Transitive permutation groups with trivial four point stabilizers

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Abstract

In this paper we analyze the structure of transitive permutation groups that have trivial four point stabilizers, but some nontrivial three point stabilizer. In particular we give a complete, detailed classification when the group is simple or quasisimple. This paper is motivated by questions concerning the relationship between fixed points of automorphisms of Riemann surfaces and Weierstraß points and is a continuation of the authors' earlier work.

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1. Introduction

In this article we classify transitive permutation groups where some nontrivial element fixes three points, but all four point stabilizers are trivial. The motivation for the study of this question is rooted in the theory of Riemann surfaces and their automorphisms. Schoeneberg proved that if an automorphism of a compact Riemann surface X of genus at least two fixes five or more points, then all of its fixed points are Weierstraß points. By definition Weierstraß points are analytically distinguished. Their significance for understanding the structure of $\mathrm{Aut}(X)$ became apparent when Schwarz used the action of $\mathrm{Aut}(X)$ on the set $\mathcal{W}(X)$ of Weierstraß points of X in order to establish the finiteness of $\mathrm{Aut}(X)$.

In the following we denote by $\mathrm{Fix}(X)$ the set of all $x \in X$ which appear as a fixed point of some nontrivial automorphism of X. This article continues the work begun in [19] whose ultimate aim is to identify the pairs $(X,\mathrm{Aut}(X))$ for which $\mathrm{Fix}(X) \not\subseteq \mathcal{W}(X)$. Our approach is to first identify the potential candidates for $\mathrm{Aut}(X)$, which is done by considering the action of $\mathrm{Aut}(X)$ from the point of view of abstract permutation groups. From this perspective, Schoeneberg's result naturally leads to the investigation of transitive permutation groups where nontrivial automorphisms have at most four fixed points.

In [19] we considered the case where at most two fixed points are allowed, which of course includes Frobenius groups. In the present paper we consider the next case, which means that all four point stabilizers are trivial, but some three point stabilizer is not. Our first two theorems classify the simple and almost simple permutation groups satisfying our hypotheses whereas the third result is a general structure theorem. The final case, where five point stabilizers are trivial and some four point stabilizer is not, is work in progress.

Theorem 1.1. *Suppose that G acts faithfully and transitively on a set Ω. Suppose that the four point stabilizers are trivial, but that some three point stabilizer is nontrivial. If G is simple and $\omega \in \Omega$, then one of the following holds:*

(i) *G_ω is not cyclic and one of the following is true:*
   (a) *$G \cong \mathcal{A}_5$, $|\Omega| = 15$ and $G_\omega \in Syl_2(G)$.*
   (b) *$G \cong \mathcal{A}_6$, $|\Omega| \in \{6, 15\}$ and G_ω is isomorphic to $\mathcal{A}_5$ or $\mathcal{S}_4$, respectively.*
   (c) *$G \cong PSL_2(7)$, $|\Omega| = 7$ and $G_\omega \cong \mathcal{S}_4$.*
   (d) *$G \cong \mathcal{A}_7$, $|\Omega| = 15$ and $G_\omega \cong PSL_2(7)$.*
   (e) *$G \cong PSL_2(11)$, $|\Omega| = 11$ and $G_\omega \cong \mathcal{A}_5$.*
   (f) *$G \cong M_{11}$, $|\Omega| = 11$ and $G_\omega \cong M_{10} \cong \mathcal{A}_6{}^{\cdot}2$.*

(ii) G_ω *is cyclic of order prime to* 6 *and one of the following is true:*
   (a) $G \cong PSL_3(q)$ *and* $|G_\omega| = q^2 + q + 1/(3, q-1)$.
   (b) $G \cong PSU_3(q)$ *and* $|G_\omega| = q^2 - q + 1/(3, q+1)$.
   (c) $G \cong PSL_4(3)$, $|\Omega| = 2^7 \cdot 3^6 \cdot 5$ *and* $|G_\omega| = 13$.
   (d) $G \cong PSU_4(3)$, $|\Omega| = 2^7 \cdot 3^6 \cdot 5$ *and* $|G_\omega| = 7$.
   (e) $G \cong PSL_4(5)$, $|\Omega| = 2^7 \cdot 3^2 \cdot 5^6 \cdot 13$ *and* $|G_\omega| = 31$.
   (f) $G \cong \mathcal{A}_7$, $|\Omega| = 360$ *and* $|G_\omega| = 7$.
   (g) $G \cong \mathcal{A}_8$, $|\Omega| = 2880$ *and* $|G_\omega| = 7$.
   (h) $G \cong M_{22}$, $|\Omega| = 2^7 \cdot 3^2 \cdot 5 \cdot 11$ *and* $|G_\omega| = 7$.

We remark that the point stbilizers in cases (i)(d) and (f) are examples of groups satisfying the main hypothesis of [19].

Theorem 1.2. *Suppose that G acts faithfully and transitively on a set Ω. Suppose that the four point stabilizers are trivial, but that some three point stabilizer is nontrivial. If G is almost simple, but not simple and if $\omega \in \Omega$, then one of the following holds:*

(i) *There is a prime p such that $G \cong Aut(PSL_2(2^p)) = Aut(SL_2(2^p))$, and Ω is the set of 1-spaces of the natural module of $SL_2(2^p)$. (This includes the example where $G \cong \mathcal{S}_5$ in its natural action on five points.)*
(ii) $G \cong PGL_3(q)$ *with* $(q-1, 3) = 3$, $|\Omega| = q^3(q^2-1)$ *and* G_ω *is cyclic of order* $(q^3-1)/(q-1)$.
(iii) $G \cong PGU_3(q)$ *with* $(q+1, 3) = 3$, $|\Omega| = q^3(q^2+1)$ *and* G_ω *is cyclic of order* $(q^3+1)/(q+1)$.

Almost 40 years ago Pretzel and Schleiermacher [20] studied an important special case of our present situation, namely they investigated transitive permutation groups in which, for a fixed prime p, every nontrivial element fixes either p or zero points. (They call these groups $(0,p)$-groups.) They stated that one would like to prove that either G contains a regular normal subgroup of index p or that G contains a normal subgroup F of index p such that F acts as a Frobenius group on its p orbits. Although our hypothesis is more general, the influence of the work of Pretzel and Schleiermacher is visible in several places in this article.
Before we state our main theorem, we recall that $H \leq G$ is said to be **strongly embedded in** G if H has even order and if, for all $g \in G \setminus H$, the intersection $H \cap H^g$ has odd order.

Theorem 1.3. *Suppose that G acts faithfully and transitively on a set Ω. Suppose that the four point stabilizers are trivial, but that some three point stabilizer is nontrivial. Then G has order divisible by 3 and if $\omega \in \Omega$, then one of the following holds:*

(i) $|G_\omega|$ *is even and one of the following is true:*
   (a) *G has a normal 2-complement.*
   (b) *G has dihedral or semidihedral Sylow 2-subgroups and 4 does not divide $|G_\omega|$. In particular G_ω has a normal 2-complement.*
   (c) *G_ω contains a Sylow 2-subgroup S of G and G has a strongly embedded subgroup.*
   (d) *$|G : G_\omega|$ is even, but not divisible by 4 and G has a subgroup of index 2 that has a strongly embedded subgroup.*
(ii) $|G_\omega|$ *is odd and one of the following is true:*
   (a) *G has a normal subgroup R of order 27 or 9, and G/R is isomorphic to $\mathcal{S}_3$, $\mathcal{A}_4$, $\mathcal{S}_4$, to a fours group or to a dihedral group of order 8.*
   (b) *G has a regular normal subgroup.*
   (c) *G has a normal subgroup F of index 3 which acts as a Frobenius group on its three orbits.*
   (d) *G has a normal subgroup N which acts semiregularly on Ω such that G/N is almost simple and G_ω is cyclic.*

This paper is structured as follows: After fixing some standard notation, we introduce examples which are typical for the situation that we analyze later on. Then we move on to proving results about the local structure of the groups under consideration and collect enough information to bring the Classification of Finite Simple Groups into action in an efficient way. In our reduction to almost simple groups it is necessary to consider normal subgroups satisfying the main hypothesis of [19], as can be seen for example in the proof of Lemma 2.23. This is one of the many places where the interaction between the individual pieces of the project outlined in the beginning of this section becomes visible.
Sections 3 to 5 deal with particular classes of simple and quasisimple groups. Then in Section 6 we collect this information for the proof of Theorems 1.1 and 1.2. Finally we give the proof of Theorem 1.3. and we explain how the possibilities arising in Theorem 1.3.2 (b) resemble the examples given in Section 2.1.

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2. Preliminaries

In this paper, by “group” we always mean a finite group, and by “permutation group” we always mean a group that acts faithfully.
In this chapter let Ω denote a finite set and let G be a permutation group on Ω.

Notation

Let $\omega \in \Omega$ and $g \in G$, and moreover let $\Lambda \subseteq \Omega$ and $H \leq G$.
Then $H_\omega := \{h \in H \mid \omega^h = \omega\}$ denotes the stabilizer of ω in H,
$\mathrm{fix}_\Lambda(H) := \{\omega \in \Lambda \mid \omega^h = \omega \text{ for all } h \in H\}$ denotes the fixed point set of H in Λ and we write $\mathrm{fix}_\Lambda(g)$ instead of $\mathrm{fix}_\Lambda(\langle g \rangle)$.
We write ω^H for the H-orbit in Ω that contains ω.
Whenever $n, m \in \mathbb{N}$, then we denote by (n, m) the greatest common divisor of n and m. Moreover we write $\mathbb{Z}_n$ (or sometimes just n) for a cyclic group of order n.

Lemma 2.1. *Suppose that G has a nontrivial proper subgroup H such that the following holds: Whenever $1 \neq X \leq H$, then $N_G(X) \leq H$.*
Then G is a Frobenius group with Frobenius complement H.

Proof. This is Lemma 2.1 in [19] . □

Lemma 2.2. *Suppose that G acts transitively on the set Ω and that $\alpha \in \Omega$. Let $1 \neq X \leq G_\alpha$. Then the following hold:*

(a) *If α is the unique fixed point of X, then $N_G(X) \leq G_\alpha$.*
(b) *If X has exactly two fixed points, then $N_{G_\alpha}(X)$ has index at most 2 in $N_G(X)$.*
(c) *If X has exactly three fixed points, then $N_{G_\alpha}(X)$ has index at most 3 in $N_G(X)$.*

Proof. Assertion (a) holds in any permutation group. As $N_G(X)$ acts on $\mathrm{fix}_\Omega(X)$, we see in (b) that $N_G(X)/N_{G_\alpha}(X)$ is isomorphic to a subgroup of $\mathcal{S}_2$. In (c) let K denote the kernel of the action of $N_G(X)$

on $\text{fix}_\Omega(X)$. Then $N_G(X)/K$ is isomorphic to a subgroup of $\mathcal{S}_3$. If this factor group is isomorphic to a proper subgroup of $\mathcal{S}_3$, then (c) holds. Otherwise we note that there is $g \in N_G(X)$ that fixes α and interchanges the other two points in $\text{fix}_\Omega(X)$. Hence $g \in G_\alpha$ and $|N_G(X) : N_{G_\alpha}(X)| = 3$. So again (c) holds. □

Lemma 2.3. *Suppose that G is a $\{2,3\}'$-group and that G acts transitively, nonregularly on a set Ω such that four point stabilizers are trivial. Then G is a Frobenius group.*

Proof. This follows from Lemmas 2.2 and 2.1. □

Hypothesis 2.4. *Suppose that (G, Ω) is such that G acts transitively, nonregularly on the set Ω, that four point stabilizers are trivial and that some three point stabilizer is nontrivial.*

Note that Hypothesis 2.4 implies that $|\Omega| \geq 5$ because nontrivial permutations on four or fewer points can have at most two fixed points.

Lemma 2.5. *If (G, Ω) satisfies Hypothesis 2.4 and $\omega \in \Omega$, then one of the following is true:*

(1) *$|G_\omega|$ is even.*
(2) *G_ω is a Frobenius group of odd order, where the Frobenius complements are three point stabilizers.*
(3) *$|\mathit{fix}_\Omega(G_\omega)| = 3$ and $|G_\omega|$ is odd.*

Proof. We suppose that $|G_\omega|$ is odd and that $|\text{fix}_\Omega(G_\omega)| \neq 3$. Thus we need to show that the statements in (2) hold, in particular that G_ω is a Frobenius group.
Hypothesis 2.4 implies that there exists a set Δ of size 3 such that $\omega \in \Delta$ and such that the point-wise stabilizer H of Δ in G is nontrivial. Let $1 \neq X \leq H$. Then X acts semiregularly on $\Omega \setminus \Delta$ and $N_G(X)$ leaves Δ invariant. Since $|G_\omega|$ is odd and $|\Delta| = 3$, this implies that $N_G(X)$ has odd order. Hence $|N_G(X) : N_H(X)| \in \{1, 3\}$ and this holds for all $1 \neq X \leq H$.
Next we observe that if there is a nontrivial subgroup X of H such that $|N_G(X) : N_H(X)| = 3$, then all $g \in N_G(X) \setminus N_H(X)$ act transitively on Δ; i.e. they fix no point of Δ.
Thus $|N_{G_\omega}(X) : N_H(X)| = 1$ for all $1 \neq X \leq H$. As $|\text{fix}_\Omega(G_\omega)| \neq 3$, we know that $H < G_\omega$ and therefore Lemma 2.1 implies that G_ω is a Frobenius group where H is a Frobenius complement. This is our claim. □

We recall that a subgroup H of G is t.i. (short for "trivial intersection“) if and only if, for all $g \in G$, either $H \cap H^g = 1$ or $H^g = H$.

Corollary 2.6. *Suppose that (G, Ω) satisfies Hypothesis 2.4 and that $|\Omega| \geq 7$. Let $\omega \in \Omega$ and suppose that G_ω is a Frobenius group of odd order with a Frobenius complement H that is a three point stabilizer. Let $\Lambda := G/H$ (with the natural action of G by right multiplication). Then (G, Λ) satisfies Hypothesis 2.4. Moreover if $h \in G^\#$ stabilizes Λ, then $|\mathit{fix}_\Lambda(h)| = 3$.*

Proof. As G is not a Frobenius group by Hypothesis 2.4 and the point stabilizers have odd order, Lemma 2.1 implies that there exists some $1 \neq X \leq H$ such that $|N_G(H) : N_H(X)| = 3$. Now X acts semiregularly on $\Omega \setminus \text{fix}_\Omega(H)$, and since $|\Omega| \geq 7$, this implies that the set-wise stabilizer of $\text{fix}_\Omega(H)$ is properly larger than H. Therefore $|N_G(H) : H| = 3$. Also if $h \in H \cap H^g$ and $H \neq H^g$, then h fixes $\text{fix}_\Omega(H) \cup \text{fix}_\Omega(H^g) \neq \text{fix}_\Omega(H)$ and hence $h = 1$ by Hypothesis 2.4. So H is t.i. and $|N_G(H) : H| = 3$, which implies our claim. □

2.1. **Examples.** Here we describe some series of examples for Hypothesis 2.4. In particular we classify all possibilities where Ω has five or six elements.

Lemma 2.7. *If (G, Ω) satisfies Hypothesis 2.4 and $|\Omega| \leq 6$, then one of the following is true:*

(1) *$|\Omega| = 5$ and $G = \mathcal{S}_5$.*

(2) $|\Omega| = 6$ *and* $G = \mathcal{A}_6$.
(3) $|\Omega| = 6$ *and* $\mathcal{A}_3 \wr \mathcal{S}_2 \leq G \leq (\mathcal{S}_3 \wr \mathcal{S}_2) \cap \mathcal{A}_6$ *(two possibilities in total).*

Proof. Hypothesis 2.4 implies that some element $g \in G$ has three fixed points on Ω and that $|\Omega| \geq 5$. In the following we view G as a subgroup of $\mathcal{S}_6$.

If $|\Omega| = 5$, then g is a 2-cycle. As 5 is prime, the hypothesis that G is transitive implies that G is primitive. Now G is a primitive permutation group on five points that contains a transposition, so $G = \mathcal{S}_5$ as stated in (1).

If $|\Omega| = 6$, then g is a 3-cycle. Without loss $g = (456)$, so g lies in the point stabilizer G_1. The 2-cycles in $\mathcal{S}_6$ have four fixed points, therefore Hypothesis 2.4 implies that $(1,2)^{\mathcal{S}_6} \cap G = \varnothing$. If G acts primitively on Ω, then it follows that $G = \mathcal{A}_6$ which leads to (2). Possibility (2) does in fact occur as an example, as an inspection of the conjugacy classes shows. If G is not primitive on Ω, then, since G contains a 3-cycle, it is a subgroup of $\mathcal{S}_3 \wr \mathcal{S}_2$. Now $|G| = 6 \cdot |G_1| \geq 18$ which implies that $\mathcal{A}_3 \wr \mathcal{S}_2 \leq G$. On the other hand $G \neq \mathcal{S}_3 \wr \mathcal{S}_2$ as G does not contain 2-cycles. Therefore $G \leq (\mathcal{S}_3 \wr \mathcal{S}_2) \cap \mathcal{A}_6$ and (3) follows. □

Having considered small examples we also look at sharply 4-transitive permutation groups. We note that any element of such a group that fixes four points is the identity element. Moreover a three point stabilizer in such a group is transitive on the set of points that are not fixed, and in particular it is nontrivial if the size of the set is at least 5. The next result is due to Jordan and can be found as Theorem 3.3 in Chapter XII of [14].

Lemma 2.8. *If G is sharply 4-transitive, then G is one of $\mathcal{S}_4, \mathcal{S}_5, \mathcal{A}_6, M_{11}$.*

Thus we see that $\mathcal{S}_5, \mathcal{A}_6, M_{11}$ in their actions on $5, 6$ or 11 points, respectively, are examples satisfying Hypothesis 2.4.

Lemma 2.9. *Suppose that P is a 3-group of order at least 27 and that $H \leq P$ is a subgroup of order 3 such that $|C_P(H)| = 9$. Let Ω denote the set of right cosets of H in P. Then (P, Ω) satisfies Hypothesis 2.4.*

Proof. Only the conjugates of elements of H have fixed points on Ω. If $h \in H^{\#}$, then $|\mathrm{fix}_\Omega(h)| = |N_P(H) : H|$. The outer automorphism group of H has order coprime to 3, therefore $N_P(H) = C_P(H)$ and our hypothesis on $|C_P(H)|$ implies that $|N_P(H) : H| = 3$. This proves our claim. □

We recall that a nonabelian p-group P is of maximal class if it possesses a p-element x such that $|C_P(x)| = p^2$. Extraspecial p-groups of order p^3 are examples of this. The 2-groups of maximal class are dihedral, quaternion or semidihedral, whereas for $p > 2$ there are many other possibilities (see [12], III.14). Lemma 2.9 implies that 3-groups of maximal class all give rise to examples for Hypothesis 2.4. The next three classes of examples are variants of those introduced in [19].

Lemma 2.10. *Let p be a prime and let A denote the additive group, M the multiplicative group, and $\mathcal{G}$ the Galois group of a finite field of order 3^p. Let G be the semidirect product $(A : M) : \mathcal{G}$ and $G_\omega := M : \mathcal{G}$. Let Ω denote the set of right cosets of G_ω in G. Then (G, Ω) satisfies Hypothesis 2.4.*

Proof. We note first that A is a regular normal subgroup of G in its action on Ω. Thus if $g \in G_\omega^{\#}$, then $\mathrm{fix}_\Omega(g) = |C_A(g)|$. Our claim follows as 1 and 3 are the only possible values for $|C_A(g)|$. □

Lemma 2.11. *Let F be a Frobenius group with kernel K and complement H and let Z be a cyclic group of order 3. Let $G := Z \times F$ and let Ω be the set of right cosets of H. Then the pair (G, Ω) satisfies Hypothesis 2.4.*

Proof. The subgroup K has three orbits on Ω which are transitively permuted by Z and fixed set-wise by elements of H. If $h \in H^{\#}$, then h fixes exactly one point on each K-orbit. Our claim follows. □

We remark that in this last example the number of fixed points of an element is either 0 or 3.

Lemma 2.12. *Let p, r be primes and let K be a field of order p^{3r}. Let A and M be the additive respectively the multiplicative group of K and let H be a subgroup of the Galois group of K of order 3. Let Ω be the set of right cosets of M in $G := (A : M) : H$. Then (G, Ω) satisfies Hypothesis 2.4.*

Proof. We first observe that A has three regular orbits in its action on Ω which are permuted transitively by H; i.e. $A \rtimes H$ acts regularly on H. If $m \in M^{\#}$ and $\alpha \in \text{fix}_\Omega(m)$, then $\alpha^H \subseteq \text{fix}_\Omega(m)$ because H normalizes M and M is cyclic. The claim follows. □

We close this section with a result by Fukushima [8], generalizing a result of Rickman [22], which leads to yet another fairly general class of examples.

