VOLUME PRESERVING SUBGROUPS OF $\mathcal{A}$ AND $\mathcal{K}$ AND SINGULARITIES IN UNIMODULAR GEOMETRY

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Abstract. For a germ of a smooth map f from $\mathbb{K}^n$ to $\mathbb{K}^p$ and a subgroup G_{Ω_q} of any of the Mather groups G for which the source or target diffeomorphisms preserve some given volume form Ω_q in $\mathbb{K}^q$ ($q = n$ or p) we study the G_{Ω_q}-moduli space of f that parameterizes the G_{Ω_q}-orbits inside the G-orbit of f. We find, for example, that this moduli space vanishes for $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$ and $\mathcal{A}$-stable maps f and for $G_{\Omega_q} = \mathcal{K}_{\Omega_n}$ and $\mathcal{K}$-simple maps f. On the other hand, there are $\mathcal{A}$-stable maps f with infinite-dimensional $\mathcal{A}_{\Omega_n}$-moduli space.

Introduction

We are going to study singularities arising in unimodular geometry. A singular subvariety of a space with a fixed volume form may be given by some parametrization or by defining equations. This leads to the following (multi-)local classification problems. (1) The classification of germs of smooth maps $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, \Omega_p, 0)$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) up to $\mathcal{A}_{\Omega_p}$-equivalence (i.e., for the subgroup of $\mathcal{A}$ in which the left coordinate changes preserve a given volume form Ω_p in the target), and also of multi-germs of such maps up to $\mathcal{A}_{\Omega_p}$-equivalence. (2) The classification of variety-germs $V = f^{-1}(0) \subset (\mathbb{K}^n, \Omega_n, 0)$ up to $\mathcal{K}_{\Omega_n}$-equivalence of $f : (\mathbb{K}^n, \Omega_n, 0) \to (\mathbb{K}^p, 0)$ (i.e., for the subgroup of $\mathcal{K}$ in which the right coordinate changes preserve a given volume form Ω_n in the source). More generally, we will consider volume preserving subgroups G_{Ω_q} of any of the Mather groups $G = \mathcal{A}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$ preserving a (germ of a) volume form Ω_q in the source (for $q = n$) or target (for $q = p$). (See the survey [50] for a discussion of the groups G and their tangent spaces LG, or see the beginning of §3 below for a brief reminder.)

These subgroups G_{Ω_q} of G fail to be geometric subgroups of $\mathcal{A}$ and $\mathcal{K}$ in the sense of Damon [11, 12], hence the usual determinacy and unfolding theorems do not hold for G_{Ω_q}. In this situation moduli and even functional moduli often appear already in codimension zero, and e.g. for $\mathcal{R}_{\Omega_n}$ this is indeed the case: a Morse function has a functional modulus (and hence infinite modality) in the volume preserving case [49]. Hence it might appear surprising that Martinet wrote 30 years ago in his book (see p. 50 of the English translation [37]) on the $\mathcal{A}_{\Omega_p}$ classification problem in unimodular geometry that the groups involved "are big enough that there is still some hope of finding a reasonable classification theorem". It turns out that Martinet was right – the results of this paper imply, for example, that over $\mathbb{C}$ the classifications of stable map-germs for $\mathcal{A}_{\Omega_p}$ and for $\mathcal{A}$ agree, and hence Mather's [40] nice pairs of dimensions (n, p). Furthermore, the classifications of simple complete

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intersection singularities agree for $\mathcal{K}_{\Omega_n}$ and for $\mathcal{K}$. Over $\mathbb{R}$ a G-orbit ($G = \mathcal{A}$ or $\mathcal{K}$) corresponds to one or two orbits in the volume preserving (hence orientation preserving) case, otherwise the results are the same.

