) *$G = R_0D$ or $G = UD$.*
(ii) *D is not a minimal AA_0-invariant normal subgroup.*

Proof. Let $L := [D, R_0][D, U]$. Then the hypothesis and (1) yield $1 \neq L \trianglelefteq G$ and therefore without loss $N \leq L$. By Coprime Action (iii) we have $L \cap H = C_L(A_0B) = C_{[D,R_0]}(A_0B)C_{[D,U]}(A_0B)$.

Assume that $R_0D \neq G \neq UD$. Then $D_1 := [R_0D, A_0]$ is a proper AA_0-invariant subgroup of G which means $H \cap D_1 \leq H_0$. Now $R_0 = [R_0, A_0] \leq D_1 \trianglelefteq R_0D$ and it follows $[R_0, D] \leq [R_0, R_0D] \leq D_1$. Therefore we have $[R_0, D] \cap H \leq D_1 \cap H \leq H_0$ and similarly $[U, D] \cap H \leq H_0$. But as $N \cap H \leq L \cap H = C_{[D,R_0]}(A_0B)C_{[D,U]}(A_0B)$, this implies $N \cap H \leq ([R_0, D] \cap H)([U, D] \cap H) \leq H_0$ contradicting (2). As a consequence we have $G = R_0D$ or $G = UD$ as stated.

To prove (ii), suppose that D is minimal. Then since G is soluble, D is elementary abelian and by (i) we have two cases to consider:

If $G = R_0D$, then $R = R_0(D \cap R)$ by Dedekind's Law. The minimality of D implies $R = R_0$ or $R = R_0D = G$. Both cases lead to a contradiction.

If $G = UD$, we recall that U centralises a hyperplane Y of A. Applying Coprime Action (iv), we can also find a hyperplane Y_D of Y such that $C_D(Y_D) \neq 1$. But $C_D(Y_D) \trianglelefteq UD = G$ and then, by minimality, D centralises Y_D. Now $Y_D \in Hyp^2(A)$ is centralised by all of G which is not possible since G is a counterexample. □

(5) $N \leq Z(G) \leq H$.

Proof. For the first inclusion, we assume that $N \not\leq Z(G)$ and apply (4)(ii). This immediately yields a contradiction. Now all the subgroups of N are normal in G which implies $H \cap N = N$ or $H \cap N = 1$. The second case is not possible by (2). Thus $N \leq H$. Applying Coprime Action (ii) to the action of A_0B on $Z(G)$ yields $Z(G) = [Z(G), A_0B] \times C_{Z(G)}(A_0B)$. Since N is contained in the second factor, we obtain $[Z(G), A_0B] = 1$ (otherwise N could be chosen in $[Z(G), A_0B]$) and finally $Z(G) \leq H$. □

We note that $Z(G) \cap R_0 = 1$ because otherwise N could be chosen in $Z(G) \cap R_0$ contradicting (3).

Now we choose $M \trianglelefteq G$ to be AA_0-invariant, contained in R and such that $M/Z(G)$ is a minimal AA_0-invariant normal subgroup of $G/Z(G)$.

(6) $G = UM$, $[M, A_0] \neq 1$ *and* M *is abelian.*

Proof. By choice, $M \not\leq Z(G)$ and thus the first statement follows from (4)(i) and the fact that $R_0M \leq R \neq G$.

Since $M/Z(G)$ is elementary abelian, M is nilpotent. First assume that $[M, A_0] = 1$. Then $[A_0, M, G] = 1 = [M, G, A_0]$ and hence $[G, M] = [G, A_0, M] = 1$ by the 3-Subgroups-Lemma, a contradiction. Now $1 \neq [M, A_0] \leq M \cap R_0$ and therefore $M \cap R_0$ is a nontrivial normal subgroup of M. This implies that $M \cap R_0 \cap Z(M) \neq 1$ because M is nilpotent, and in particular $Z(M) \cap R_0 \neq 1$.

Assume that $Z(M) \cap R_0 \leq Z(G)$. Then $Z(M) \cap R_0 \leq Z(G) \cap R_0 = 1$, a contradiction (see above). So $Z(M) \cap R_0$ is not contained in $Z(G)$ and in particular $1 \neq Z(M) \not\leq Z(G)$. The choice of M forces $Z(M) = M$. □

(7) G *centralises a subgroup of* A *of index* p^2*.*

Proof. We recall that U centralises a hyperplane Y of A. Now Coprime Action (iv), applied to the action of Y on $M/Z(G)$, gives a hyperplane Y_M of Y such that $C_{M/Z(G)}(Y_M) \neq 1$. Since, by (6), M is abelian, this forces $[M, Y_M] < M$. But $G = UM$ implies that $[M, Y_M]$ is normal in G. By the minimal choice of M, we have $[M, Y_M] \leq Z(G)$. With $X := Y \cap Y_M$, we see $[G, X] = [UM, X] = [M, X] \leq Z(G)$ and therefore $[X, G, A_0] \leq [Z(G), A_0] = 1$. But $[A_0, X, G] = 1$ and then the 3-Subgroups-Lemma yields $[G, X] = [G, A_0, X] = 1$. By definition X has index p^2 in A. □

Now (7) contradicts the fact that G is a counterexample. This final contradiction proves the theorem.

□

A natural way to generalise the above theorem is to try and replace $Hyp^2(A)$ by $Hyp(A)$. However, this more general version does not hold, as the following example illustrates:

Let p, q and r be primes such that p divides $r - 1$ and $q - 1$ is divisible by both r and p. This choice is possible, e.g. $p = 3$, $r = 7$ and $q = 43$. Then let R be a cyclic group of order r and suppose that the cyclic group P of order p acts non-trivially on R. Moreover let V be a p-dimensional vectorspace over $GF(q)$ such that R and P act on V, $V = [V, R]$ and $\dim(C_V(P)) = 1$. These choices are possible because of the particular way we picked the primes. We set $G := VR$. Now since $R = [R, P] \leq [G, P] \trianglelefteq G$, we have $\langle R^G \rangle \leq [G, P]$. On the other hand, it follows $V = [V, R] \leq \langle R^G \rangle \leq [G, P]$ and therefore $G = [G, P]$.

Next we construct an elementary abelian p-group A which acts on G. We understand PR as a subgroup of $GL(V)$ and let Z be a cyclic group of order p of $Z(GL(V))$. Then Z centralises PR and acts as a non-trivial group of scalar automorphisms on V. Finally let U be a cyclic group of order p centralising Z, P, R and V. Set $A_0 := P$, $A := U \times Z \times A_0$ and $B = 1$. Now A is an elementary abelian group of order p^3 and the central product $AA_0 = A$ acts coprimely on the soluble p'-group G. As we have seen above, $[G, A_0] = G$. Let $H := C_G(A_0B) = C_G(A_0)$. Then we show

$$\langle [C_G(X), A_0] \cap H \mid X \in Hyp(A) \rangle = 1.$$

(But clearly $H \neq 1$.)

Assume that there exists an $X \in Hyp(A)$ such that $[C_G(X), A_0] \cap H \neq 1$. Then in particular $[C_G(X), A_0] \neq 1$ and thus $A_0 \not\leq X$. This implies $A = X \times A_0$ and it follows that $[C_G(X), A_0] \cap H \leq C_G(X) \cap C_G(A_0) \leq C_G(A)$. But by construction $C_G(A) = 1$, a contradiction.

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