Lemma 2.13. *Let H be a finite group and $\alpha \in Aut(H)$ of odd prime order. If the order of α is coprime to $|H|$ and if $C_H(\alpha)$ is a 3-group, then H is solvable and more specifically $H = O_{3,3'}(H)C_H(\alpha)$. If $G := H \rtimes \langle\alpha\rangle$, if moreover Ω is the set of right cosets of $\langle\alpha\rangle$ in G and $|C_H(\alpha)| = 3$, then the pair (G, Ω) satisfies Hypothesis 2.4.*

Proof. The first statement is the combined content of Theorem 1 and Proposition 3 of Fukushima [8], whereas the second is a corollary of the first. □

2.2. More general properties following from our hypothesis.

Lemma 2.14. *Suppose that Hypothesis 2.4 holds. Then $|Z(G)| \in \{1, 3\}$.*

Proof. Let $\alpha \in \Omega$. As G acts faithfully on Ω, we know that $Z(G)$ intersects G_α trivially. Let $x \in G_\alpha$ be an element with exactly three fixed points. Then $Z(G) \leq C_G(x)$ and hence Lemma 2.2 (c) implies that $|Z(G)| \in \{1, 3\}$. □

Lemma 2.15. *Suppose that Hypothesis 2.4 holds and let $\alpha \in \Omega$. Then the following hold:*

(a) *If some 2-element in G_α has exactly three fixed points on Ω, then G_α contains a Sylow 2-subgroup of G.*
(b) *If some 3-element in G_α has exactly three fixed points, then 3 divides $|\Omega|$. In particular, in this case, G_α does not contain a Sylow 3-subgroup of G.*
(c) *For all primes $p \geq 5$ that divide $|G_\alpha|$, some Sylow p-subgroup of G is contained in G_α.*

Proof. Suppose that $x \in G_\alpha$ is a 2-element with exactly three fixed points. As x has orbits of 2-power lengths on the set of points that are not fixed, it follows that $|\Omega|$ is odd. Therefore $|G : G_\alpha|$ is odd and G_α contains a Sylow 2-subgroup of G.
For (b) suppose that $y \in G_\alpha$ is a 3-element with exactly three fixed points on Ω. The remaining orbits of y on Ω have 3-power lengths and therefore $|\Omega|$ is divisible by 3. This means that $|G : G_\alpha|$ is divisible by 3 and in particular G_α does not contain a Sylow 3-subgroup of G.
Finally suppose that $p \in \pi(G_\alpha)$ is such that $p \geq 5$. Let $x \in G_\alpha$ be an element of order p and let $x \in P \in \text{Syl}_p(G)$. Then Lemma 2.2 (b) implies first that $Z(P) \leq G_\alpha$ and then that $P \leq G_\alpha$. This finishes the proof. □

Lemma 2.16. *Suppose that Hypothesis 2.4 holds and that $N \trianglelefteq G$ is such that all N-orbits on Ω have size 2. Let $\tilde{\Omega}$ denote the set of N-orbits of Ω and let K denote the kernel of the action of G on $\tilde{\Omega}$. Then $|\Omega| \leq 6$ and (G, Ω) is as in Lemma 2.7.*

Proof. By hypothesis N is a 2-group. If N has order 2, then $N \leq Z(G)$ and this contradicts Lemma 2.14. Hence N has order at least 4. We set $m := |\tilde{\Omega}|$ and we simplify notation by denoting the elements of Ω by $1, ..., 2m$ and by expressing elements of G as elements from $\mathcal{S}_{2m}$. We write $\tilde{\Omega} = \{\{1, 2\}, ..., \{2m-1, 2m\}\}$. Now it is sufficient to prove that $m \leq 3$. Hence we assume otherwise. Our fixed point hypothesis tells us that all elements from $N^{\#}$ are a product of at least $m - 1$ disjoint transpositions. Suppose that $t \in N^{\#}$

induces $(1,2)\cdots(2m-1,2m)$ on Ω and let $s \in N^{\#}$ be such that $s \neq t$. On each element of $\tilde{\Omega}$, only one nontrivial action of s is possible, namely the action of the corresponding transposition. If t and s both induce a transposition on $\{1,2\}$, then $s \cdot t$ fixes 1 and 2. Otherwise s fixes 1 and 2 and we have the same two possibilities on $\{3,4\}$. As $|\Omega|$ is even, all elements from $N^{\#}$ can only have zero or two fixed points, so looking at the remaining elements of $\tilde{\Omega}$ yields that s or $s \cdot t$ fixes at least four points on Ω. This is impossible. A similar argument applies if we choose t to already have two fixed points on Ω. Hence $m \leq 3$ as stated. □

Lemma 2.17. *Suppose that Hypothesis 2.4 holds. Let $S \in Syl_2(G)$ and $\alpha \in \Omega$. Then one of the following holds:*

(1) *G_α has odd order.*
(2) *S is dihedral or semidihedral and $|S_\alpha| = 2$. In particular G_α has a normal 2-complement.*
(3) *$|S| \geq 4$, there is a unique S-orbit on Ω of length 2, and all other S-orbits have length $|S|$. Then $O_2(G) = 1$ or $O_2(G)$ is a fours group and $|\Omega| \leq 6$.*
(4) *$|\Omega|$ is odd.*

Proof. Suppose that (1) does not hold. Then with Sylow's Theorem we may suppose that $S_\alpha \neq 1$. Let $\Delta := \alpha^S$ and let $n, m \in \mathbb{N}_0$ be such that $|S_\alpha| = 2^n$ and $|S : S_\alpha| = 2^m$. First suppose that $m \geq 2$. Let d denote the number of fixed points of S_α on Δ and choose $a \in \mathbb{N}_0$ such that $|\Delta| = d + a \cdot 2^n$. As $n \geq 1$ and $|\Delta| = 2^m \geq 4$, we see that $d = 2$ and hence $2^m = 2 \cdot (1 + a \cdot 2^{n-1})$. This implies that $n = 1$ and that $a = 2^{m-1} - 1$, so Lemma 2.2 (b) forces $|C_S(S_\alpha)| \leq 4$. Thus either S is of order 2 or of maximal class. For (2) we assume that S is quaternion. Then $|S| \geq 8$ and $|S_\alpha| = 2$, in particular G_α contains the unique involution in S. But then Lemma 2.2 forces a subgroup of index 2 of S to be contained in G_α, which is impossible. Now 11.9 in [12] yields that S is dihedral or semidihedral. Moreover S_α has order 2 which means that G_α has cyclic Sylow 2-subgroups and hence a normal 2-complement. This is (2).
Now we suppose that $m \leq 1$. Then (4) holds or S_α has index exactly 2 in S. We look at the second case more closely. By Lemma 2.2 we know that there exists $\beta \in \Omega$ such that $\alpha \neq \beta$, $S_\alpha = S_\beta$ and all elements in $S \backslash S_\alpha$ interchange α and β. As S_α already has two fixed points and $|\Omega|$ is even in this case, it follows that S_α has exactly two fixed points and hence it has regular orbits on the remaining points of Ω. It follows that $\Delta := \{\alpha, \beta\}$ is the unique S-orbit of length 2 and all other orbits have length $|S|$. As $|\Omega| > 2$ by hypothesis, there exists a regular S-orbit of Ω and this means that we may choose $g \in G$ such that $\Delta \cap \Delta^g = \varnothing$. Then $D := S \cap S^g$ stabilizes the set $\Delta \cup \Delta^g$ of size 4. Moreover D acts faithfully on this set by Hypothesis 2.4 and it fixes the subsets Δ and Δ^g. Thus $|D| \leq 4$ and in particular $O_2(G)$ has order at most 4. The point stabilizers have index 2 in $O_2(G)$ and hence $O_2(G)$ has orbits on Ω of length 2. Now Lemmas 2.16 and 2.14 imply all the remaining details of (3). □

Lemma 2.18. *Suppose that Hypothesis 2.4 holds. Let $\alpha \in \Omega$ and suppose further that G is simple. Then one of the following holds:*

(1) *G_α has odd order.*
(2) *G_α contains a Sylow 2-subgroup of G. In particular G_α contains an involution from every conjugacy class.*
(3) *G has dihedral or semidihedral Sylow 2-subgroups, in particular G is isomorphic to $\mathcal{A}_7$ or M_{11} or there exists an odd prime power q such that $G \cong PSL_2(q)$, $PSU_3(q)$ or $PSL_3(q)$.*

Proof. We go through the cases in Lemma 2.17 with the special hypothesis that G is simple. The cases (1) and (4) from Lemma 2.17 give exactly the conclusions (1) and (2) here. If (2) from the lemma holds, then we use the classification of the corresponding groups by Gorenstein-Walter and Alperin-Brauer-Gorenstein, respectively (see [10] and [1]). This gives the possibilities in (3), so it is only left to prove that Case (3) of Lemma 2.17 does not occur in a simple group.

Assume otherwise and let $S \in \mathrm{Syl}_2(G)$ be such that $S_\alpha \neq 1$ and S has order at least 4. Moreover we assume that S has a unique orbit of length 2 on Ω and all other orbits have length $|S|$. We choose $\beta \in \Omega$ such that $\{\alpha, \beta\}$ is the S-orbit of length 2, in particular $S_\alpha = S_\beta$ has index 2 in S.
Let $t \in S \setminus S_\alpha$. Then t interchanges α and β and it fixes all orbits of length $|S|$. As S is not cyclic by Burnside's Theorem (recall that G is simple), it follows that t acts as an even permutation on each S-orbit and hence on $\Omega \setminus \{\alpha, \beta\}$. Thus t acts as an odd permutation on Ω. This means that G possesses a normal subgroup of index 2. But $|G| \geq 4$ and G is simple, so this is impossible. □

Lemma 2.19. *Suppose that Hypothesis 2.4 is satisfied and that $|\Omega|$ is odd. Then one of the following holds:*

(1) *G has odd order and $3 \in \pi(G)$.*
(2) *G has a strongly embedded subgroup.*
(3) *G has a normal 2-complement. In particular G is solvable.*
(4) *G has a normal subgroup G_0 of index 2 that has a strongly embedded subgroup.*

In particular, if G is simple, then G is isomorphic to $\mathcal{A}_7$, to M_{11} or there exists a prime power q such that G is isomorphic to $PSL_2(q)$, to $Sz(q)$, to $PSU_3(q)$ or to $PSL_3(q)$ (with q even).

Proof. Let $\alpha \in \Omega$. Then the transitivity of G on Ω yields that $|\Omega| = |\alpha^G| = |G : G_\alpha|$ and hence $|G| = |\Omega| \cdot |G_\alpha|$. In particular G_α contains a Sylow 2-subgroup of G.
Suppose that G_α has odd order. Then G has odd order, but it is not a Frobenius group and therefore Lemma 2.3 forces $3 \in \pi(G)$. This is (1).
Next suppose that G_α has even order and let $S \in \mathrm{Syl}_2(G)$ be contained in G_α. We look at the orbits of S on $\Gamma := \Omega \setminus \{\alpha\}$. As $|\Omega|$ is odd, there are three possibilities: S fixes two points on Γ or every element in $S^\#$ is fixed point free on Γ or S has a unique orbit of length 2 on Γ. Suppose that every element of $S^\#$ fixes only α and let $H := G_\alpha$. Let $g \in G \backslash H$ and suppose that $x \in H \cap H^g$ is a 2-element. Then x has at least two fixed points on Ω, namely α and α^g, and in the present case this forces $x = 1$ (because without loss $x \in S$). It follows that $H \cap H^g$ has odd order and hence H is a strongly embedded subgroup of G. This is (2).
Next suppose that S fixes three points. Let Δ denote this fixed point set. Let M_0 denote the point-wise stabilizer of Δ and let $M := N_G(M_0)$. We show that $N_G(M)$ is strongly embedded:
First Lemma 2.2 and the fact that $S \leq M_0$ yield that M has index at most 3 in $N_G(M)$. Moreover $|\Omega|$ is odd, so in particular M does not have two orbits of length 3 on Ω, but it has a unique orbit of length 3 on Ω. (Otherwise S has too many fixed points.) Therefore $N_G(M)$ stabilizes Δ and is hence contained in M.
Next we let $g \in G \setminus M$ and we choose a 2-element $t \in M \cap M^g$. Without loss $t \in S$. Then t stabilizes Δ and Δ^g. These sets have size 3 and therefore t has a fixed point on both of them, moreover it fixes Δ point-wise. But also $t \in S^g$ and therefore t fixes Δ^g point-wise. The previous paragraph showed that $\Delta \neq \Delta^g$, hence t fixes at least four points and this forces $t = 1$. Now we have that $M = N_G(M)$ is strongly embedded in G.
The last case is that S has a unique orbit $\{\beta, \gamma\}$ of length 2 on Γ. Then a subgroup of index 2 of S fixes three points and therefore the orbit lengths of S on Ω are 1, 2 and $|S|$.
If S is cyclic, then by Burnside's Theorem (3) holds. So we suppose that S is not cyclic. Then, in the action on $\Omega \setminus \{\alpha, \beta, \gamma\}$, the elements of S are even permutations. Thus the elements of $S^\#$ are odd permutations in their action on Γ (and on Ω), which means that G has a subgroup G_0 of index 2. Let $S_0 := S \cap G_0$. If $\Omega \neq \alpha^{G_0}$, then G_0 makes two orbits on Ω which are interchanged by an element in $N_G(S_0) \backslash G_0$. But $S \cap G_0$ has different numbers of fixed points on these orbits, which is impossible. Thus $\Omega = \alpha^{G_0}$ and so (G_0, Ω) satisfies Hypothesis 2.4. Moreover S_0 fixes three points of Ω and we already showed that this implies that G_0 has a strongly embedded subgroup.

If G is simple, then G is nonabelian because of its nonregular action on Ω and hence only case (2) is possible. Then the main result in [3] leads to the groups listed. $\square$

Lemma 2.20. *Suppose that Hypothesis 2.4 holds and that $P \in Syl_3(G)$. Let $\alpha \in \Omega$. Then one of the following holds:*

1. *G_α is a $3'$-group.*
2. *P is of maximal class, $|P_\alpha| = 3$ and P_α fixes three points.*
3. *$|P : P_\alpha| = 3$, P has order at least 9 and P has exactly one orbit of size 3 on Ω, all remaining orbits have size $|P|$. Moreover, in this case, $O_3(G)$ is elementary abelian of order at most 9.*
4. *3 does not divide $|\Omega|$.*

Proof. Suppose that (1) does not hold. Then $3 \in \pi(G_\alpha)$ and so we may suppose that $P_\alpha \neq 1$.
Set $\Delta = \alpha^P$ and let $n \in \mathbb{N}$ be such that $|\Delta| = 3^n$. Let $m \in \mathbb{N}$ be such that $|P_\alpha| = 3^m$. First we suppose that $n \geq 2$. We set $d := |\mathrm{fix}_\Omega(P_\alpha)|$ and note that $3^n = d + a3^m$ for some integer a. The fact that $\alpha \in \mathrm{fix}_\Omega(P_\alpha)$ together with Hypothesis 2.4 implies that $1 \leq d \leq 3$. Thus $d = 3$ as P is a 3-group, and this means that P_α fixes three points of Ω. We obtain that $3^n = 3 + a3^m$ and thus $n = 2$, $m = 1$ and $a = 3^{n-1} - 1$.
It follows that $|P_\alpha| = 3$ and now Lemma 2.2 implies that $|N_P(P_\alpha)| \leq 9$. As stated after Lemma 2.9, we now have that P has maximal class. So we proved (2) in this case.
Next suppose that $n \leq 1$. Then $|P : P_\alpha| \leq 3$ which means that G_α contains a subgroup of index at most 3 of P. Therefore (3) or (4) holds, and it is left to analyze Case (3) more closely. First we notice that $|P| \geq 9$ and P_α fixes three points, so P has one orbit Δ of size 3 (consisting of these three points) and all other orbits are of size $|P|$. As $|\Omega| \geq 5$, there exists a regular P-orbit and hence we may choose $g \in G$ such that $\Delta \cap \Delta^g = \varnothing$. Then $D := P \cap P^g$ stabilizes the set $\Delta \cup \Delta^g$ of size 6 and it acts faithfully on it by Hypothesis 2.4. It follows that D is isomorphic to a subgroup of $\mathcal{S}_6$ and hence it is elementary abelian of order at most 9. As $O_3(G) \leq D$, all statements in (3) are proved. $\square$

Lemma 2.21. *Suppose that Hypothesis 2.4 holds and let $\alpha \in \Omega$. If $E(G) \neq 1$, then $E(G) \cap G_\alpha \neq 1$.*

Proof. Assume that $E(G) \neq 1$, but $E(G) \cap G_\alpha = 1$ and let E be a component of G. Let $x \in G_\alpha$ be of prime order p. First we show that x normalizes E:
Assume otherwise and let $E_1, ..., E_p$ denote the x-conjugates of E, where $E = E_1$. Then $L := E_1 \cdots E_p$ is an x-invariant product of components. Let $e \in E$. Then $e \cdots e^{x^{p-1}} \in C_L(x)$. By Lemma 2.2 a subgroup of index 2 or 3 of $C_L(x)$ is contained in G_α and so, by assumption, we see that e has order 2 or 3. It follows that E is a $\{2, 3\}$-group. But this is a contradiction because E is not solvable.
Thus x normalizes E and Lemma 2.2 yields that G_α contains a subgroup of index 2 or 3 of $C_E(x)$. By assumption (and as E is not nilpotent) $C_E(x)$ has order 2 or 3. If the order is 2, then [9] implies that E is solvable, which is a contradiction. Hence $|C_E(x)| = 3$. If $o(x) \neq 3$, then the main result in [22] yields that E is solvable again, which is impossible.
We deduce that $o(x) = 3$ and now Theorem 2 in [4] yields that E is solvable, which is again a contradiction. $\square$

Lemma 2.22. *Suppose that Hypothesis 2.4 holds and that $E(G) \neq 1$. Then G has a unique component.*

Proof. We assume otherwise. Let E denote a component of G and let L be a product of components such that $E(G) = E \cdot L$. With Lemma 2.21 we let $\alpha \in \Omega$ and $1 \neq e \in E(G) \cap G_\alpha$. Let $a \in E$ and $b \in L$ be such that $e = a \cdot b$. Lemma 2.2 implies that a subgroup of index at most 3 of $C_E(e) = C_E(a)$ and of $C_L(e) = C_L(b)$, respectively, lies in G_α. Moreover G_α does not contain any normal subgroup of G and hence G_α does not contain a component, again with Lemma 2.2. As G has more than one component by assumption, it follows that all components intersect G_α trivially. In particular $a, b \notin G_\alpha$ and the groups $C_E(a)$ and $C_L(b)$ have order 2 or 3. The first case is impossible by Burnside's Theorem, and in the second case the main result in [7] forces $E \cong L \cong$PSL$_2(7)$ and in particular $G_\alpha \cap E(G) = \langle e \rangle$ and e fixes three points of Ω. From the structure of PSL$_2(7)$, there is an involution $t \in EL$ that inverts e and hence fixes one of the three fixed points of e. Let γ denote this fixed point and let $g \in G$ be such that $\alpha^g = \gamma$. Then $G_\gamma \cap E(G)$ contains elements of order 3 and 2, which is impossible. □

Lemma 2.23. *Suppose that Hypothesis 2.4 holds and that E is a component of G. Then one of the following holds:*

(a) *There exists a power q of 2 such that $E \cong$PSL$_2(q)$, $|G : E|$ is prime and every element from $G \setminus E$ induces a field automorphism on E. For all $\alpha \in \Omega$, we have that $|G_\alpha| = q \cdot (q-1) \cdot |G : E|$ and moreover E_α does not contain any elements that fix three points. This includes the special case where $E \cong \mathcal{A}_5$ and $G \cong \mathcal{S}_5$.*

(b) *There exists $\alpha \in \Omega$ such that (E, α^E) satisfies Hypothesis 2.4.*

Proof. As $E(G) \neq 1$ by hypothesis, we know from Lemmas 2.21 and 2.22 that E is the unique component of G and that E intersects the points stabilizers nontrivially. Hence let $\alpha \in \Omega$ and $\Delta := \alpha^E$. Then $E_\alpha \neq 1$ and therefore E acts transitively and nonregularly on Δ. Moreover E acts faithfully because $E \trianglelefteq G$. As E is a component and thus not solvable, we know that $|\Delta| \geq 5$ and therefore (E, Δ) satisfies Hypothesis 2.4.
Suppose that E does not have any element that fixes three points on Δ. Then (E, Δ) satisfies Hypothesis 1.1 from [19] and in particular $Z(E) = 1$ by Lemma 2.8 in [19] and Lemma 2.14. Thus E is simple and Theorem 1.2 in [19] applies. We refer to Theorem 5.6 in the same paper for details on the possible action of E on Δ. We also note that Lemmas 2.2 (a) and (b) and 2.14 force $F(G) = 1$.

Case 1: $E \cong \mathcal{A}_5$.
We know that $E = F^*(G)$ and hence G acts faithfully on E. As (E, Δ) does not satisfy Hypothesis 2.4, but (G, Ω) does, it follows that $G \neq E$ and hence $G \cong \mathcal{S}_5$ as stated in (a).

Case 2: $E \cong$PSL$_3(4)$.
Here the only possibility for the action is that E_α has order 5. In particular E_α is a Sylow subgroup of E. A Frattini argument yields that $G = EN_G(E_\alpha)$. As $G \neq E$ and $|N_E(E_\alpha)| = 10$, Lemma 2.2 implies that some $g \in G \setminus E$ is contained in G_α. Therefore 2 or 3 is contained in $\pi(G_\alpha)$. If $2 \in \pi(G_\alpha)$, then by Lemma 2.17 an index 2 subgroup of a Sylow 2-subgroup of G is contained in G_α. But this is impossible because E_α has odd order. If $3 \in \pi(G_\alpha)$, then also $2 \in \pi(G_\alpha)$ by Lemma 2.2. (For information about Aut(PSL$_3(4)$) see for example [5].) We already excluded this.

Case 3: $E \cong$PSL$_2(7)$.
We recall that $E_\alpha \cong \mathcal{A}_4$. The point stabilizers in PGL$_2(7)$ grow by a factor of either 2 or 1. Inspection of the maximal subgroups of PGL$_2(7)$ shows that the former case does not happen. In the latter case the centralizer order of the inner involution grows by a factor of 2 while the order of the point stabilizer does not. This implies that the number of fixed points of the involution on Ω is 4, and this violates Hypothesis 2.4.

Case 4: There exists a prime power q such that $E \cong$PSL$_2(q)$.
Using Hypothesis 2.4 choose $x \in G_\alpha$ such that x fixes three points on Ω and induces an outer automorphism on E. Lemma 2.2 implies that a subgroup of index at most 3 of $C_E(x)$ is contained in E_α.
First suppose that x induces a field automorphism. Then it follows from the possible structure of point stabilizers that $C_E(x)$ is a solvable subfield subgroup and we see that $2 \in \pi(E_\alpha)$. Moreover q is a power of 2 or of 3. If q is odd, then E_α contains a fours group from $C_E(x)$ and this is impossible. If q is even,

then E_α has order $q \cdot (q-1)$. Moreover x induces an automorphism of prime order. Hence (b) holds in this case.
Next suppose that x induces a diagonal automorphism. Then G_α contains an involution that fixes three points, and hence Lemma 2.15 (a) forces G_α to contain a Sylow 2-subgroup of G, and in particular of E. This is impossible because E_α does not contain a Sylow 2-subgroup of E.

Case 5: There exists a prime power q such that $E \cong \mathrm{Sz}(q)$.
Let $x \in G \setminus E$ be such that $x \in G_\alpha$ and x fixes three points on Ω. Then x induces a field automorphism on E and hence $C_E(x)$ is a subfield subgroup. Now any subfield group contains $\mathrm{Sz}(2)$, a group of order 20, and then Lemma 2.2 implies that E_α has an element of order 5. But we know from Theorem 5.6 in [19] that $|E_\alpha|_{2'} = (q-1)$. Since $(q-1)$ is not divisible by 5 (because q is a power of 2 with odd exponent), we see that E cannot be a Suzuki group.

These are all possible cases by [19], hence the proof is complete. □

Theorem 2.24. *Suppose that Hypothesis 2.4 holds and that N is a minimal normal subgroup of G. Let $\alpha \in \Omega$. Then one of the following holds:*

(a) *All Sylow subgroups of G_α have rank 1.*
(b) *N is a 2-group. Moreover N is a fours group whose involutions act without fixed points on Ω or $|N : N_\alpha| = 2$ and N_α fixes two points.*
(c) *N is a 3-group. Moreover G has Sylow 3-subgroups of maximal class or $|N : N_\alpha| = 3$, N_α fixes three points and $|N| \leq 9$.*
(d) *$N = E(G)$ and (N, α^N) satisfies Hypothesis 2.4 or $N \cong \mathcal{A}_5$ or there exists a 2-power q such that $N \cong PSL_2(q)$.*

Proof. The faithful action of G on Ω yields that $N \not\leq G_\alpha$. We begin with the case where N is elementary abelian. Let r be a prime such that N is an r-group and suppose that (a) does not hold. Let $p \in \pi(G_\alpha)$ and suppose that G_α contains an elementary abelian subgroup X of order p^2.
If $r \geq 5$, then Lemma 2.15 (c) yields that $r \notin \pi(G_\alpha)$ and hence the coprime action of X on N yields that $N = \langle C_N(x) \mid x \in X^\# \rangle$. It follows with Lemma 2.2 that $N \leq G_\alpha$. This is a contradiction. Therefore $r \in \{2, 3\}$.
First suppose that $r = 2$. Then $|N| \geq 4$ by Lemma 2.14. If $p = 2$, then $2 \in \pi(G_\alpha)$. If p is odd, then $N = \langle C_N(x) \mid x \in X^\# \rangle$ by coprime action and so, applying Lemma 2.2, it follows again that $2 \in \pi(G_\alpha)$. Let $S \in \mathrm{Syl}_2(G)$ and suppose that $|N : N_\alpha| \neq 2$. Then Lemma 2.17 yields that S is dihedral or semidihedral. As G has no normal subgroup of order 2, we see that N is not cyclic, so it follows that N is a fours group, that the involutions in N act without fixed points on Ω and that $G/C_G(N)$ is isomorphic to $\mathcal{S}_3$. This is one of the cases in (b). Otherwise $|N : N_\alpha| = 2$ and we let $t \in N$ be such that $t \notin G_\alpha$. As t normalizes N_α, but does not fix α, there must be a second point $\beta \in \Omega$ that is fixed by N_α and such that t interchanges α and β. This is the other case in (b).

Next suppose that $r = 3$. If $|N| = 3$, then the second case in (c) holds. So we suppose that $|N| \geq 9$ and we argue as in the previous paragraph. If $p = 3$, then $3 \in G_\alpha$, and if $p \neq 3$, then $3 \in G_\alpha$ by coprime action and Lemma 2.2. Now Lemma 2.20 yields the possibilities in (c). We note that, if $|N : N_\alpha| = 3$ and $y \in N$ is such that $y \notin N_\alpha$, then N_α must fix three points and y interchanges these three points in a 3-cycle.
This concludes the case where N is solvable.

Next suppose that N is a product of components. Then $E(G) \neq 1$ and hence Lemmas 2.22 and 2.21 yield that N is the unique component of G. Then (d) holds by Lemma 2.23. □

Our preliminary results enable us to prove one of the statements in Theorem 1.3:

Lemma 2.25. *Suppose that Hypothesis 2.4 holds. Then* $3 \in \pi(G)$.

Proof. Assume otherwise, choose G to be a minimal counter-example and let $\alpha \in \Omega$. First we consider the case where G_α has odd order. Let $1 \neq H \leq G_\alpha$ be a three point stabilizer, fixing the distinct points α, β and γ of Ω. Let $1 \neq X \leq H$ and $g \in N_G(X)$. As $o(g)$ is coprime to 3 by assumption, the fixed points of X cannot be interchanged by g in a 3-cycle. But the fact that point stabilizers have odd order also implies that g cannot interchange two of the points α, β, γ and fix the third. Thus it fixes them all and is hence contained in H. Now Lemma 2.1 forces G to be a Frobenius group, contrary to Hypothesis 2.4.
We conclude that G_α has even order.

(1) If G_α contains a Sylow 2-subgroup of G, then G has cyclic or quaternion Sylow 2-subgroups.

Proof. Suppose that G_α contains a Sylow 2-subgroup. Then $O_2(G) = 1$, and moreover $O_3(G) = 1$ by assumption. If E is a component of G, then one of the cases from Lemma 2.23 holds. The first two cases are impossible by the assumption that $3 \notin \pi(G)$, and in the third case the main result of [24] yields that $E/Z(E)$ is a Suzuki group. But this contradicts Lemma 4.3. Hence $E(G) = 1$ and $F^*(G) = F(G)$ is a $\{2,3\}'$-group. Looking at Theorem 2.24, we deduce that (a) holds and therefore our claim follows. □

(2) G has a subgroup M of index 2.

Proof. First suppose that G_α contains a Sylow 2-subgroup of G and let T be a 2-subgroup of G. Then T is cyclic or quaternion by (1) and therefore $N_G(T)/C_G(T)$ is a 2-group (recall that $3 \notin \pi(G)$). So Frobenius' Theorem implies that G has a normal 2-complement and hence a subgroup of index 2.

Now two cases from Lemma 2.17 remain, namely (2) and (3). First suppose that $S \in \mathrm{Syl}_2(G)$ is dihedral or semidihedral. Then Frobenius' Theorem is applicable again and G has a normal 2-complement, in particular a subgroup of index 2.

Finally suppose that Lemma 2.17 (3) holds and let $\beta \in \Omega$ be such that $S_\alpha = S_\beta$. Let $s \in S \setminus S_\alpha$. We already treated the case where S is cyclic, so we may suppose that $o(s) \neq |S|$. Then s induces a product of an even number of cycles of 2-power length on each regular S-orbit. Moreover s interchanges α and β and therefore it induces an odd permutation on Ω. So again G has a subgroup of index 2. □

(3) Let M be as in (2). Then M acts transitively on Ω.

Proof. Assume otherwise. Then M has two orbits on Ω which we denote by Δ_1 and Δ_2. Then the elements in $G \setminus M$ interchange Δ_1 and Δ_2, so they have no fixed points. By Hypothesis 2.4 we find $y \in M_\alpha$ such that y fixes three points on Ω. We may choose y of prime order p and we may suppose that $\alpha \in \Delta_1$. If α is the unique fixed point of y on Δ_1, then $|\Delta_1| \equiv 1$ modulo p and it follows that y also has a unique fixed point on Δ_2. But then y cannot have three fixed points in total, so this is impossible. With similar arguments it follows that, if y has two fixed points on Δ_1, then it has two or zero fixed points on Δ_2, which again gives a contradiction.

Thus the only remaining possibility is that all fixed points of y are contained in Δ_1. In particular $|\Delta_1| \equiv 3$ modulo p. Then y acts without fixed points on Δ_2 and it follows that $|\Delta_2| \equiv 0$ modulo p. As $|\Delta_1| = |\Delta_2|$, this forces $p = 3$, which is impossible. This proves our claim that M acts transitively on Ω. □

Let M be as in (2) and (3). Since $3 \notin \pi(M)$ and G is a minimal counter-example, we know that (M, Ω) does not satisfy Hypothesis 2.4. In particular the three point stabilizers in M are trivial, which forces

$G \setminus M$ to contain elements with three fixed points. As $|G : M| = 2$, this implies that some involution $t \in G$ fixes exactly three points and hence $|\Omega|$ is odd by Lemma 2.15 (a). Now (1) yields that G has cyclic or quaternion Sylow 2-subgroups, and this forces $\langle t \rangle \in \mathrm{Syl}_2(G)$. In particular M has odd order. It follows with Lemmas 2.1 and 2.2 that M acts regularly on Ω or that M is a Frobenius group. In the first case $3 = |\mathrm{fix}_\Omega(t)| = |C_M(t)|$, contrary to the fact that $3 \notin \pi(M)$. In the second case we let K denote the Frobenius kernel of M. Then K acts regularly on Ω and t normalizes it, so we have the same contradiction as above. □

When we study simple groups satisfying Hypothesis 2.4 (using the Classification of Finite Simple Groups), we adapt some of Aschbacher's notation from Section 9 of [2]. We introduce it here and use it throughout the following sections.

Definition 2.26. *Suppose that $p, q \in \pi(G)$ are prime numbers and let $H \leq G$ be a point stabilizer in G.*

- *We write $p \vdash q$ if and only if one of the following holds:*
  - *$q \geq 5$ and there exists a nontrivial p-subgroup $X \leq G$ such that $q \in \pi(N_G(X))$.*
  - *$q = 2$ and and there exists a nontrivial p-subgroup $X \leq G$ such that 4 divides $|N_G(X)|$.*
  - *$q = 3$ and and there exists a nontrivial p-subgroup $X \leq G$ such that 9 divides $|N_G(X)|$.*
- *We write $\rightarrow$ for the transitive extension of $\vdash$.*

Lemma 2.27. *Suppose that Hypothesis 2.4 holds and that $H \leq G$ is a point stabilizer. Suppose further that $q \in \pi(G)$ and $p \in \pi(H)$. If $p \geq 5$ and $p \rightarrow q$, then $q \in \pi(H)$.*

Proof. By definition of $\rightarrow$ it suffices to consider the case where $p \vdash q$. Lemma 2.15 (c) gives that H contains a Sylow p-subgroup of G. Then by Sylow's Theorem there exists a nontrivial p-subgroup X of H such that q (or 4 or 9) divides $|N_G(X)|$ and therefore Lemma 2.2 yields that $q \in \pi(H)$. □

3. Alternating Groups

In this chapter we discuss what alternating or symmetric groups appear as examples for Hypothesis 2.4 and if so, then with what actions. We begin with some small cases and then bring Lemma 2.18 into play. We use the notation that has been introduced at the end of the previous section.

Lemma 3.1. *Suppose that G is isomorphic to $\mathcal{A}_4$ or to $\mathcal{S}_4$. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Assume otherwise and let $\alpha \in \Omega$ and $x \in G_\alpha^\#$ be such that $|\mathrm{fix}_\Omega(x)| = 3$. If x is a 2-element, then Lemma 2.2 (a) yields that $|\Omega|$ is odd and hence $|\Omega| = 3$. This is too small for Hypothesis 2.4. If x is a 3-element, then G_α contains a Sylow 3-subgroup of G (because this has only order 3) and this contradicts Lemma 2.15 (b). □

Lemma 3.2. *Suppose that G is isomorphic to $\mathcal{A}_5$ or $\mathcal{S}_5$ and that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4. Then $|\Omega| = 15$ and the action of G is as on the set of cosets of a Sylow 2-subgroup, or $G \cong \mathcal{S}_5$, $|\Omega| = 5$ and G acts naturally.*

Proof. The action of G on the set of cosets of a Sylow 2-subgroup satisfies Hypothesis 2.4, as does the natural action of $\mathcal{S}_5$ on a set with five elements, so we need to show that these are the only possibilities. Suppose that (G, Ω) satisfies Hypothesis 2.4 and let $\alpha \in G_\alpha$ and $x \in G_\alpha^\#$ be such that $|\mathrm{fix}_\Omega(x)| = 3$. Assume that x is a 5-element. The nontrivial orbits of x have lengths divisible by 5 and hence $|\Omega| \equiv 3$ modulo 5. The only divisor of $|G|$ satisfying this property is 3, but then Ω is too small. The Sylow

3-subgroups of G have order 3 and hence Lemma 2.15 (b) yields that x does not have order 3. Thus x is a 2-element.
It follows from Lemma 2.15 (a) that $|\Omega|$ is odd, hence G_α has order 4 or 12 in the $\mathcal{A}_5$-case and order 8 or 24 in the $\mathcal{S}_5$-case. If $G \cong \mathcal{S}_5$ and $|G_\alpha| = 24$, then this is the natural action of $\mathcal{S}_5$.
Assume that $G \cong \mathcal{A}_5$ and that $|G_\alpha| = 12$. Then we first note that every double transposition in G has exactly three fixed points on Ω. As $|\Omega| = 5$, there are only 10 possibilities for fixed point sets for x. But there are 15 double transpositions in G and hence we find an involution $y \in G$ such that $x \neq y$ and $\text{fix}_\Omega(x) = \text{fix}_\Omega(y)$. Then x and y interchange the remaining two points and hence xy fixes all of Ω point-wise, which is a contradiction.
Therefore, if G_α is not a 2-group, then the only example is $\mathcal{S}_5$ in its natural action. If G_α is a 2-group, then it is a Sylow 2-subgroup and G acts as stated. □

Lemma 3.3. *Suppose that $G \cong \mathcal{A}_6$ and that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4. Then $|\Omega| \in \{6, 15\}$. The action of G is natural as $\mathcal{A}_6$ on six points in the first case, and G acts as on the set of cosets of a subgroup of order 24 in the second case, respectively.*

Proof. Let $\alpha \in \Omega$ and let $x \in G_\alpha$ be such that $|\text{fix}_\Omega(x)| = 3$. If x is a 5-element, then the subgroup structure of G allows G_α to be of order 5, 10 or 60. However, this means that $|\Omega| \in \{72, 36, 6\}$ and these numbers are not congruent to 3 modulo 5.
Next suppose that x is a 3-element. Then Lemmas 2.2 (c) and 2.15 (b) imply that G_α has even order and that $|\Omega|$ is divisible by 3. Applying Lemma 2.2 to a 2-element in G_α yields that $|G_\alpha|$ is divisible by 4, hence by 12. This leads to the cases $G_\alpha \cong \mathcal{A}_4, \mathcal{S}_4$ or $\mathcal{A}_5$. Hence $|\Omega| \in \{30, 15, 6\}$. However the first case is impossible as an element of order 3 will fix six points on Ω. The other two possibilities give the examples in the conclusion.
If x is a 2-element, then G_α has order 8 or 24 by Lemma 2.15 (a). The former case leads to the possibility that $|\Omega| = 45$. However in this case an involution fixes five points, which is impossible. The second case is that $G_\alpha \cong \mathcal{S}_4$, which is one of our conclusions. □

Lemma 3.4. *Suppose that G is almost simple, but not simple and that $F^*(G) \cong \mathcal{A}_6$. There does not exist a set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Let $E := F^*(G)$. Then Lemma 2.23 is applicable and we see that (a) and (b) cannot hold. So (c) holds and we let $\alpha \in \Omega$ be such that (E, α^E) satisfies Hypothesis 2.4. In particular we know that $H := E_\alpha \cong \mathcal{A}_5$ or $\mathcal{S}_4$ from Lemma 3.3.
In the former case, $|\Omega| = 6$ or 12 whereas in the second case $|\Omega| = 15$ or 30. If the action is on 6 or 15 points, then $G \cong S_6$ and one of the outer involutions has too many fixed points.
If the action is on 12 or 30 points, then an inner involution has four respectively six fixed points, ruling out these possibilities as well. □

Lemma 3.5. *Suppose that $G \cong \mathcal{A}_7$ and that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4. Then either $|\Omega| = 15$ and the action of G is as on the set of cosets of a subgroup isomorphic to $PSL_2(7)$, or $|\Omega| = 360$ and G acts on the set of cosets of a Sylow 7-subgroup. In the first case the three point stabilizer contains a Sylow 2-subgroup of G.*

Proof. Let $\alpha \in \Omega$ and $x \in G_\alpha$ be such that $|\text{fix}_\Omega(x)| = 3$. First assume that x has order 7. Then $|\Omega| \equiv 3$ modulo 7 and, as $|\Omega| \geq 4$, this only leaves the possibilities 10, 24, 45 or 360. There are no subgroups of G of index 10, 24 or 45, ruling out these cases. The normalizer of a Sylow 7-subgroup has index 360 and this yields the second example.
Next assume that x has order 5. Then Lemma 2.2 yields that some point stabilizer contains a subgroup of order 20, so we may suppose that 20 divides $|G_\alpha|$. Moreover $|\Omega| \equiv 3$ modulo 5 and $7 \notin \pi(G_\alpha)$ by the subgroup structure of $\mathcal{A}_7$. In particular 7 divides $|\Omega|$. This only leaves the possibility $|\Omega| = 63$ and $|G_\alpha| = 40$. But G does not have a subgroup of this order.

We continue with the case where x has order 3. Then Lemmas 2.2 and 2.15 (b) yield that G_α has even order and that $|\Omega|$ is divisible by 3. From the centralizer of an involution in G_α and Lemma 2.2 we obtain that G_α contains a subgroup isomorphic to $\mathcal{A}_4$. Thus G_α is isomorphic to $\mathcal{A}_4$, $\mathcal{S}_4$, $\mathcal{A}_5$, $\mathcal{S}_5$ or $\mathrm{PSL}_2(7)$. Correspondingly, $|\Omega| \in \{210, 105, 42, 21, 15\}$.
Assume that $G_\alpha \simeq \mathcal{A}_4$ and $|\Omega| = 210$. Let $V \leq G_\alpha$ be a fours group. Then $N_G(V)$ contains a subgroup of order 9 of which a subgroup A of order 3 centralises V. Each involution in V has exactly two fixed points, hence A fixes these two points and therefore $A \leq G_\alpha$. It follows that 9 divides G_α, which is contradiction. With the same argument we exclude the case where $G_\alpha \simeq \mathcal{A}_5$ and $|\Omega| = 42$.
Next assume that $G_\alpha \simeq \mathcal{S}_4$ and $|\Omega| = 105$. Then every involution has one or three fixed points. The second case will be treated below. In the first case Lemma 2.2 forces G_α to contain a Sylow 2-subgroup of G, which is impossible. With the same argument we exclude the case where $G_\alpha \simeq \mathcal{S}_5$ and $|\Omega| = 21$.
Finally suppose that x is a 2-element. Then G_α contains a Sylow 2-subgroup of G by Lemma 2.15 (a) and hence $3 \in \pi(G_\alpha)$ by Lemma 2.2. This means that 24 divides $|G_\alpha|$ and the only new case is $G_\alpha \cong \mathcal{A}_6$. But then G acts as it does naturally on seven points; this is impossible because one conjugacy class of 3-elements has four fixed points in this action.
It follows that the only possibility is that $G_\alpha \cong \mathrm{PSL}_2(7)$. Then Lemma 2.2 and the fact that 9 does not divide $|G_\alpha|$ imply that the three point stabilizer contains a Sylow 2-subgroup of G. □

Corollary 3.6. *Suppose that $G \cong \mathcal{S}_7$. Then there is no set Ω is a set such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Let $E := F^*(G) \cong \mathcal{A}_7$. Then Lemma 2.23 is applicable and we see that (c) holds. Let $\alpha \in \Omega$ be such that (E, α^E) satisfies Hypothesis 2.4. Then Lemma 3.5 yields that $E_\alpha \simeq \mathrm{PSL}_2(7)$ and that a Sylow 2-subgroup of E is contained in a three point stabilizer, or that E_α is a Sylow 7-subgroup of G. In the first case, as $|G : E| = 2$, Lemma 2.2 implies that a Sylow 2-subgroup of G is contained in a point stabilizer. Therefore $|G_\alpha| = 2 \cdot |E_\alpha| = 2^4 \cdot 3 \cdot 7$. Let $t \in G_\alpha \setminus E_\alpha$ be an involution. Then $|C_G(t)|$ is divisible by 5, and this contradicts Lemma 2.2 because $5 \notin \pi(G_\alpha)$.
In the second case, as $|G : E| = 2$, Lemma 2.2 implies that G_α contains an involution t. However then Lemma 2.2 yiels that $C_G(t) \cap E_\alpha \neq 1$, contradicting the fact that $E_\alpha \in \mathrm{Syl}_7(G)$.
□

Lemma 3.7. *Suppose that Hypothesis 2.4 holds and that G is an alternating group of degree at least 8. Let $\alpha \in \Omega$. Then G_α has odd order or it contains a Sylow 2-subgroup of G.*

Proof. This follows immediately from Lemma 2.18. □

Lemma 3.8. *Suppose that $G \cong \mathcal{A}_8$ and that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4. Then $|\Omega| = 2880$. The action of G is as on the set of cosets of a Sylow 7-subgroup.*

Proof. Let $\alpha \in \Omega$ and $x \in G_\alpha$ be such that $|\mathrm{fix}_\Omega(x)| = 3$. First we suppose that G_α has odd order and we choose x of prime order p. Then $p \not\vdash 2$ by Lemma 2.27. If $p = 7$, then $|\Omega| \equiv 3$ modulo 7 and $|\Omega| \geq 5$, so in this case we only have the possibilities that $|G_\alpha| = 45$ or $|G_\alpha| = 2880$. The former is impossible, whereas the latter yields our example.
As $5 \vdash 2$ and $3 \vdash 2$, we see that $p \neq 5$ and $p \neq 3$, so this case is finished.
Using Lemma 3.7 we now have that G_α contains a double transposition t. Then Lemma 2.2 yields that 32 divides $|G_\alpha|$. Now we look at the normalizer of a fours group in G_α and deduce that $3 \in \pi(G_\alpha)$. This gives two possibilities: G_α is contained in a subgroup isomorphic to $2^3 : \mathrm{PSL}_3(2)$ or to $2^4 : (\mathcal{S}_3 \times \mathcal{S}_3)$. Hence $|\Omega| \in \{35, 105, 210\}$. However in all of these cases the involutions have at least six fixed points, so this does not occur. □

Corollary 3.9. *Suppose that $G \cong \mathcal{S}_8$. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Let $E := F^*(G) \cong \mathcal{A}_8$. First we note that Lemma 2.23 (c) holds and we let $\alpha \in \Omega$ be such that (E, α^E) satisfies Hypothesis 2.4. Moreover $G_\alpha \cap E$ is a Sylow 7-subgroup of G.
Thus, as $|G : E| = 2$, Lemma 2.2 implies that G_α contains an involution t. However then Lemma 2.2 implies that $C_G(t) \cap E_\alpha \neq 1$, contradicting the fact that $E_\alpha \in \mathrm{Syl}_7(G)$. □

Lemma 3.10. *Suppose that G is isomorphic to $\mathcal{A}_9$ or $\mathcal{S}_9$. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. First suppose that $G \cong \mathcal{A}_9$ and assume that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4. We let $\alpha \in \Omega$ and begin as follows:

$(*)$ G_α does not contain a 3-cycle.

Proof. Assume otherwise. Then G_α contains an $\mathcal{A}_6$ (by Lemma 2.2), in particular Lemma 3.7 yields that G contains involutions from both conjugacy classes. Then Lemma 2.2 implies that $2^5 \cdot 3^3 \cdot 5$ divides $|G_\alpha|$. But there is no maximal subgroup of G that could contain G_α now. □

Suppose first that G_α has odd order. Let $x \in G_\alpha$ be of prime order p and such that $|\mathrm{fix}_\Omega(x)| = 3$. We will use that $p \not\vdash 2$ by Lemma 2.27. Then $p \neq 7$ because $7 \vdash 2$ and similarly $p \neq 5$. Hence $p = 3$ and x is not the product of two 3-cycles, by Lemma 2.2. If x is the product of three 3-cycles, then x is 3-central and therefore G_α contains a subgroup of order 3^3. In particular G_α contains a 3-cycle, contrary to $(*)$.
Lemma 3.7 yields that G_α contains a double transposition. Applying Lemma 2.2 to its centralizer gives that G_α has a subgroup isomorphic to $\mathcal{A}_5$, contrary to $(*)$.
Now suppose that $G \cong \mathcal{S}_9$ and let $E := F^*(G) \cong \mathcal{A}_9$. Then by Lemma 2.23 there is some $\alpha \in \Omega$ such that (E, α^E) also satisfies Hypothesis 2.4. But this is impossible by the previous paragraph. □

Lemma 3.11. *Suppose that G is isomorphic to $\mathcal{A}_{10}$ or $\mathcal{S}_{10}$. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Assume otherwise and let $\alpha \in \Omega$.
As in the previous lemma, we begin with the case where $G \cong \mathcal{A}_{10}$. The special role of 3-cycles will be key once more.

$(*)$ G_α does not contain a 3-cycle.

Proof. Assume otherwise. Then G_α contains a subgroup $H \cong \mathcal{A}_7$ (by Lemma 2.2). In particular $|\Omega| \leq 2^4 \cdot 3^2 \cdot 5$. Let $\beta \in \Omega$ be such that $\beta \neq \alpha$ and let $\Delta := \beta^H$. In its action on Δ, every nontrivial element of H has at most two fixed points, and moreover H does not act regularly. But we proved in Lemma 3.5 in [19] that $\mathcal{A}_7$ does not allow such an action. Hence this is impossible. □

Now we suppose that G_α has odd order and we let $x \in G_\alpha$ be of prime order p. Then $p \not\vdash 2$ by Lemma 2.27. In particular $p \neq 7$ and $p \neq 5$. If $p = 3$, then we first look at the case where x is a product of three 3-cycles. Here x is 3-central and therefore G_α contains a subgroup of order 3^3. In particular G_α contains a 3-cycle, contrary to $(*)$.
If x is the product of two 3-cycles, then Lemma 2.2 yields that G_α has even order, contrary to our assumption in this case. By $(*)$ x is not a 3-cycle. So this case is finished and by Lemma 3.7 it remains to consider the case where G_α contains a Sylow 2-subgroup of G. Then Lemma 2.2, applied to a double transposition, yields that G_α contains an $\mathcal{A}_6$. But this is impossible by $(*)$.
If $G \cong \mathcal{S}_9$, then the previous paragraph and Lemma 2.23 give the result. □

Lemma 3.12. *Suppose that $n \geq 11$, that $G \cong \mathcal{S}_n$ or $\mathcal{A}_n$ and that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4.*
Then the order of a point stabilizer in G is not divisible by 3.

Proof. Assume otherwise and let $\alpha \in \Omega$. We show that our hypothesis implies that G_α contains a 3-cycle. Throughout we use that, if $m \geq 5$, then $\mathcal{A}_m$ does not have subgroups of index 2 or 3. This will play a role when applying Lemma 2.2.
We first note that G_α contains a 3-cycle if it contains a double transposition, by Lemma 2.2, because the centralizer of a double transposition in G contains $\mathcal{A}_7$. Thus it is left to prove our statement in the case where G_α has odd order, by Lemma 3.7.
Let $x \in G_\omega$ be an element of order 3 and suppose that $k \geq 2$ is such that x is a product of k cycles of length 3. If $n - 3 \cdot k \geq 4$, then $C_G(x)$ contains a fours group or a subgroup isomorphic to $\mathcal{A}_5$, which is impossible. Therefore $n - 3 \cdot k \leq 3$. The structure of $C_{\mathcal{A}_n}(x)$ is $((3^k : \mathcal{S}_k) \times \mathcal{S}_{n-3\cdot k}) \cap \mathcal{A}_n$ and thus, if $k \geq 4$, then again $C_G(x)$ contains a fours group. Thus $k \leq 3$ and we obtain that $n \leq 3 + 3k \leq 12$.
If $n = 12$, then $C_G(x)$ contains a subgroup of structure $((3^3 : \mathcal{S}_3) \times \mathcal{S}_3) \cap \mathcal{A}_{12}$ and hence Lemma 2.2 implies that G_α contains a 3-cycle or a double 3-cycle. In the second case we change x to such a double 3-cycle. Its centralizer contains an $\mathcal{A}_5$, so this is a contradiction. If $n = 11$, then $C_G(x)$ still contains a subgroup of structure $(3^3 : \mathcal{S}_3)$ and thus, with Lemma 2.2, it follows once more that G_α contains a 3-cycle.
As G contains a subgroup isomorphic to $\mathcal{A}_{11}$, it is ninefold transitive, and so we may suppose that $x = (1, 2, 3)$. It follows from Lemma 2.2 that G_α contains a subgroup isomorphic to $\mathcal{A}_{n-3}$ and hence, without loss, the 3-cycle $y := (4, 5, 6)$. The same argument yields that $C_G(y) \leq G_\alpha$. Now we deduce that $G_\alpha \geq \langle C_G(x), C_G(y)\rangle \cong \mathcal{A}_n$, which contradicts the fact that G acts faithfully on Ω. □

Theorem 3.13. *Suppose that $n \geq 11$ and that $G \cong \mathcal{A}_n$ or $\mathcal{S}_n$. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Assume that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4. Let $\alpha \in \Omega$, let p be a prime and let $x \in G_\alpha$ be a p-element. Then there exists $k \in \mathbb{N}$ such that x is a product of k cycles of length p. Now $C_G(x)$ contains a subgroup of structure $p^k : \mathcal{S}_k \times \mathcal{A}_{n-p\cdot k}$ if p is odd and of structure $(2^k : \mathcal{S}_k \times \mathcal{S}_{n-2\cdot k}) \cap \mathcal{A}_n$ otherwise.
Assume that $n - p \cdot k \geq 3$. Then Lemma 2.2 implies that $C_G(x) \cap G_\alpha$ contains a 3-cycle, contrary to Lemma 3.12.
Therefore $n - p \cdot k \leq 2$, so $11 \leq n \leq 2 + p \cdot k$. First we assume that $p = 2$. Then G_α contains a double transposition t by Lemma 3.7. As $n \geq 11$, we see that $C_G(t)$ contains a subgroup isomorphic to $\mathcal{A}_7$, which is a perfect group of order divisible by 3. Together with Lemma 2.2 this contradicts Lemma 3.12. This means that G_α has odd order.
With Lemma 3.12 it follows that $p > 3$. Then Lemma 2.15 (c) implies that $G_\alpha \cap \mathcal{A}_n$ contains a full Sylow p-subgroup P of G. Thus $G_\alpha \cap \mathcal{A}_n$ contains a p-cycle, say y. If $n - p > 3$, then $C_{G_\alpha}(y)$ contains a double transposition by Lemma 2.2, and this contradicts the fact that G_α has odd order.
Therefore $n - p \leq 3$ and this property holds for all prime divisors p of $|G_\omega|$.
As $n \geq 11$, the above property forces $p \geq 8$. But p is prime and so we have that $p \geq 11$. In particular $|N_G(\langle x\rangle) : \langle x\rangle| \geq \frac{p-1}{2} \geq 5$ and it follows that $|G_\omega \cap N_G(\langle x\rangle)|$ is divisible by a prime r such that $2 \cdot r \leq p - 1 \leq n$. This implies $r \leq n - r$. We know that $r \neq 2$ and $r \neq 3$ (by Lemma 3.12 and because G_α has odd order), so $5 \leq r \leq n - r$. We proved in the previous paragraph that r satisfies $n - r \leq 2$. Now this is impossible. □

The next result collects all the information from this chapter.

Theorem 3.14. *Let $n \in \mathbb{N}$ and suppose that G is isomorphic to $\mathcal{A}_n$ or to $\mathcal{S}_n$. If Ω is a set such that (G, Ω) satisfies Hypothesis 2.4, then $n \in \{5, 6, 7, 8\}$ and one of the following holds:*

(1) *$n = 5$, $G \cong \mathcal{A}_5$, $|\Omega| = 15$ and the action of G is as on the set of cosets of a Sylow 2-subgroup.*
(2) *$n = 5$, $G \cong \mathcal{S}_5$, $|\Omega| = 5$ and G acts naturally.*
(3) *$n = 6$, $G \cong \mathcal{A}_6$, $|\Omega| = 6$ and G acts naturally.*

(4) $n = 6$, $G \cong \mathcal{A}_6$, $|\Omega| = 15$ *and* G *acts as on the set of cosets of a subgroup of order* 24.
(5) $n = 7$, $G \cong \mathcal{A}_7$, $|\Omega| = 15$ *and the action of* G *is as on the set of cosets of a subgroup isomorphic to* $PSL_2(7)$.
(6) $n = 7$, $G \cong \mathcal{A}_7$, $|\Omega| = 360$ *and the action of* G *is as on the set of cosets of a Sylow* 7*-subgroup.*
(7) $n = 8$, $G \cong \mathcal{A}_8$, $|\Omega| = 2880$ *and the action of* G *is as on the set of cosets of a Sylow* 7*-subgroup.*

Proof. Theorem 3.13 and Lemma 3.1 imply that $n \in \{5, 6, 7, 8\}$. Moreover $\mathcal{S}_7$ and $\mathcal{S}_8$ do not occur by Lemmas 3.4, 3.6 and 3.9.
The possibilities are then listed in Lemmas 3.2, 3.3, 3.5 and 3.8. □

4. Lie type groups

We organize our analysis around Lemma 2.18 and begin with the almost simple groups where the normalizers of Sylow 2-subgroups are strongly embedded. Then we consider groups with dihedral or semidihedral Sylow 2-subgroups and finally those groups where we know from the outset that $|G_\omega|$ is odd.
We record a general lemma, which is a consequence of Lemma 2.1 for use in this section.

Lemma 4.1. *Suppose that* (G, Ω) *satisfies Hypothesis 2.4 with* $|\Omega| \geq 7$ *and suppose that* $\alpha, \beta, \gamma \in \Omega$ *are pair-wise distinct and such that* $1 \neq H := G_\alpha \cap G_\beta \cap G_\gamma$. *Then there exists a subgroup* $1 \neq X \leq H$ *and an element* $g \in N_G(X) \setminus X$ *such that* 3 *divides* $o(g)$, *or* G_α *has even order.*

Proof. The nonidentity subgroups X of H fix the elements of $\Delta := \{\alpha, \beta, \gamma\}$ and act semiregularly on $\Omega \setminus \Delta$. Thus for every such X we see that $N_G(X)$ acts on Δ with kernel $N_H(X)$, and $|N_G(X) : N_H(X)| \leq 3$ by Lemma 2.2 (c).
Suppose, for all nontrivial subgroups X of H, that $(|N_G(X) : N_H(X)|, 3) = 1$.
If $1 \neq X \leq H$ is such that $|N_G(X) : N_H(X)| = 2$, then $N_G(X)$ has even order and a fixed point on Δ which is one of our conclusions.
If H has no nontrivial subgroup X such that $|N_G(X) : N_H(X)| = 2$, then for all these subgroups $N_G(X) \leq H$. Since $1 \neq H \neq G$, it follows with Lemma 2.1 that G is a Frobenius group. But this contradicts Hypothesis 2.4. □

4.1. **Groups with strongly embedded Sylow 2-subgroup normalizers.** The simple groups of Lie type considered in this section are those where the normalizers of Sylow 2-subgroups are strongly embedded. We consider them in individual lemmas.

Lemma 4.2. *Suppose that* q *is power of* 2 *and that* $G = PSL_2(q)$. *If* $q \neq 4$, *then there is no set* Ω *such that* (G, Ω) *satisfies Hypothesis 2.4.*

Proof. We note that $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathcal{A}_5$ (which has been treated in Lemma 3.2) and that $\mathrm{PSL}_2(2) \cong \mathcal{S}_3$ (which does not satisfy Hypothesis 2.4). Therefore we may suppose that $q \geq 8$. We assume that the lemma is false and let Ω be such that (G, Ω) satisfies Hypothesis 2.4. Let $\omega \in \Omega$.
First we suppose that $|\Omega|$ is odd. Then G_ω contains a Sylow 2-subgroup S of G. Now $|N_G(S)/S| = q - 1 \geq 7 > 3$, so Lemma 2.2 implies that G_ω contains an element x of order $(q-1)/(q-1, 3)$. If $(q - 1, 3) = 1$, then $G_\omega = N_G(S)$, as $N_G(S)$ is maximal in G. If $(q - 1, 3) \neq 1$, then we note that $N_G(\langle x \rangle)$ is dihedral of order $2(q - 1)$. Lemma 2.2 implies that either $|G_\omega \cap N_G(\langle x \rangle)| = q - 1$, in which case $G_\omega = N_G(S)$, or $|G_\omega \cap N_G(\langle x \rangle)| = 2(q - 1)/3$, in which case G_ω contains an involution which does not lie in S. As S together with any involution $t \notin S$ generates G, we see that the latter cannot happen and that $G_\omega = N_G(S)$. Thus (G, Ω) appears in the conclusion of Theorem 1.2 of [19], and in particular no nonidentity element of G has three fixed points on Ω. This is a contradiction.

Thus we may now suppose that $|\Omega|$ is even. If $S \in \mathrm{Syl}_2(G)$, then S is elementary abelian of order at least 8 and thus Lemma 2.18 implies that G_ω has odd order. Inspection of the maximal subgroups of G yields that G_ω is cyclic of order dividing $q-1$ or $q+1$. This means that, if $x \in G_\omega$, then

$$|F_\Omega(x)| = |N_G(\langle x \rangle) : G_\omega| \geq |N_G(G_\omega) : G_\omega| \in \{2 \cdot \frac{q-1}{|G_\omega|}, 2 \cdot \frac{q+1}{|G_\omega|}\}.$$

But $|F_\Omega(x)| \leq 3$ and therefore $|G_\omega| \in \{q-1, q+1\}$. This means that (G, Ω) appears in the conclusion Theorem 1.2 of [19]. In particular no nonidentity element of G has three fixed points on Ω, contrary to our assumption. □

As a corollary of Lemma 2.25 we obtain:

Lemma 4.3. *Let q be a power of 2 such that $q \geq 8$, and suppose that $G = Sz(q)$. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Prior to proving our next lemma we note that the group $\mathrm{PSU}_3(2)$ is a Frobenius group of order 72, and in particular it does not lead to any examples for Hypothesis 2.4.

Lemma 4.4. *Let $q \geq 4$ be a power of 2 and let $G = PSU_3(q)$.*
Let Λ be the set of cosets of a cyclic subgroup of order $q^2 - q + 1/(3, q+1)$ of G. Then (G, Λ) satisfies Hypothesis 2.4, and this is the unique example for G.

Proof. Let Ω be such that (G, Ω) satisfies Hypothesis 2.4. We show that the point stabilizers are cyclic of order $q^2 - q + 1/(3, q+1)$ and that the action described in the lemma does in fact give an example. Let $\omega \in \Omega$. We first consider the situation where $|\Omega|$ is odd; i.e. $S \leq G_\omega$ for some $S \in \mathrm{Syl}_2(G)$.
The group $N_G(S)/S$ is cyclic of order $q^2 - 1$, which implies by Lemma 2.2 that G_ω also contains a subgroup of order $((q+1)/(q+1, 3))^2 \neq 1$. However $N_G(S)$ is strongly embedded in G, so the proper overgroups of S in G are contained in $N_G(S)$. Thus $G_\omega = G$, which is impossible. Now $|\Omega|$ is even.
Next we note that S is neither dihedral nor semidihedral, so Lemma 2.18 implies that G_ω has odd order. The elements of G of odd order are conjugate to elements of tori of orders $q^2 - 1$, $(q+1)^2/(q+1, 3)$ or $(q^2 - q + 1)/(q+1, 3)$.
First suppose that $p \in \pi(G_\omega)$ is such that p divides $q-1$. Then $p \vdash r$ for all divisors r of $q + 1/(3, q+1)$ and Lemma 2.27 yields that all these primes r divide $|G_\omega|$. Thus if $p \in \pi(G_\omega)$ divides $(q^2 - 1)$, then Lemma 2.2 implies that $((q+1)/(q+1, 3))^2$ divides $|G_\omega|$. This means that G_ω contains an element y with $C_G(y)$ of structure $(q+1) \times \mathrm{PSU}_2(q)$ and hence G_ω contains a subgroup isomorphic to $\mathrm{PSU}_2(q)$. This contradicts the fact that $|G_\omega|$ is odd.
Thus no $p \in \pi(G_\omega)$ divides $q^2 - 1$, which implies that all $p \in \pi(G_\omega)$ divide $q^2 - q + 1/(3, q+1)$. Now if $x \in G_\omega$ has prime order p, then $C_G(x)$ is cyclic of order $q^2 - q + 1/(3, q+1)$, and $|N_G(\langle x \rangle) : C_G(x)| = 3$. As 3 divides $q^2 - 1$, but not $|G_\omega|$, this yields that $G_\omega = C_G(x)$.

The previous arguments show that there is at most one possibility for the action of G on Ω. Now let Λ be the set of cosets of $C_G(x)$ in G. We show that this actually gives an example. Since $(q+1, q^3+1) = 3(q+1)$, we see that $C_G(y) = C_G(x)$ for all $y \in C_G(x)^\#$.
Therefore $|\mathrm{fix}_\Lambda(y)| = |N_G(\langle y \rangle) : C_G(x)| = |N_G(C_G(x)) : C_G(x)| = 3$, which shows that (G, Λ) satisfies Hypothesis 2.4 as claimed. □

4.2. **Groups with dihedral or semidihedral Sylow 2-subgroups.** The simple groups of Lie type considered in this section are those whose Sylow 2-subgroups are dihedral or semidihedral. Again we look at the corresponding series of groups in individual lemmas.

Lemma 4.5. *Suppose that q is a power of an odd prime and that $G = PSL_2(q)$. Then (G, Ω) satisfies Hypothesis 2.4 if and only if one of the following is true:*

(i) $G \cong PSL_2(7) \cong PSL_3(2)$ *with* $|\Omega| = 7$ *and* $G_\omega \cong \mathcal{S}_4$.
(ii) $G \cong PSL_2(7) \cong PSL_3(2)$ *with* $|\Omega| = 24$ *and* G_ω *is cyclic of order* 7.
(iii) $G \cong PSL_2(11)$ *with* $|\Omega| = 11$ *and* $G_\omega \cong \mathcal{A}_5$.

Proof. We note that $\mathrm{PSL}_2(5) \cong \mathcal{A}_5$, $\mathrm{PSL}_2(9) \cong \mathcal{A}_6$ and that $\mathrm{PSL}_2(3) \cong \mathcal{A}_4$ does not give rise to any example by Lemma 3.1. Therefore we may assume that $q = 7$ or $q \geq 11$. We also assume that (G, Ω) satisfies Hypothesis 2.4.
The full table of marks of $\mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(11)$ is available in GAP (see [23]) and these confirm our claim. Thus we may assume that $q \geq 13$. Let $\omega \in \Omega$.
If $r \in \pi(G_\omega)$ is a divisor of $(q+1)/2$ and if $x \in G_\omega$ has order r, then $N_G(\langle x \rangle)$ is dihedral of order $q+1$. As in the proof of Lemma 4.2 this implies that G_ω is cyclic of order $(q+1)/2$ or $(q-1)/2$. This action occurs in the conclusion of Theorem 1.2 of [19], contradicting Hypothesis 2.4.
Now if $r \in \pi(G_\omega)$ and $(q, r) \neq 1$, then $r \vdash p$ for all divisors p of $(q-1)/2$. Thus we assume that G_ω contains an element x of order dividing $(q-1)/2$. As $N_G(\langle x \rangle)$ is dihedral of order $(q-1)$, Lemma 2.2 implies that G_ω contains a subgroup of index at most 3 of this normalizer. Now assume that G_ω contains an involution t inverting x. Then $C_{G_\omega}(x)$ and $C_{G_\omega}(t)$ generate G (by the subgroup structure of G). This is impossible. The only overgroups of $\langle x \rangle$ are conjugates of B, the Borel subgroup of G, or the dihedral group of order $(q-1)$. The latter possibility is ruled out because no involution in G_ω inverts x, and the possibility $G_\omega = B$ is ruled out because the action of G on the set of cosets of B occurs in the conclusion of Theorem 1.2 of [19]. Thus G_ω is cyclic of order $(q-1)/2$ but again this possibility occurs in the conclusion of the main theorem of [19]. This proves that $\mathrm{PSL}_2(q)$ for $q \geq 13$ does not yield examples satisfying Hypothesis 2.4. □

Lemma 4.6. *Suppose that* p *is an odd prime, that* $q = p^a$ *with* $a \in \mathbb{N}$ *and that* $G = PSU_3(q)$. *Let* Λ *be the set of cosets of a cyclic subgroup of order* $q^2 - q + 1/(3, q+1)$ *of* G. *Then* (G, Λ) *satisfies Hypothesis 2.4, and this is the unique example for* G.

Proof. Let Ω be such that (G, Ω) satisfies Hypothesis 2.4. We show that the point stabilizers are cyclic of order $q^2 - q + 1/(3, q+1)$ and that the action described in the lemma does in fact give an example. Let $\omega \in \Omega$. For $q = 3$ our claim follows from inspection of the table of marks in GAP (see [23]). So we may suppose that $q \geq 5$.
If $t \in G_\omega$ is an involution, then $C_G(t)$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$. But $\mathrm{SL}_2(q)$ is perfect because $q \geq 5$, and therefore Lemma 2.2 implies that G_ω has a subgroup isomorphic to $\mathrm{SL}_2(q)$. Let $P \leq G_\omega$ be such that P is isomorphic to a Sylow p-subgroup of $\mathrm{SL}_2(q)$. Then G_ω contains an index three subgroup of $N_G(P)$, again by Lemma 2.2.
If $p > 3$, then Lemma 2.15 (c) implies that G_ω contains a Sylow p-subgroup of G, and if $p = 3$, then a straightforward computation shows that a torus in $N_G(P)$ acts transitively on the commutator factor group of $\mathrm{Syl}_p(N_G(P))$. In this case G_ω contains a Sylow p-subgroup of $N_G(P)$, and hence of G, again. This is impossible because a subgroup of G isomorphic to $\mathrm{SL}_2(q)$ together with a Sylow p-subgroup generates all of G.
So we may now suppose that $|G_\omega|$ is odd. If $p \in \pi(G_\omega)$, then $p \vdash r$ for every prime divisor r of $q^2 - 1/(9, q^2 - 1)$ and hence Lemma 2.27 implies that all these primes r divide $|G_\omega|$. From the existence of tori of order $(q^2-1)/(3, q+1)$ and $(q+1)^2/(3, q+1)$ it follows that $(\frac{(q+1)}{(3,q+1)})^2$ divides $|G_\omega|$, whenever p or a divisor of $q^2 - 1$ divides $|G_\omega|$. Thus there exist commuting elements $x_1, x_2 \in G_\omega$ with centralizers containing a subgroup isomorphic to $\mathrm{SL}_2(q)$ and such that $G = \langle C_G(x_1)', C_G(x_2)' \rangle$. However, Lemma 2.2 then forces $G = G_\omega$. This is a contradiction.
We deduce that no prime $p \in \pi(G_\omega)$ divides $q^2 - 1$. This means that they all divide $q^2 - q + 1/(3, q+1)$. Let $x \in G_\omega$ be of prime order p. Then $C_G(x)$ is cyclic of order $q^2 - q + 1/(3, q+1)$, and $|N_G(\langle x \rangle) : C_G(x)| = 3$. But 3 does not divide G_ω, so this implies that $G_\omega = C_G(x)$.

These arguments show that there is at most one possibility for the action of G on Ω. Now let Λ be the set of cosets of $C_G(x)$ in G.
As $(q+1, q^3+1) = 3(q+1)$, it follows that $C_G(y) = C_G(x)$ for all $y \in C_G(x)^\#$. Therefore $|\mathrm{fix}_\Lambda(y)| = |N_G(\langle y \rangle) : C_G(x)| = |N_G(C_G(x)) : C_G(x)| = 3$ which shows that (G, Λ) satisfies Hypothesis 2.4. □

Lemma 4.7. *Let $G =$PSL$_3(q)$ with q odd. If (G, Ω) satisfies Hypothesis 2.4, then for all $\omega \in \Omega$ the group G_ω is cyclic of order $(q^2+q+1)/(3, q-1)$. Moreover $|N_G(G_\omega)| = 3 \cdot |G_\omega|$ and $(|G_\omega|, 3) = 1$.*

Proof. Inspection of the table of marks in GAP establishes our claim for $q = 3$. Thus we may assume that $q \geq 5$. Let $\omega \in \Omega$.
If $r \in \pi(G_\omega)$ and r is a divisor of $q(q^2-1)$, then $r \vdash s$ for all prime divisors s of $q-1$. Thus in every such case a subgroup of index at most 3 of a split torus T of order $(q-1)^2/(3, q-1)$ will be contained in G_ω. But this implies, as in the proof of Lemma 4.6, that G_ω has commuting elements x_1, x_2 with centralizers containing a subgroup isomorphic to $\mathrm{SL}_2(q)$ and so that $G = \langle C_G(x_1)', C_G(x_2)' \rangle$. But then Lemma 2.2 forces $G = G_\omega$, which is a contradiction.
Thus the only possibilities for $r \in \pi(G_\omega)$ are divisors of $(q^2+q+1)/(3, q-1)$. If $x \in G_\omega$ has order r dividing $(q^2+q+1)/(3, q-1)$, then $C_G(x)$ is cyclic of order $q^2+q+1/(3, q+1)$, and $|N_G(\langle x \rangle) : C_G(x)| = 3$. As 3 divides $p(q^2-1)$, but not $|G_\omega|$, this implies that $G_\omega = C_G(x)$. Moreover $(q-1, q^3-1) = 3(q-1)$ and so we see that $C_G(y) = C_G(x)$ for all $y \in C_G(x)^\#$. Thus $|\mathrm{fix}_\Omega(y)| = |N_G(\langle y \rangle) : G_\omega| = |N_G(G_\omega) : G_\omega| = 3$. This shows all assertions of the lemma. □

4.3. **Point Stabilizers of odd order.** The groups treated in the previous sections were those whose Sylow 2-subgroups fell into conclusions (2) or (3) of Lemma 2.18. In what follows, we therefore work under the following hypothesis:

Hypothesis 4.8. *Suppose that (G, Ω) satisfies Hypothesis 2.4 and that G is a simple group of Lie type, but none of the groups PSL$_2(q)$, Sz(q) or PSU$_3(q)$ where q is even, or PSL$_2(q)$, PSU$_3(q)$ or PSL$_3(q)$ where q is odd. Moreover we suppose that G_ω has odd order.*

Lemma 4.9. *If (G, Ω) satisfies Hypothesis 4.8, then $G \ncong$Sp$_4(3)$.*

Proof. We first observe that $5 \vdash 2$ and thus G_ω is a 3-group by Lemma 2.27. The centralizers of elements of order 3 in G have order divisible by 27, so Lemmas 2.2 and 2.20 imply that $|G_\omega| \geq 27$. Moreover the Sylow 3-subgroups of G are isomorphic to a wreath product $3 \wr 3$, therefore we see that G_ω contains 3-central elements whose centralizer order is divisible by 4. Together with Lemma 2.2 this contradicts Hypothesis 4.8. □

Lemma 4.10. *Suppose that (G, Ω) satisfies Hypothesis 4.8 and let $\alpha, \beta, \gamma \in \Omega$ be pair-wise distinct and such that $1 \neq H := G_\alpha \cap G_\beta \cap G_\gamma$. Then $|N_G(X) : N_H(X)| \in \{1, 3\}$ for all $1 \neq X \in H$ and there exists a nontrivial subgroup X of H such that $|N_G(X) : N_H(X)| = 3$.*

Proof. Hypothesis 4.8 implies that $|\Omega| \geq 7$. For all nontrivial subgroups X of H, we know by Lemma 2.2 that $|N_G(X) : N_H(X)| \leq 3$. There exists some $1 \neq X \leq H$ such that $N_G(X) \nleq H$ by Lemma 2.1, and for this subgroup $|N_G(X) : N_H(X)| \in \{2, 3\}$. However, index 2 cannot occur because otherwise some 2-element in $N_G(X)$ fixes one of α, β, γ, contrary to Hypothesis 4.8. □

We recall that, in a simple group G of Lie type of characteristic p, an element g is called semisimple if and only if its order is coprime to p. A semisimple element is called regular semisimple if and only if $(|C_G(g)|, p) = 1$. We note that the centralizer of a nonregular semisimple element contains a subgroup which is isomorphic to either $\mathrm{SL}_2(q)$ or $\mathrm{PSL}_2(q)$ and is generated by root elements of G. Recall that $\mathrm{SL}_2(q)$ is a perfect group when $q \geq 4$, and hence does not contain subgroups of index less than or equal to 4. Moreover $\mathrm{SL}_2(3) \cong Q_8 : 3$ and $\mathrm{SL}_2(2) = \mathrm{PSL}_2(2) \cong \mathcal{S}_3$.

Lemma 4.11. *If (G, Ω) satisfies Hypothesis 4.8, then all non-identity elements in point stabilizers are regular semisimple elements.*

Proof. Let $\omega \in \Omega$ and suppose that some non-identity element $g \in G_\omega$ is not regular and semisimple. Then either g is semisimple and $C_G(g)$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$ or $\mathrm{PSL}_2(q)$ and is generated by root elements of G, or g is not semisimple.
If $g \in G_\omega$ is not semisimple, then g powers to a p-element, so Hypothesis 4.8 implies that p is odd. If $p \neq 3$, then Lemma 2.15 yields that G_ω contains a full Sylow p-subgroup of G. Thus G_ω contains a long root element r and also $C_G(r)$ which, under Hypothesis 4.8, is a perfect group containing a subgroup isomorphic to $SL_2(q)$. Thus G_ω has even order which is a contradiction to Hypothesis 4.8.
If $p = 3$ and G_ω contains a 3-element, then Lemma 2.20 yields that G_ω either contains an index 3 subgroup of a Sylow 3-subgroup of G, or the Sylow 3-subgroup of G is of maximal class. The latter is excluded by Hypothesis 4.8, so we may assume that G_ω contains an index 3 subgroup of a Sylow 3-subgroup P of G. The Chevalley commutator relations imply that any index 3 subgroup of P must contain $Z(P)$ and thus $Z(P) \leq G_\omega$. If $G \not\cong {}^2G_2(q)$, then it follows from Lemma 2.2 that an index 3 subgroup of $C_G(Z(P))$ is contained in G_ω. But $|C_G(Z(P))|_2 \geq 8$, so now $|G_\omega|$ has even order, contradicting Hypothesis 4.8. Finally if $G \cong {}^2G_2(q)$, then $N_G(Z(P))$ has structure $P : (q-1)$ which implies that an element h of order $(q-1)/2$ lies in G_ω. Now $|N_G(\langle h \rangle)|_2 = 4$ which by Lemma 2.2 forces $|G_\omega|$ to be even, again contradicting Hypothesis 4.8.
Thus we have shown that the elements of G_ω are semisimple.
If $q \geq 4$ and g is semisimple, but not regular, then $C_G(g)$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$ or $\mathrm{PSL}_2(q)$ which is perfect. Hence Lemma 2.2 forces $|G_\omega|$ to be even, which violates Hypothesis 4.8.
If $q = 3$ and g is semisimple, but not regular, then $C_G(g)$ contains a subgroup isomorphic to $\mathrm{SL}_2(3)$. As $|\mathrm{SL}_2(3)|_2 = 8$, Lemma 2.2 implies that G_ω has even order, contradicting Hypothesis 4.8.
If $q = 2$, and g is semisimple, but not regular, then $C_G(g)$ has a subgroup isomorphic to $\mathrm{SL}_2(2)$. Since this group has order 6, Lemma 2.2 shows that $(|G_\omega|, 6) \neq 1$. So under Hypothesis 4.8 this means that G_ω contains a 3-element whose centralizer contains the centralizer R of a root subgroup $\mathrm{SL}_2(2)$. Hypothesis 4.8 implies that G is not of rank 2, because $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, $\mathrm{Sp}_4(2) \cong \mathrm{PSL}_2(9) \cong \mathcal{A}_6$, $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, and $G_2(2)' \cong \mathrm{PSU}_3(2)$. Therefore G has rank at least 3 and we see that $|R|_2 \geq 4$. But this means that $|G_\omega|$ is even, again contradicting Hypothesis 4.8. □

Having established that every element in a point stabilizer is regular, we now consider centralizers of regular semisimple elements in groups satisfying Hypothesis 4.8. We note that such a centralizer is a torus. Moreover the order of a torus is a polynomial in q of degree equal to the untwisted Lie rank of G.

Lemma 4.12. *If q is a prime power, $q > 3$ and $G = {}^2G_2(q)$, then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Assume otherwise and let $\omega \in \Omega$. If $g \in G_\omega^\#$, then $|C_G(g)| \in \{q-1, q+\sqrt{3q}+1, q-\sqrt{3q}+1\}$. If $|C_G(g)| = q-1$, then $|N_G(\langle g \rangle)|_2 = 4$ which implies that $|G_\omega|$ is even, a contradiction to Hypothesis 4.8. If $|C_G(g)| = q+\sqrt{3q}+1$ or $q-\sqrt{3q}+1$, then Lemma 2.2 implies that $C_G(g) \leq G_\omega$. Next we recall that $|N_G(C_G(g))/C_G(g)| = 6$, so Lemma 2.2 yields that 2 or 3 divides $|G_\omega|$. The former contradicts Hypothesis 4.8 whereas the latter contradicts Lemma 4.11. □

Hypothesis 4.13. *From now on until the end of this subsection we suppose that (G, Ω) satisfies Hypothesis 4.8 and that G is of Lie rank at least 2, but not isomorphic to $PSL_3(2)$, $G_2(2)$, $Sp_4(2)$, $PSU_4(2) \cong Sp_4(3)$ or $PSL_4(2) \cong \mathcal{A}_8$. Moreover all nonidentity elements of G_ω are regular and semisimple.*

We denote the natural module of G by N. By $\phi_d(x)$ we denote the irreducible cyclotomic polynomial dividing $x^d - 1$, but not $x^k - 1$ for all $k < d$.

Lemma 4.14. *Suppose that (G, Ω) satisfies Hypothesis 4.13 and let $\omega \in \Omega$. If $g \in G_\omega^\#$, then the following are true:*

(1) *$C_G(g)$ is a maximal torus of G.*
(2) *If G is a classical group, then $\dim(N) - \dim([N, g^i]) \le 2$ for all $i < o(g)$. Moreover if $\phi_d(q)$ is a divisor of $|C_G(g)|$, then $|N_G(T)/T|$ is divisible by d.*
(3) *If G is a classical group, then $(3, |G_\omega|) = 1$.*
(4) *If G is an exceptional group and $C_G(g)$ is not a 3-group, then 4 divides $|N_G(C_G(g))/C_G(g)|$.*
(5) *If G is an exceptional group, then for all 3-elements the centralizer has order divisible by 8.*

Proof. The conclusion of Lemma 4.11 is that g is regular and semisimple. This implies that $C_G(g)$ is a maximal torus; i.e. (1) follows.
Suppose now that G is classical and that $C_G(g)$ is not cyclic. Recall that N denotes the natural module of G. If $\phi_d(q)$ is a divisor of $|G|$, then G possesses an element x_d such that $\dim([N, x_d]) \in \{d, 2d\}$ and $[N, x_d]$ is nondegenerate with respect to the form defining G.
This is clear if $G = \mathrm{SL}(N)$ as there exists a $d \times d$ matrix with characteristic polynomial $\phi_d(x)$. For $\mathrm{Sp}(N)$ and $\mathrm{SU}(N)$ we embed the element via the overfield groups $SL_2(q^d) : d$, and if G is orthogonal, then we use the overfield groups $O_2^{\pm 1}(q^d) : d$. The embeddings show that d is a divisor of $N_G(\langle x_d \rangle)/C_G(x_d)$. Next we note that, if G is not orthogonal and $\dim(C_N(x_d)) \ge 2$, or if G is orthogonal and $\dim(C_N(x_d)) \ge 3$, then $|C_G(x_d)|$ is divisible by 4.
So if $r > 3$ is a prime divisor of $(|C_G(g)|, \phi_d(q))$, then Lemma 2.2 implies that G_ω contains a Sylow r-subgroup of G and thus a conjugate of a suitable power of the element x_d above. In light of Lemma 2.2 we must have that $|C_G(x_d)|_2 \le 2$ which implies that $\dim(N) - \dim([N, x_d]) = 0$ if G is symplectic, $\dim(N) - \dim([N, x_d]) \le 1$ if G is linear or unitary, and $\dim(N) - \dim([N, x_d]) \le 2$ if G is orthogonal. If $\dim(N) - \dim([N, g^i]) > 2$ for some proper power g^i of g, then the element g^i is not regular, contradicting Hypothesis 4.13. Thus (2) is proved.
If $r = 3$ and G_ω contains an element t of order 3, then Hypothesis 4.13 implies that t is semisimple, and hence $(3, q) = 1$. Thus if $q \equiv 1 \bmod 3$, then t is contained in a maximal split torus T^+ of G, and if $q \equiv -1 \bmod 3$, then t is contained in a torus T^- of order $(q+1)^{\dim(N)/2}$. If $q \equiv 1 \bmod 3$ and $q - 1 > 3$, then Lemma 2.2 implies that $T^+ \cap G_\omega$ contains every element of order $(q-1)/3$ of T^+. If $q \ne 4$, then, since the rank of G is at least 2, some element of T^+ of order $(q-1)/3$ is not regular and contained in G_ω. This contradicts Hypothesis 4.13. Similarly if $q \equiv -1 \bmod 3$ and $q + 1 > 3$, then Lemma 2.2 implies that $T^+ \cap G_\omega$ contains every element of order $(q+1)/3$ of T^+. If $q \ne 2$, then, as the rank of G is at least 2, some element of T^+ of order $(q+1)/3$ is not regular and contained in G_ω. Again this contradicts Hypothesis 4.13.
If $q = 2$, then either $|T^-| \ge 27$ and $N_G(T^-)$ has a subgroup isomorphic to $\mathcal{S}_3 \wr \mathcal{S}_{[\dim(N)/2]}$ or G is $\mathrm{PSL}_5(2)$. In all cases $|N_G(\langle t \rangle)|_2 \ge 4$ for every element $t \in T^-$ and hence Lemma 2.2 implies that $|G_\omega|$ is even, which is a contradiction.
If $q = 4$, then either $|T^+| \ge 27$ or the rank of G is 2. In the former case G_ω contains elements t with $|N_G(\langle t \rangle)|_2 \ge 4$ (choose t in a suitable rank 2 subgroup of T^+). So Lemma 2.2 forces that $|G_\omega|$ is even, again contradicting Hypothesis 4.13. The classical groups of rank 2 over the field of 4 elements are $\mathrm{PSL}_3(4)$, $\mathrm{PSp}_4(4)$, $\mathrm{PSU}_4(4)$ and $\mathrm{PSU}_5(4)$. The group $\mathrm{PSp}_4(4)$ is a subgroup of $\mathrm{PSU}_4(4) = \mathrm{SU}_4(4)$ and $\mathrm{SU}_4(4)$ is isomorphic to a subgroup of $\mathrm{PSU}_5(4)$. As $|\mathrm{PSp}_4(4)|_3 = |\mathrm{PSU}_4(4)|_3 = |\mathrm{PSU}_5(4)|_3 = 9$ we see that every 3-element of $\mathrm{PSU}_4(4)$ and $\mathrm{PSU}_5(4)$ fuses to a 3-element in $\mathrm{PSp}_4(4)$.
If t is a 3-element in $\mathrm{PSp}_4(4)$, then its centralizer in $\mathrm{PSp}_4(4)$ has order divisible by 4.
If $G = \mathrm{PSL}_3(4)$, then for every 3-element $t \in G$ we have that $|N_G(\langle t \rangle)| = 18$.
Now we suppose that $t \in G_\omega$ is of order 3. Then Lemma 2.2 implies that $|G_\omega|$ is even (see previous paragraph) or that G_ω contains a Sylow 3-subgroup of G. The first case contradicts Hypothesis 4.13. If, in the second case, T is a Sylow 3-subgroup of G with $T \le G_\omega$, then, as $|N_G(T)/T|_2 = 4$, Lemma 2.2 implies that $|G_\omega|$ is even. This is again a contradiction. We conclude that $3 \notin \pi(G_\omega)$ and hence (3) is proved.

Statement (4) can be deduced from Tables 5.1 and 5.2 of [17], whereas Statement (5) can be deduced from the tables in [18]. □

Corollary 4.15. *If (G, Ω) satisfies Hypothesis 4.13, then G is not an exceptional group.*

Proof. Assume otherwise and let $\omega \in \Omega$. If $g \in G_\omega^\#$, then Lemma 4.14 (1) says that $T := C_G(g)$ is a maximal torus. Let r be an odd prime and let R be a Sylow r-subgroup of T. If $r \neq 3$, then $R \leq G_\omega$ by Lemma 2.15 (c) and thus $|N_G(R) : (G_\omega \cap N_G(R))| \leq 3$ by Lemma 2.2. It follows with Lemma 4.14 (4) that $|N_G(R)|$ is divisible by 4 and hence G_ω has even order, contrary to Hypothesis 4.13. If $r = 3$, then G_ω contains a 3-element and so Lemma 4.14 (5) implies that $|G_\omega|$ is even again. This is a contradiction. □

Lemma 4.16. *If (G, Ω) satisfies Hypothesis 4.13 and $\omega \in \Omega$, then one of the following is true:*

(1) *$G = PSL_3(q)$ with q even, G_ω is cyclic of order $(q^2+q+1)/(3, q-1)$ (in particular of order coprime to 3) and $|\Omega| = (q-1)^2(q+1)q^3$. Moreover $|N_G(G_\omega) : G_\omega| = 3$.*
(2) *$G = PSL_4(3)$, G_ω is cyclic of order 13, and $|\Omega| = 2^7 \cdot 3^6 \cdot 5$.*
(3) *$G = PSL_4(5)$, G_ω is cyclic of order 31, and $|\Omega| = 2^7 \cdot 3^2 \cdot 5^6 \cdot 13$.*
(4) *$G = PSU_4(3)$, G_ω is cyclic of order 7, and $|\Omega| = 2^7 \cdot 3^6 \cdot 5$.*

Proof. By (1) of Lemma 4.14 $C_G(g)$ is a maximal torus for every $1 \neq g \in G_\omega$. Also Lemma 2.2 and (3) of Lemma 4.14 imply that $|C_G(g)|_3 \leq 3$. Let $d := \dim(N)/2$.
If G is symplectic, then the proof of (2) of Lemma 4.14 showed that $C_N(g) = 0$ for every $g \in G_\omega^\#$. On the other hand, using the fact that g is contained in a subgroup of G isomorphic to $\mathrm{SL}_2(q^d) : d$, we see that $|N_G(\langle g \rangle)/C_G(g)| = 2d$.
It follows from Hypothesis 4.13 that $|N_G(\langle g \rangle) : G_\omega \cap N_G(\langle g \rangle)|$ is even, which implies that G_ω contains an element h of order d which induces a Galois automorphism of order d on $\langle g \rangle$. Now $\dim(C_N(h)) = 2$, which means that h is not regular, contrary to Hypothesis 4.13. So G is not symplectic.
We observe that Lemma 4.10 yields that 3 must divide $N_G(\langle X \rangle)$ for some $X \leq G_\omega$. It follows with Lemma 2.2 that X lies inside some three point stabilizer H.
Now if for all $g \in H^\#$ and all $h \in \langle g \rangle^\#$ we have that $N_G(\langle h \rangle) \leq H$, then Lemma 2.1 implies that G is a Frobenius group, contrary to our main hypothesis. Therefore we find some $g \in H^\#$ such that $|N_G(\langle g \rangle) : N_H(\langle g \rangle)| = 3$.
If G is linear or unitary, then the elements $g \in G_\omega^\#$ satisfy $\dim(N) - \dim([N, g^i]) \leq 1$. Thus the order of every such g is a divisor of $q^{\dim(N)} - 1$ or $(q^{\dim(N)-1} - 1)$. Now using the fact that some nontrivial subgroup of H has to have a normalizer whose order is divisible by 3 implies that either $\dim(N) \equiv 0 \bmod 3$ or $\dim(N) - 1 \equiv 0 \bmod 3$. On the other hand Lemma 4.14 shows that if $o(g)$ is a divisor of $q^{\dim(N)} + 1$, of $q^{\dim(N)} - 1$ or of $(q^{\dim(N)-1} \pm 1)$, then $|N_G(\langle g \rangle)/C_G(g)|$ is divisible by $\dim(N)$ or $\dim(N) - 1$, respectively.
An element $h \in N_G(\langle g \rangle) \setminus C_G(g)$ whose order is one of $\dim(N)$ or $\dim(N) - 1$ and is divisible by 3 has the property that $\dim(C_N(h^3)) \geq 3$. Therefore it does not lie in G_ω by Hypothesis 4.13. Thus $|N_G(\langle g \rangle) : G_\omega \cap N_G(\langle g \rangle)| \geq \dim(N) - 1$ which implies that $\dim(N) \leq 4$ and $o(g)$ is a divisor of $q^3 + 1$ or $q^3 - 1$. Then also $C_G(g) \leq G_\omega$.
If $\dim(N) = 4$ and G is linear, then $|C_G(g)| = (q^3 - 1)/(4, q-1)$. If $q \notin \{3, 5\}$, then $C_G(g)$ contains elements of order dividing $(q-1)$ whose centralizer in G contains a subgroup isomorphic to $\mathrm{SL}_3(q)$. But then Lemma 2.2 implies that G_ω contains such a subgroup, which is a contradiction. If $q \in \{3, 5\}$, then $C_G(g)$ is cyclic of order $(q^3 - 1)/(4, q-1) = q^2 + q + 1$ and $N_G(\langle g \rangle)/C_G(g)$ is cyclic of order 3. Thus we obtain examples (2) and (3) from the lemma.
If $\dim(N) = 3$ and G is linear, then q is even, $|C_G(g)| = (q^2 + q + 1)/(3, q-1)$ and $N_G(\langle g \rangle)/C_G(g)$ is cyclic of order 3.

As 3 divides (q^2-1), but not $|C_G(g)|$, this implies that $G_\omega = C_G(g)$. Moreover $(q-1, q^3-1) = 3(q-1)$ and so we see that $C_G(y) = C_G(g)$ for all $y \in C_G(g)^\#$.
Thus $|F_\Omega(y)| = |N_G(\langle y\rangle) : G_\omega| = |N_G(G_\omega) : G_\omega| = 3$ as in (1).
If G is unitary, then $\dim(N) \geq 4$ and $q > 2$ by Hypothesis 4.13, and hence $\dim(N) = 4$. In this case $|C_G(g)| = (q^3+1)/(4, q+1)$. If $q > 3$, then $C_G(g)$ contains elements of order dividing $(q+1)$ whose centralizer in G has a subgroup isomorphic to $\mathrm{SU}_3(q)$. But then Lemma 2.2 implies that G_ω contains a subgroup isomorphic to $\mathrm{SU}_3(q)$, which is a contradiction. Note that $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$ does not give any examples by Lemma 4.9. Finally if $G = \mathrm{PSU}_4(3)$, then $|C_G(g)| = (3^3+1)/(4, 3+1) = 7$ and $|N_G(\langle g\rangle)/C_G(g)| = 3$, which yields example (4) in the conclusion of our lemma.
If G is orthogonal, then $\dim(N) \geq 7$ and $\dim([N, g])$ is even. Therefore $|N_G(\langle g\rangle)/C_G(g)| \geq 6$, which implies that $G_\omega = C_G(g)$. This means that G_ω contains an involution, contradicting our hypothesis that $|G_\omega|$ is odd. □

4.4. **Summary.**

Theorem 4.17. *Suppose that G is simple and of Lie type and that G is not isomorphic to an Alternating Group. Suppose further that (G, Ω) satisfies Hypothesis 2.4. Then one of the following is true:*

1. *$G = PSL_3(q)$, G_ω is cyclic of order $(q^2+q+1)/(3, q-1)$, and $|\Omega| = (q-1)^2(q+1)q^3$.*
2. *$G = PSL_4(3)$, G_ω is cyclic of order 13, and $|\Omega| = 2^7 \cdot 3^6 \cdot 5$.*
3. *$G = PSL_4(5)$, G_ω is cyclic of order 31, and $|\Omega| = 2^7 \cdot 3^2 \cdot 5^6 \cdot 13$.*
4. *$G = PSU_3(q)$ with $q \geq 3$, G_ω is cyclic of order $(q^2-q+1)/(3, q+1)$ and $|\Omega| = (q-1)(q+1)^3q^3$.*
5. *$G = PSU_4(3)$, G_ω is cyclic of order 7, and $|\Omega| = 2^7 \cdot 3^6 \cdot 5$.*
6. *$G = PSL_2(7) \cong PSL_3(2)$ with $|\Omega| = 7$ and $G_\omega \cong \mathcal{S}_4$.*
7. *$G = PSL_2(11)$ with $|\Omega| = 11$ and $G_\omega \cong \mathcal{A}_5$.*

We note that in (1) and (4) the point stabilizers have order coprime to 6.

Proof. The groups with strongly 2-embedded subgroups were considered in Lemmas 4.2, 4.3 and 4.4. The only examples arising here are the groups $\mathrm{PSU}_3(q)$ where q is even, as described in (4).
The groups with dihedral or semidihedral Sylow 2-subgroups were considered in Lemmas 4.5, 4.6 and 4.7. The examples arising here are the groups $\mathrm{PSU}_3(q)$ with q odd, which are accounted for in (4), the groups $\mathrm{PSL}_3(q)$ with q odd, which appear in (1), and the groups $\mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(11)$ which are listed in (6) and (7).
The groups for which the normalizer of a Sylow 2-subgroup is not strongly embedded and where the Sylow 2-subgroups are neither semidihedral nor dihedral satisfy Hypothesis 4.8. In fact all but $\mathrm{PSp}_4(3)$ and ${}^2G_2(q)$ satisfy Hypothesis 4.13. Lemma 4.9 shows that the group $\mathrm{PSp}_4(3)$ does not give any example and Lemma 4.12 shows the same for the groups ${}^2G_2(q)$.
The exceptional groups of Lie type which satisfy Hypothesis 4.13 do not lead to examples, as we have seen in Corollary 4.15. The classical groups of Lie type which satisfy Hypothesis 4.13 are treated in Lemma 4.16 and here the examples involving $\mathrm{PSL}_3(q)$ with q even, $\mathrm{PSL}_4(3)$, $\mathrm{PSL}_4(5)$, $\mathrm{PSU}_4(3)$ arise. These are accounted for in (1), (2), (3) and (5), respectively. □

For convenience we remind the reader that $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$ gives rise to two examples which are listed in (1) and (6) above.
Before analyzing the almost simple groups with socle $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$ (which play a role for Theorems 1.2 and 1.3), we need a preparatory lemma.

Lemma 4.18. *Suppose that p is a prime and let $a \in \mathbb{N}$ and $q := p^a > 4$, and let $E := PSL_3(q)$. Then $|Out(E)| = 2a \cdot (q-1, 3)$.*
Now suppose that G is a group such that $E < G \leq Aut(E)$ and that (G, Ω) satisfies Hypothesis 2.4. Let $\omega \in \Omega$ and suppose that E_ω is cyclic of order $q^2+q+1/(q-1, 3)$. Then the following are true:

1. *$N_G(E_\omega)/N_E(E_\omega) \cong G/E$.*

(2) *If $q > 4$, then $(q-1,3) = 3$, G/E is cyclic of order 3, and no element of $G \setminus E$ induces a field automorphism on E.*

Proof. The order of the outer automorphism group of E is well known and is as claimed. The outer automorphisms are diagonal, field or graph automorphisms and their products. All of this can be found in Theorem 2.5.12 of [11].
Now (1) follows from a Frattini argument, using the fact that G acts transitively on the set of point stabilizers in E.

We know from Lemmas 4.7 and 4.16 that $|N_E(E_\omega)| = 3 \cdot |E_\omega|$ and that $(|E_\omega|, 3) = 1$. Thus (1) implies that $|G_\omega : E_\omega| = |G/E|$.
To prove (2) we suppose to the contrary that $1 \neq |N_G(E_\omega)/N_E(E_\omega)| =: b$. Then Lemma 2.2 implies that G_ω contains a subgroup of order b. Now if $(b,3) = 1$, then all elements $h \in N_G(E_\omega) \setminus N_E(E_\omega)$ of prime order dividing b are either graph, field or graph-field automorphisms. Thus, as $q > 4$, it follows that $C_E(h)' \cong \mathrm{PSO}_3(q)$, $\mathrm{PSL}_3(q_0)$ or $\mathrm{PSU}_3(q_0)$, where q_0 divides q. As no proper subgroup of E contains both E_ω and $C_E(h)'$, we see that $N_G(E_\omega)/E_\omega$ is a 3-group.
Next we note that if $(3, q-1) = 1$, then $\mathrm{PGL}_3(q) \cong \mathrm{PSL}_3(q) = E$ and hence every element t of order 3 in $G \setminus E$ is a field automorphism such that $C_E(t)' \cong \mathrm{PSL}_3(q_0)$. Now (1) forces a conjugate of t into G_ω. However, as no proper subgroup of E can contain E_ω and $C_E(t)'$ we see that $3 = (q-1,3)$.
Thus $N_G(E_\omega)/E_\omega$ is a 3-group and $(3, q-1) = 3$ and, with l denoting the highest power of 3 dividing a (from our hypothesis), we see that G/E is a 3-group of order at most $3 \cdot l$.
If G/E contains field automorphisms of order 3, then (1) implies that G_ω contains a field automorphism t of order 3 such that $C_E(t) \cong \mathrm{PSL}_3(q_0)$. As before Lemma 2.2 forces $E \leq G_\omega$, which is impossible. The fact that no element of $G \setminus E$ is allowed to induce a field automorphism of E implies that $|G/E| = 3$, which is our claim. □

Lemma 4.19. *Suppose that G is almost simple and not simple and that $E = F^*(G) \cong PSL_3(q)$. If (G,Ω) satisfies Hypothesis 2.4, then $(3, q-1) = 3$, $G = PGL_3(q)$ and G_ω is cyclic of order $(q^3-1)/(q-1)$.*

Proof. If $F^*(G) = \mathrm{PSL}_3(q)$ with $q \leq 4$, then the table of marks for the almost simple groups of this type are in [23]. Inspection of these tables yields exactly our claimed example; i.e. $\mathrm{PGL}_3(4)$ acting on the cosets of a cyclic group of order 21.
So without loss we may assume that $q > 4$. Let $\omega \in \Omega$. First we note that E_ω is cyclic of order $(q^2+q+1)/(3, q-1)$ by Theorem 4.17 (1). Moreover by Lemma 4.18 we know that $3 = (q-1, 3)$ and either $G \cong \mathrm{PGL}_3(q)$ or $q = q_0^3$ and $G \cong \mathrm{PGL}_3^*(q) = \langle E, d \rangle$ where d induces a diagonal-field automorphism on E.
In the latter case we see, by direct computation, that any $g \in \mathrm{PGL}_3^*(q) \setminus E$ has order divisible by 9. Thus if $g \in G_\omega \setminus E_\omega$, then g has order 9 which implies that 3 divides $|E_\omega|$, contradicting Theorem 4.17. Hence $G \cong \mathrm{PGL}_3(q)$. We note that $G_\omega \leq N_G(E_\omega)$, but that $G_\omega \not\leq E$ by Lemma 4.18 (1).

We also note that a Singer cycle in $\mathrm{GL}_3(q)$ has order (q^3-1) and maps via the natural projection to a cyclic subgroup C of order $(q^3-1)/(q-1) = q^2+q+1$ of $\mathrm{PGL}_3(q)$. It follows from the subgroup structure of $\mathrm{PGL}_3(q)$ that $C \cap E$ is conjugate to E_ω. (They have the same order and are both cyclic.) So we may suppose that $E_\omega \leq C \leq N_G(E_\omega) = \langle C, t \rangle$ where $t \in N_E(E_\omega)$ is an element of order 3. We note that $N_G(E_\omega)/E_\omega$ is elementary abelian of order 9 and now we let $d \in C$ be of order 3 and such that $\langle d, t \rangle$ is a Sylow 3-subgroup of $N_G(E_\omega)$. In particular $C = \langle E_\omega, d \rangle$.

There are four possibilities for G_ω ($\leq N_G(E_\omega)$) because $\langle d, t \rangle$ has four subgroups of order 3.
The first possibility is that $G_\omega = \langle E_\omega, t \rangle$. But this is impossible because $t \in E$ and $G_\omega \not\leq E$. Now we assume that $G_\omega \in \{\langle E_\omega, dt \rangle, \langle E_\omega, d^{-1}t \rangle\}$.

Let $h \in \{dt, d^{-1}t\}$ (depending on G_ω) and choose $g \in K :=$GL$_3(q)$ to be a 3-element that projects onto h. Then $|C_K(g)| \geq (q-1)^2$, which implies that $|C_G(h)| \geq (q-1) > 3$ (because $q > 4$). Now $h \in G \setminus E$ and so $N_G(\langle h \rangle) = C_G(h)$. Hence $|N_G(\langle h \rangle)| = |C_G(\langle h \rangle)| = 3|C_G(h)| \geq 3(q-1) > 9$ and Lemma 2.2 implies that $N_G(\langle h \rangle) \leq N_G(C)$. It follows that $C_G(h) = C_H(h)$. On the other hand $|C_C(h)| = 3$ because t acts fixed point freely on E_ω. Therefore $9 = |C_H(h)| < |C_G(h)|$, which is a contradiction.
Now there is only one possibility left, namely that $G_\omega = \langle E_\omega, d \rangle = C$.

Finally we observe that the possibility $G_\omega = C$ leads to an example. To see this it suffices to observe that $N_G(\langle c \rangle) \leq N_G(C)$ for all $1 \neq c \in C \setminus E_\omega$. The latter is clear as $C_G(c) \leq C_G(d) = C$. □

Lemma 4.20. *Suppose that q is a prime power, $q \neq 2$, and that G is almost simple, but not simple, with $F^*(G) \cong PSU_3(q)$. If (G, Ω) satisfies Hypothesis 2.4, then $(3, q+1) = 3$, $G = PGU_3(q)$ and G_ω is cyclic of order $(q^3+1)/(q+1)$.*

Proof. Theorem 4.17 in combination with Theorem 1.2 of [19] implies that the only possible action for $F^*(G)$ is the action on the set of cosets of a cyclic group C of order $(q^2-q+1)/(3, q+1)$. By observing that GU$_3(q)$ lies naturally in GL$_3(q^2)$ such that the group C lies naturally in the cyclic group $E_\omega \leq$PSL$_3(q^2)$ of order $(q^4+q^2+1)/(3, q^2-1)$ from Lemma 4.19 above, we may use the computations from Lemma 4.18 and Lemma 4.19 to establish our claim. We omit the details. □

Lemma 4.21. *Suppose that (G, Ω) satisfies Hypothesis 2.4 and that G is almost simple such that $F^*(G)$ is one of $PSL_4(3)$, $PSL_4(5)$ or $PSU_4(3)$. Then G is simple.*

Proof. Let $\omega \in \Omega$ and suppose that $F^*(G)$ is one of PSL$_4(3)$, PSL$_4(5)$ or PSU$_4(3)$. Then $P := G_\omega \cap F^*(G)$ is a Sylow 13-, 7- or 31-subgroup of G, respectively. Now we note that $P < N_{F^*(G)}(P)$ by Theorem 4.17, but also $G = F^*(G) \cdot N_G(P)$ by Frattini. Hence Lemma 2.2 forces an involution $t \in N_G(P)$ into G_ω. Then the structure of $C_{F^*(G)}(t)$ and Lemma 2.2 imply that $F^*(G) \cap G_\omega \neq P$, which is a contradiction. □

5. The sporadic simple groups

In this section we adapt the notation in the ATLAS ([5]) for the names of the sporadic groups.

Lemma 5.1. *Suppose that G is M_{11} or M_{12} and that (G, Ω) satisfies Hypothesis 2.4. Then $G = M_{11}$ in its 4-fold sharply transitive action on 11 points.*

Proof. Let $\alpha \in \Omega$ and $H := G_\alpha$. Let $x \in H$ and $X := \langle x \rangle$. For maximal subgroups of G and information about local subgroups we refer to Tables 5.3a and 5.3b in [11].
First assume that x has order 11. Then $N_G(X)$ has order $11 \cdot 5$ and therefore Lemma 2.2 yields that H contains a subgroup Y of order 5. In both cases, $|N_G(Y)|$ is divisible by 4 and hence H has even order, again by Lemma 2.2. Then let $t \in H$ be an involution. Lemma 2.2 implies that H contains a subgroup of index at most 3 of $C_G(t)$. As $|H|$ is also divisible by 11 and by 5, the lists of maximal subgroups yield that $H = G$. This is impossible.
If x has order 5, then $N_G(X)$ has order divisible by 4 and hence H contains a subgroup of index at most 3 of an involution centralizer (applying Lemma 2.2 twice). This is possible if $G = M_{11}$ and Ω has 11 elements, and we already know that this is in fact an example for Hypothesis 2.4. In M_{12}, we see that H lies in the centralizer of an involution from class $2A$ and hence H contains a 3-element. Lemma 2.2 implies that 9 divides $|H|$, but this is false.
So from now on we consider the case where H is a $\{2, 3\}$-group.
Let us assume that x is an involution and that $|\text{fix}_\Omega(x)| \in \{1, 3\}$. Then all involutions have an odd number of fixed points and hence Lemma 2.15 (or 2.2 (a)) yields that H has odd index. In M_{12} we

immediately have $3 \in \pi(H)$ via $C_G(x)$ and Lemma 2.2. In M_{11} we look at a fours group in H and apply Lemma 2.2 to it in order to see that $3 \in \pi(H)$. Let $Y \leq H$ be a subgroup of order 3.
If $G = M_{11}$, looking at the list of maximal subgroups, we see that H does not contain a Sylow 3-subgroup of G in this case. So we may suppose that $|\mathrm{fix}_\Omega(Y)| = 3$ by Lemma 2.2 (a) and (b). It follows that $H = C_G(x)$. Let $a \in H$ be an element of order 8. As $|\Omega| = 165$, we see that x has either one fixed point, one orbit of length 4 and regular orbits or three fixed points, one orbit of length 2 and regular orbits on Ω. In both cases a^4 is an involution that has too many fixed points.
If G is M_{12}, then H contains a full involution centralizer. This implies that $5 \in \pi(H)$, which is a contradiction. Suppose that $o(x) = 3$. Then $N_G(X)$ has order divisible by 4 and hence Lemma 2.2 yields that H has even order. Let $t \in H$ be an involution. We already treated the case where some involution in H has one or three fixed points, so $|\mathrm{fix}_\Omega(t)| = 2$ and in particular $|\Omega|$ is even. Lemma 2.2 (b) yields that H contains an index two subgroup of an involution centralizer, which in the case M_{12} implies that H contains involutions from all conjugacy classes (see Lemma 2.18). In particular H contains a Sylow 2-subgroup of G, contrary to the fact that $|\Omega|$ is even. If $G = M_{11}$, then H contains subgroups of structure $\mathrm{SL}_2(3)$ and $\mathcal{S}_3 \times 2$, which is also impossible.
This finishes the proof. □

The remaining sporadic groups do not have dihedral or semidihedral Sylow 2-subgroups. This makes Lemma 2.18 very useful again.

Lemma 5.2. *Suppose that Hypothesis 2.4 holds and that G is a sporadic group, but not M_{11}. Let $\alpha \in \Omega$. Then G_α contains a Sylow 2-subgroup of G or it has odd order. In the second case there exists no prime $p \in G_\alpha$ such that $p \vdash 2$.*

Proof. This is a combination of Lemmas 2.18 and 2.27. □

Lemma 5.3. *Suppose that G is M_{22}, M_{23} or M_{24} and that Ω is a set such that (G, Ω) satisfies Hypothesis 2.4. Then $G = M_{22}$ and the action of G on Ω is as the action of G on the set of cosets of a subgroup of order 7.*

Proof. Let $\alpha \in \Omega$, let $x \in H := G_\alpha$ and $X := \langle x \rangle$. We may suppose that x has prime order p. For maximal subgroups of G and information about local subgroups we refer to Tables 5.3c-e in [11].
We first suppose that H has odd order, in particular p is odd and $p \not\vdash 2$ by Lemma 5.2. In all groups considered here, $11 \vdash 5$ and $5 \vdash 2$, so p is neither 11 nor 5. Moreover $23 \vdash 11$ and hence $p \neq 23$. If $p = 7$, then this either leads to M_{22} and the example that is stated in the lemma, or, in the larger groups, we have that $7 \vdash 3$. But also $3 \vdash 2$, so this leads to a contradiction. Lemma 5.2 leaves the case where H contains a Sylow 2-subgroup of G. Looking at centralizers of involutions (and in M_{22}, also at the normalizer of an elementary abelian subgroup of order 8), we see that $3 \in \pi(H)$ by Lemma 2.2.
If $G = M_{22}$, then H lies in a maximal subgroup of structure $2^4 : \mathcal{A}_6$ or $2^4 : \mathcal{S}_5$, so by Lemma 2.2 it is equal to one of these groups. But this does not agree with Lemma 2.2.
If G is M_{23} or M_{24}, then H contains a full involution centralizer. In M_{23} this means that H is a maximal subgroup of structure $2^4 : \mathcal{A}_7$. Then by congruence modulo 3, all 3-elements must have a unique fixed point and Lemma 2.2 forces H to contain a subgroup of structure $(3 \times \mathcal{A}_5) \cdot 2$. This is a contradiction. In M_{24} we see that H contains a Sylow 2-subgroup of G, hence $|\Omega|$ is odd. It is also coprime to 5 and 7, and in the only remaining possible case it follows that elements of order 5 in H have too many fixed points. □

Lemma 5.4. *Suppose that G is a Janko Group. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Assume that Ω is such a set, let $\alpha \in \Omega$ and $H := G_\alpha$. For information about local subgroups of G we refer to Tables 5.3f-i in [11] whereas we use the lists of maximal subgroups of G from Tables 5.4 and 5.11 in [25].

First suppose that H has odd order and let $x \in H$ be of prime order p. We note that $3 \vdash 2$ and $5 \vdash 2$, so $p \geq 7$ by Lemma 5.2. Then the tables yield that also $p \notin \{7, 19\}$. Moreover $11 \vdash 5$, $17 \vdash 2$, $23 \vdash 11$, $29 \vdash 7$ and $43 \vdash 7$. The only remaining primes are 31 and 37, but they are also impossible because $31 \vdash 5$ and $37 \vdash 2$. Hence this case does not occur at all.
With Lemma 5.2 we know that H contains involutions from all conjugacy classes. In particular $3, 5 \in \pi(H)$ whence, by Lemma 2.15, we see a Sylow 5-subgroup of G in H.
If $G = J_1$, then Lemma 2.2 yields that H contains a subgroup of shape $3 \times D_{10}$ or $\mathcal{S}_3 \times 5$ and a subgroup isomorphic to $\mathcal{A}_5$. There is no maximal subgroup that could contain H now.
If $G = J_2$, then H is contained in a maximal subgroup of structure $\mathcal{A}_5 \times D_{10}$ or $5^2 : D_{12}$ (by its index in G). Both cases are impossible because 9 divides $|H|$ by Lemma 2.2.
If $G = J_3$, then there is only one type of maximal subgroup that contains a Sylow 5-subgroup and a subgroup of order 3^3, and it has structure $(3 \times \mathcal{A}_6) : 2$. But its order is only divisible by 2^4 and not by 2^6, so it cannot contain H. In the last case $G = J_4$, we see that the centralizer of an involution involves the group M_{22}. Hence Lemma 2.2 yields that $|H|$ is divisible not only by $2, 3$ and 5, but also by 7 and 11, hence by 11^3 (using Lemma 2.15 (c)). There are only two types of maximal subgroups that have order divisible by 11^3, and in both cases their order is not divisible by 7. This is a contradiction. □

Lemma 5.5. *Suppose that G is a Conway Group. Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.*

Proof. Assume otherwise, let Ω denote such a set, let $\alpha \in \Omega$ and $H := G_\alpha$. For information about local subgroups of G we refer to Tables 5.3j-l in [11] and for lists of maximal subgroups of G and their indices we use [5]. The tables yield that for all prime divisors p of G, we have that $p \to 2$. Hence it is impossible that H has odd order, by Lemma 2.27. Lemma 5.2 implies that H contains involutions from all conjugacy classes. This yields that $3, 5 \in \pi(H)$. In particular H contains a full Sylow 5-subgroup of G by Lemma 2.15 (c). Inspection of the lists of maximal subgroups of G shows that all maximal subgroups have index divisible by 5 or by 2, which is a contradiction. □

The proof of the previous lemma indicates a general approach for most of the remaining sporadic groups.

Theorem 5.6. *Suppose that G is one of the following sporadic simple groups:*
$HS, McL, Suz, He, Ly, Ru, O'N, Fi_{22}, Fi_{23}, F'_{24}, HN, Th, BM$.
Then there is no set Ω such that (G, Ω) satisfies Hypothesis 2.4.

Proof. Assume otherwise and let Ω be such that (G, Ω) satisfies Hypothesis 2.4. Let $\alpha \in \Omega$ and $H := G_\alpha$. For information about local subgroups of G we refer to Tables 5.3m-y in [11] and for lists of maximal subgroups of G and their indices we use [5] unless stated otherwise.

(1) $2, 3, 5 \in \pi(H)$.

Proof. In all groups we see that for all odd $p \in \pi(G)$, we have that $p \to 2$ and hence H has even order. It contains involutions from all conjugacy classes by Lemma 5.2 and so we see that also $3, 5 \in \pi(H)$ by Lemma 2.2. □

(2) H is contained in a maximal subgroup of index that is odd and coprime to 5.

Proof. We know from (1) and from Lemma 2.15 (c) that H contains a Sylow 5-subgroup of G. Moreover H contains a Sylow 2-subgroup of G by (1) and Lemma 5.2. The same holds for a maximal subgroup containing H and hence the statement about the index follows. □

We inspect the lists of maximal subgroups of the groups and in particular their indices. In most cases, this already contradicts (2). For lists of maximal subgroups of the Fischer sporadic simple groups we refer to Table 5.5 in [25] (particularly because there is a mistake in the list of subgroups of Fi_{23} in [5]). For BM, we refer to Table 5.7 in [25].

For Th, there is one maximal subgroup missing in the ATLAS, namely $\mathrm{PSL}_3(3)$ (see Table 5.8 in [25]). Its index is divisible by 2^{11} and by 5^3, so this possibility contradicts (2). For Fi_{24}', we also note that the maximal subgroups of structure $\mathrm{PSU}_3(3):2$ and $\mathrm{PGL}_2(13)$ cannot contain H because of (2).
However, there are a few exceptions.
If $G = O'N$, then H could be contained in a maximal subgroup of structure $4^{\cdot}\mathrm{PSL}_3(4):2$. Then H contains subgroups of order 5 and 7, so by Lemma 2.15 (c) it follows that H contains a Sylow 7-subgroup of G. This has order 7^3, which is impossible.
If $G = \mathrm{Fi}_{23}$, then H could be contained in an involution centralizer of structure $2\mathrm{Fi}_{22}$. In particular H contains a subgroup of order 3^9 and hence a 3-central element of G. Lemma 2.2 implies that 3^{12} divides $|H|$, but this is false. □

Lemma 5.7. *There is no set Ω such that (M, Ω) satisfies Hypothesis 2.4.*

Proof. Assume otherwise and let G denote the Monster sporadic group M. Let Ω be such that (G, Ω) satisfies Hypothesis 2.4, let $\alpha \in \Omega$ and let $H := G_\alpha$. We refer to Table 5.3z in [11] for information about local subgroups and to Table 5.6 in [25] for the list of known maximal subgroups of G.
First we show that H has even order. This follows easily because, if p is any odd prime divisor of G, then inspection of the tables shows that $p \to 2$. Then we use Lemma 2.27. It follows from Lemma 5.2 that H contains involutions from both conjugacy classes, so looking at the involution centralizers in Table 5.3z in [11], Lemma 2.2 tells us that H contains a subgroup isomorphic to BM and to Co_1. Checking the list of known maximal subgroups of G, we already see that this does not occur.
On page 258 in [25] it is noted (quoting work of Holmes and Wilson) that if U is any other maximal subgroup of G, then there exists a group E isomorphic to one of $\mathrm{PSL}_2(13)$, $\mathrm{PSU}_3(4)$, $\mathrm{PSU}_3(8)$, $\mathrm{Sz}(8)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_2(16)$ or $\mathrm{PSL}_2(27)$ such that $E \leq U \leq \mathrm{Aut}(E)$. Checking the possibilities for U with these constraints, we see that U does not have a subgroup isomorphic to BM or to Co_1 and therefore H cannot be contained in a maximal subgroup U of G of this kind. □

All results of this section together yield the following:

Theorem 5.8. *Suppose that G is a sporadic simple group and that Ω is such that (G, Ω) satisfies Hypothesis 2.4. Then $G = M_{11}$ and $|\Omega| = 11$ or $G = M_{22}$ and $|\Omega| = 2^7 \cdot 3^2 \cdot 5 \cdot 11$.*

6. Proofs of the main results

Lemma 6.1. *Suppose that N is an elementary abelian normal subgroup of G and that H is a t.i. subgroup of G of order coprime to 6. Suppose further that $|N_G(X) : N_H(X)| = 3$ for all subgroups $1 \neq X \leq H$ and that $|C_N(H)| = 3$. Then H has a normal complement K in G.*

Proof. Our hypotheses imply that N is a 3-group. As $3 \notin \pi(H)$, it follows that H acts coprimely on N and therefore $N = C_N(H) \times [N, H]$. Moreover $|N_G(H) : H| = 3$, again by hypothesis. Now $C_N(H)$ has order 3 and it is not contained in H, so we have that $N_G(H) = C_N(H) \times H$. Moreover $[N, H]$ is an H-invariant subgroup of N, in particular $[N, H]H$ is a subgroup of NH. Let $h \in H^\#$ and let $x \in [N, H]$ be such that $x^h = x$. Then $h \in H \cap H^x$, so $H = H^x$ because H is a t.i. subgroup. This means that $[H, x] \leq H \cap N = 1$ and therefore $x \in C_N(H) \cap [N, H]$. This forces $x = 1$ and we deduce that $[N, H]H$ is a Frobenius group with complement H. As $|H|$ is odd, the Sylow subgroups of H are cyclic and in particular H is metacyclic (see 8.18 in [12]). Also we see that $Z(NH) = C_N(H)$.
Let $n \in \mathbb{N}$ and let $p_1, .., p_n$ be pair-wise distinct prime numbers such that $\pi(H) = \{p_1, ..., p_n\}$ and $p_1 < \cdots < p_n$. Let $P_1 \in \mathrm{Syl}_{p_1}(H)$. We recall that H is a $\{2, 3\}'$-group, so we know that $p_1 \geq 5$ and hence $P_1 \in \mathrm{Syl}_{p_1}(G)$ by Lemma 2.15 (c).

As P_1 is cyclic and p_1 is the smallest element in $\pi(H)$ we see that $|\mathrm{Aut}(P)|_{p_1'} = p_1 - 1$ is strictly smaller than the numbers $p_2, ..., p_n$. This means that $N_H(P_1) = P_1$. Moreover $|N_G(P_1) : N_H(P_1)| = 3$ by hypothesis and it follows that $N_G(P_1) = C_N(H) \times N_H(P_1) = C_N(H) \times P_1$. Burnside's p-complement theorem implies that P_1 has a normal p_1-complement M_1 in G. We recall that $p_1 \geq 5$ and hence $N \leq M_1$. Moreover $H_1 := H \cap M_1$ is characteristic in H and so $N_G(H_1) = C_N(H) \times H$.
We show that M_1, H_1 and N satisfy the hypotheses of the lemma instead of G, H and N. Of course N is an elementary abelian normal subgroup of M_1 and H_1 is a $\{2,3\}'$-group. Let $g \in M_1$ be such that $H_1 \cap H_1^g \neq 1$. Then $1 \neq H \cap M_1 \cap H^g$, in particular $H \cap H^g \neq 1$. This forces $H = H^g$ because H is a t.i. subgroup by hypothesis. Therefore $H_1 \cap H_1^g = H \cap M_1 \cap H^g = H \cap M_1 = H_1$, which means that H_1 is a t.i. subgroup of M_1. If $1 \neq Y \leq H_1$, then $N_G(Y) = N_H(Y) \times C_N(H)$ by hypothesis and hence $N_{M_1}(Y) = N_{H_1}(Y) \times C_N(H)$. In particular $|N_{M_1}(Y) : N_{H_1}(Y)| = 3$.
We continue in this way: $p_2 \geq 7$ and hence H_1 contains a Sylow p_2-subgroup P_2 of G, hence of M_1 (by Lemma 2.15 (c)). Arguing for M_1, H_1 and P_2 as for G, H and P_1 before, we find a normal p_2-complement M_2 in M_1. Then M_2 is characteristic in G, in fact $M_2 = O_{\{p_1,p_2\}'}(G)$ and M_2 contains N, so we may repeat these arguments until we reach the largest prime divisor of $|H|$. This way we find a normal complement for H in G, namely $O_{\pi(H)'}(G)$. □

In light of the results of the previous sections, the proofs of Theorem 1.1 and 1.2 are basically an application of the Classification of Finite Simple Groups (CFSG). The main point of this section is to prove Theorem 1.3, which requires a bit more work.

Proof of Theorem 1.1. Let Ω be a set such that (G, Ω) satisfies Hypothesis 2.4 and such that G is simple. Then we apply the CFSG and Theorems 3.14, 4.17 and 5.8. This gives exactly the possibilities that are listed in Theorem 1.1. □

Proof of Theorem 1.2. Let Ω be a set such that (G, Ω) satisfies Hypothesis 2.4 and suppose that G is almost simple, but not simple. Then Lemma 2.23 implies that either $F^*(G) \cong \mathrm{PSL}_2(2^p)$ with p a prime, which is conclusion (1), or $(F^*(G), \Omega)$ satisfies Hypothesis 2.4. If $F^*(G)$ is an Alternating Group, then Theorem 3.14 yields that $\mathcal{S}_5$ acting on 5 points is the only example. But in light of the isomorphism $\mathcal{S}_5 \cong \mathrm{Aut}(\mathrm{PSL}_2(4))$ we see that this example is a special case of conclusion (1). If $F^*(G)$ is of Lie type, then Lemmas 4.19, 4.20 and 4.21 show that (2) and (3) are the only possible examples.
Finally if $F^*(G)$ is sporadic, then $F^*(G)$ is isomorphic to M_{11} or to M_{22}. Our hypothesis that G is not simple implies that only the latter case can occur and in fact $G \cong \mathrm{Aut}(M_{22})$. Let $\omega \in \Omega$. Then G_ω contains a Sylow 7-subgroup S of G and $N_{F^*(G)}(S) \cap G_\omega = S$. Now $|N_G(S)/N_{F^*(G)}(S)| = 2$ and thus Lemma 2.2 forces an involution t into G_ω. However $|C_{F^*(G)}(t)| = 1344$ and then Lemma 2.2 gives that $N_{F^*(G)}(S) \cap G_\omega \neq S$, which is a contradiction. □

We already proved in Section 2 that, if (G, Ω) satisfies Hypothesis 2.4, then $3 \in \pi(G)$. For the additional details in our main results, we split our analysis in two parts.

Proposition 6.2. *Let Ω be a set such that (G, Ω) satisfies Hypothesis 2.4 and let $\omega \in \Omega$. If $|G_\omega|$ is even, then one of the following is true:*

(1) *G has a normal 2-complement.*
(2) *G has dihedral or semidihedral Sylow 2-subgroups and 4 does not divide $|G_\omega|$. In particular G_ω has a normal 2-complement.*
(3) *G_ω contains a Sylow 2-subgroup S of G and G has a strongly embedded subgroup.*
(4) *$|G : G_\omega|$ is even, but not divisible by 4 and G has a subgroup of index 2 that has a strongly embedded subgroup.*

Proof. By hypothesis one of the cases (2), (3) or (4) from Lemma 2.17 holds. Case (2) leads to possibility (2) of our proposition. In Case (4) we apply Lemma 2.19, where one of the possibilities (2), (3) or (4)

holds. They lead to the cases (3), (1) and (4) of our proposition. Finally we suppose that Lemma 2.17 (3) holds. Then either S is cyclic, which leads to (1), or some elements of $S^\#$ act as odd permutations on Ω and hence G has a subgroup G_0 of index 2. Let $S_0 := G_0 \cap S$. Then S_0 fixes exactly two points α, ω on Ω. Let M denote the set-wise stabilizer of $\{\alpha, \omega\}$ in G_0.
Let $g \in G_0$ and let $x \in M \cap M^g$ be a nontrivial 2-element, without loss $x \in S_0$. Then x fixes α and ω and it is contained in a Sylow 2-subgroup of M^g, so we may suppose that it fixes α^g and ω^g. Lemma 2.15 (a) implies that x does not have three fixed points, so $\{\alpha, \omega\} = \{\alpha^g, \omega^g\}$ and therefore $g \in M$. This shows that M is a strongly embedded subgroup of G_0 as stated in (4). □

We say that G **is a** $(0,3)$**-group** on Ω (as in [20]) if and only if G acts as a transitive permutation group on Ω and all elements in $G^\#$ fix either 0 or 3 points.

Proposition 6.3. *Let Ω be a set such that (G, Ω) satisfies Hypothesis 2.4 and let $\omega \in \Omega$. Suppose that $|G_\omega|$ is odd. If $|fix_\Omega(G_\omega)| = 3$, then one of the following is true:*

(1) *G has a normal subgroup R of order 27 or 9, and G/R is isomorphic to $\mathcal{S}_3$, $\mathcal{A}_4$, $\mathcal{S}_4$, to a fours group or to a dihedral group of order 8.*
(2) *G has a regular normal subgroup.*
(3) *G has a normal subgroup F of index 3 which acts as a Frobenius group on its three orbits.*
(4) *G has a normal subgroup N which acts semiregularly on Ω such that G/N is almost simple and G_ω is cyclic.*

Proof. If G_ω is not t.i., then the main theorem of [21] implies that G has a regular normal subgroup of order 27 or 9. The structure of G/R as described in (1) is given in the corollary to the main theorem of [21].
On the other hand if G_ω is t.i. and $3 \in \pi(G_\omega)$, then Proposition 6.5 of [20] implies that G has a normal subgroup N of index 3. If the action of N on Ω is transitive, then by induction over the order of an example we can see that N contains a regular normal subgroup N_0, or a normal index 3 Frobenius group F_0. In the first case a Frattini argument implies that $G = N_0 G_\omega = G_\omega N_0$ and thus N_0 is normal in G, proving that G possesses a regular normal subgroup. In the second case the Frobenius kernel K_0 of F_0 is a characteristic subgroup of F_0, and is hence also normal in G. The number of F_0-orbits on Ω is equal to 3, thus the orbit stabilizer G_0 in G of one of the F_0-orbits acts as a Frobenius group on its fixed orbit, and hence on all F_0 orbits. Thsi means that G_0 is a Frobenius group of index 3 in G. As G/F_0 has order 9, every index three subgroup of G/F_0 is normal. Thus $G_0 \trianglelefteq G$, which is one of our possible conclusions.
Finally we consider the case where G_ω is still a t.i. subgroup and moreover $|G_\omega|$ is coprime to 6. If G is solvable, then Proposition 3.1 of [20] shows that either (2) or (3) holds. Thus we may assume that G is not solvable.
Suppose that r is a prime and that N is a minimal normal subgroup of G which is an elementary abelian r-group. If $r \notin \{2, 3\}$, then $N \cap G_\omega = 1$ by Lemma 2.15. Otherwise $N \cap G_\omega = 1$ because $|G_\omega|$ is coprime to 6. In both cases N acts semiregularly on Ω. If $r \neq 3$, then Lemma 1.9 in [20] implies that G_ω has at most one fixed point on ω^N. If $r = 3$, then N is an elementary abelian 3-group and thus so is $C_N(G_\omega)$. As $|C_N(G_\omega)|$ is the number of fixed points of N on ω^N, we see that either $\mathrm{fix}_\Omega(G_\omega) \cap \omega^N = \{\omega\}$ (as desired) or that $|\mathrm{fix}_\Omega(G_\omega) \cap \omega^N| = 3$ and thus $|C_N(G_\omega)| = 3$.
If $|C_N(G_\omega)| = 3$, then Lemma 6.1 implies that G_ω has a normal complement K in G. As $|K||G_\Omega| = |G| = |\Omega||G_\omega|$, we obtain that $K \cap G_\omega = 1$ and thus that K is a regular normal subgroup. This is one of our conclusions.
So if H does not possess a normal complement in G, then every abelian minimal normal subgroup N of G acts semiregularly on Ω and $\mathrm{fix}_\Omega(G_\omega)$ intersects an N-orbit in at most one point. If $r \neq 3$ and conclusion (3) does not hold, then Lemma 1.9 of [20] asserts that the action of G/N on $\tilde{\Omega}$, the set of

N-orbits on Ω, is faithful. Let $\tilde{\omega}$ denote the element of $\tilde{\Omega}$ containing ω. Then $G_{\tilde{\omega}} \cong G_\omega$ and every $x \in G_{\tilde{\omega}}$ fixes either three or no points of $\tilde{\Omega}$. So $(G/N, \tilde{\Omega})$ satisfies Hypothesis 2.4 and it is a $(0,3)$-group.
If $r = 3$, then we saw that $C_N(x) = 1$ for all $x \in G_\omega^\#$. Thus x fixes exactly 3 orbits of N. On each of these the action of NG_ω is Frobenius. If $|\tilde{\Omega}| = 3$, then NG_ω is an index three Frobenius subgroup of G and G acts on $\tilde{\Omega}$ as a cyclic group or as $\mathcal{S}_3$. The latter case cannot happen because G_ω has odd order. So NG_ω is the kernel of the action of G on $\tilde{\Omega}$ and hence it is normal in G. So if $|\tilde{\Omega}| = 3$, then conclusion (3) holds. Thus we may assume that $|\tilde{\Omega}| > 3$. The kernel of the action of G on $\tilde{\Omega}$ lies in the stabilizer of ω^N which is NG_ω. As NG_ω is a Frobenius group with complement G_ω the kernel of the action must lie inside N, which implies that G/N acts faithfully on $\tilde{\Omega}$. Also $G_{\tilde{\omega}} \cong G_\omega$ and every $x \in G/N$ fixes either three or no points of $\tilde{\Omega}$.
Thus if conclusion (3) does not hold, then $(G/N, \tilde{\Omega})$ satisfies Hypothesis 2.4 and G/N acts as a $(0,3)$-group. If neither (2) nor (3) holds for G and if N has an abelian minimal normal subgroup, then by induction on $|G|$ we may conclude that (4) holds for $(G/N, \tilde{\Omega})$. In turn this implies that conclusion (4) holds for G. On the other hand if neither (2) nor (3) holds for G and N does not have an abelian minimal normal subgroup, then by Theorem 2.24 we see that G is almost simple and the action on Ω must satisfy Hypothesis 2.4. (The case with $F^*(G) = \mathrm{PSL}_2(2^p)$ implies that $|G_\omega|$ is even, hence it is not allowed here.) Inspection of the simple and almost simple examples now yields that G_ω is cyclic. Thus again conclusion (4) holds and our proof is complete. □

Proof of Theorem 1.3. Let Ω be a set such that (G, Ω) satisfies Hypothesis 2.4. Then G has order divisible by 3 by Lemma 2.25. If $\omega \in \Omega$, then we first consider the case where G_ω has even order. Then Lemma 6.2 gives exactly the possibilities in Theorem 1.3 (i). Next we suppose that G_ω has odd order. Then Corollary 2.6 reduces our situation to the case of $(0,3)$-groups, so Proposition 6.3 is applicable. It yields the details in Theorem 1.3 (ii). □

We now consider the situation where G_ω is a Frobenius group of odd order. We note that Corollary 2.6 implies that G acts as a $(0,3)$-group on the set of cosets of a nontrivial three point stabilizer H and therefore one of the conclusions of Proposition 6.3 holds. Conclusion (4) is impossible because G_ω is a Frobenius group by hypothesis, so in particular it is not cyclic. Conclusions (1) and (3) pin down the structure of G as best as possible and thus we now consider the situation where G has a regular normal subgroup.

Lemma 6.4. *Let Ω be a set such that (G, Ω) satisfies Hypothesis 2.4 and let $\omega \in \Omega$. Suppose further that $|G_\omega|$ is odd and that $|fix_\Omega(G_\omega)| \neq 3$. If G has a normal subgroup N that acts regularly on the set of cosets of a three point stabilizer H, then G is solvable. More precisely it is one of the groups from Lemma 2.10.*

Proof. As $|\mathrm{fix}_\Omega(G_\omega)| \neq 3$, Lemma 2.5 implies that G_ω is a Frobenius group with complement H. We denote the Frobenius kernel by K and we remark that $K \cap H = 1$, so that all elements of $K^\#$ fix at most two points in Ω. By hypothesis N acts transitively on G/H and therefore $G = N \cdot H$ by a Frattini argument. In particular $K \leq N$.
We will show that N is a Frobenius group with complement K, using Lemma 2.1. Hence assume that $1 \neq X \leq K$ is such that $N_N(X) \not\leq K$. Then Lemma 2.2 implies that X fixes exactly two points and $|N_N(X) : N_K(X)| = 2$. In particular there exists a 2-element $t \in N \setminus K$ that interchanges the two fixed points of X. We recall that $K \trianglelefteq G_\omega$ and $G = N \cdot H = N\dot{G}_\omega$, so $N_G(K) = N_N(K) \cdot G_\omega = N_N(K) \cdot K \cdot H = N_N(K) \cdot H$.
...
It follows that $t \in N_G(H)$.
...

Therefore $N_N(X) \leq K$ for all $1 \neq X \leq K$, establishing that N is a Frobenius group. This also implies that every nontrivial element of K fixes a unique point in Ω.
As K is a Frobenius complement in N of odd order (because $K \leq G_\omega$) and Frobenius kernels are nilpotent, it follows that N is solvable. Then $G = N \cdot H$ is also solvable. Let $\omega_1, \omega_2 \in \Omega$ be such that $\mathrm{fix}_\Omega(H) = \{\omega, \omega_1, \omega_2\}$.
Let Γ be a nonregular G_ω-orbit on Ω such that $|\Gamma| \geq 2$. Then there are $y \in G_\omega^\#$ and $\gamma \in \Gamma$ such that y fixes γ. Since y also fixes ω and all elements of $K^\#$ have a unique fixed point, we see that $y \notin K$. Without loss $y \in H$, which means that $\gamma \in \{\omega, \omega_1, \omega_2\}$. We recall that $G_\omega = K \cdot H$, so $\omega_1^{G_\omega} = \omega_1^K$ and $\omega_2^{G_\omega} = \omega_2^K$. We deduce that $\{\omega\}$, ω_1^K and ω_2^K are the only nonregular G_ω-orbits.
Thus G satisfies the hypotheses of Lemma 4.3 in [21], and this lemma implies that G has a normal subgroup A which is isomorphic to the additive group of a finite field of order 3^p (where p is prime). Moreover K is a subgroup of the multiplicative group of this field and H is the Galois group of the field. This coincides with the series of examples in Lemma 2.10. □

The final two lemmas give additional information for conclusion (2) of Proposition 6.3.

Lemma 6.5. *Let Ω be a set such that (G, Ω) satisfies Hypothesis 2.4. Let $\omega \in \Omega$ and suppose that $|G_\omega|$ is odd. If $|fix_\Omega(G_\omega)| = 3$ and G contains a regular normal subgroup N, and if moreover G_ω is not a 3-group, then N is solvable and $N = O_{3,3'}(N)C_N(x)$ for some $x \in G_\omega$ with $|C_N(x)| = 3$.*

Proof. As G_ω has odd order and is not a 3-group, there exists $x \in G_\omega$ of prime order $p > 3$. Now Lemma 2.15 implies that G_ω contains a Sylow p-subgroup of G and hence N is a p'-group. In particular $o(x)$ and $|N|$ are coprime, so the hypotheses of Lemma 2.13 are satisfied and this implies our conclusion. □

Lemma 6.6. *Suppose that Hypothesis 2.4 holds and let $\omega \in \Omega$. Suppose that G_ω is a 3-group and that $|fix_\Omega(G_\omega)| = 3$. If G has a regular normal subgroup N, then G_ω is cyclic, N is solvable and $G = O_{3,3'}(N)G_\omega$.*

Proof. We set $H := G_\omega$ and, by hypothesis, we let $P \in \mathrm{Syl}_3(G)$ be such that $H \leq P$. As N acts regularly on Ω, a Frattini argument implies that $G = N \rtimes H = N \cdot P$. We also note that $|\mathrm{fix}_\Omega(H)| = 3$ and therefore $|C_N(H)| = 3$. In particular $3 \in \pi(N)$ and therefore $P \cap N \neq 1$. This implies that $P \neq H$. Lemma 2.15 tells us that $|P : H| \leq 3$ or that $|H| = 3$ and P has maximal class. In the first case $|P : H| = 3$ and therefore $P \cap N$ (a Sylow 3-subgroup of N) has order 3. We recall that $H \leq P$ and that, therefore, H normalizes $N_N(P \cap N)$ and $C_N(P \cap N)$.
Assume that $C_N(P \cap N) \neq N_N(P \cap N)$. Then $|N_N(P \cap N)/C_N(P \cap N)| = 2$ and therefore H centralizes this quotient. This means, conversely, that $C_N(H)$ has even order. But any non-trivial 2-element of $C_N(H)$ has a fixed point on $\mathrm{fix}_\Omega(H)$, contrary to our hypothesis that H is a 3-group.
It follows that $C_N(P \cap N) = N_N(P \cap N)$, so by Frobenius' p-complement Theorem N has a normal 3-complement K. This means that $G = N \cdot P = K \cdot P$. Since $|N : K| = 3$ and N acts regularly on Ω, we obtain that K has three orbits on Ω and that $K \cdot H$ is a Frobenius group with Frobenius complement H. As H is a 3-group, we see that H is cyclic and that $N = O_{3,3'}(N)$, which is our conclusion. This finishes the first case.
In the second case $|H| = 3$ and P has maximal class. Lemma 1.9 of [20] implies that $O_{3'}(N)$ acts semiregularly on Ω and that the action of $G/O_{3'}(N)$ is faithful on the set $\tilde{\Omega}$ of $O_{3'}(N)$-orbits. Now, since no almost simple group can satisfy Hypothesis 2.4 with point stabilizer a 3-group (see Theorem 1.2), we deduce that $F^*(G/O_{3'}(N)) = F(G/O_{3'}(N)) = O_3(G)(G/O_{3'}(G))$ and that $O_3(G)(G/O_{3'}(G))$ acts semiregularly on $\tilde{\Omega}$. Thus one of the following could happen:
$O_3(G)(G/O_{3'}(G))$ acts regularly on $\tilde{\Omega}$, or it acts semiregularly with at least three orbits on $\tilde{\Omega}$.
The latter possibility does not occur because a 3-group never acts fixed point freely on a 3-group and hence H fixes three points on any H-invariant $O_3(G)(G/O_{3'}(G))$-orbit. Thus the first possibility occurs,

which means that $N/O_{3'}(G)$ acts regularly on $\tilde{\Omega}$. Then $G/O_{3'}(G)$ is a 3-group because $|G/O_{3'}(G)| = |\tilde{\Omega}| \cdot |H| = |N/O_{3'}(G)| \cdot |H|$. Again our claim follows. $\square$

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