We will now summarize our main results. For any of the above Mather groups G, let G_f denote the stabilizer of a map-germ f in G and let G_e, as usual, denote the extended pseudo group of non-origin preserving diffeomorphisms. The differential of the orbit map of f (sending $g \in G$ to $g \cdot f$) defines a map $\gamma_f : LG \to LG \cdot f$ with kernel LG_f. Let LG_f^q be the projection of LG_f onto the source (for $q = n$) or the target factor (for $q = p$). Notice that, for example, the group $G = \mathcal{R}$ can be viewed as a subgroup $\mathcal{R} \times 1$ of $\mathcal{A}$ with Lie algebra $L\mathcal{R} \oplus 0$ – allowing such trivial factors 1 enables us to define the projections LG_f^q for all Mather groups G, which will be convenient for the uniformness of the exposition. For a given volume form Ω_q in $(\mathbb{K}^q, 0)$ we have a map div : $\mathcal{M}_q \cdot \theta_q \to C_r$ sending a vector field (vanishing at 0) to its divergence, where $r = q$ for all G_{Ω_q} except $\mathcal{K}_{\Omega_p}$ (we use here the following standard notation: C_q denotes the local ring of smooth function germs on $(\mathbb{K}^q, 0)$ with maximal ideal $\mathcal{M}_q$, and θ_q denotes the C_q module of vector fields on $(\mathbb{K}^q, 0)$). For $\mathcal{K}_{\Omega_p}$ we consider linear vector fields in $(\mathbb{K}^p, 0)$ with coefficients in C_n, the divergence of such a vector field is an element of C_n. We will show that for the (infinitesimal) G_{Ω_q} moduli space $\mathcal{M}(G_{\Omega_q}, f)$ we have the following isomorphism

$$\mathcal{M}(G_{\Omega_q}, f) := \frac{LG \cdot f}{LG_{\Omega_q} \cdot f} \cong \frac{C_r}{\mathrm{div}(LG_f^q)}.$$

For $\mathcal{K}_{\Omega_n}$ the vector space on the right is in turn isomorphic to the nth cohomology group of a certain subcomplex of the de Rham complex associated with any finitely generated ideal $\mathcal{I}$ in C_n (defined in Section 4), taking $\mathcal{I} = \langle f_1, \dots, f_p \rangle$ (the ideal generated by the component functions f_i of f). For $\mathcal{A}_{\Omega_p}$ we obtain an analogous isomorphism by taking the vanishing ideal $\mathcal{I}$ of the discriminant (for $n \geq p$) or the image (for $n < p$) of f, provided $L\mathcal{A}_f^p$ (also known as Lift(f)) is equal to Derlog of the discriminant or image of f.

Furthermore, if LG has the structure of a C_r-module (this is the case for all G_{Ω_q} except $\mathcal{A}_{\Omega_n}$) then $\dim \mathcal{M}(G_{\Omega_q}, f)$ is equal to the number of G_{Ω_q} moduli of f (for $\mathcal{A}_{\Omega_n}$ this equality becomes a lower bound). This will be shown in the following way. The notion of G_{Ω_q}-equivalence of maps f and g (for a given volume form Ω_q) is easily seen to be equivalent to the following notion of G_f^q-equivalence of volume forms Ω_q and Ω'_q (for a given map f): $\Omega'_q \sim_{G_f^q} \Omega_q$ if and only if for some $h \in G_f^q$ we have that $h^*\Omega'_q = \Omega_q$. It then turns out that a pair Ω_q and Ω'_q (that in the case of $\mathbb{R}$ defines the same orientation) can be joined by a path of G_f^q equivalent volume forms if and only if $\Omega'_q - \Omega_q = d(\xi \rfloor \Omega)$ for some $\xi \in LG_f^q$ and any volume form Ω in $(\mathbb{K}^q, 0)$. And the number of G_f^q moduli of volume forms (and hence of G_{Ω_q} moduli of f) is given by the dimension of the space $\Lambda^q / \{d(\xi \rfloor \Omega) : \xi \in LG_f^q\}$ (here Λ^q denotes the space of q-forms in $(\mathbb{K}^q, 0)$), which turns out to be equal to $\dim C_q / \mathrm{div}(LG_f^q)$.

If, furthermore, $\mathcal{M}(G_{\Omega_q}, f) = 0$ then, over $\mathbb{C}$, we have at the formal level (and also in the smooth category, provided the sufficient vanishing condition w.q.h. for $\mathcal{M}(G_{\Omega_q}, f)$ below holds)

$$G_{\Omega_q} \cdot f = G \cdot f$$

Over $\mathbb{R}$, the orbit $G \cdot f$ consists of one or two G_{Ω_q}-orbits, due to orientation as mentioned above. More precisely, if G^+ denotes the subgroup of G for which the elements of the q-factor of G are orientation-preserving then $G_{\Omega_q} \cdot f = G^+ \cdot f$.

For the most interesting groups G_{Ω_q} we have the following sufficient conditions for the vanishing of $\mathcal{M}(G_{\Omega_q}, f)$, namely certain weak forms of quasihomogeneity. We call f *weakly quasihomogeneous for* G_{Ω_q} if f is q.h. for weights $w_i \in \mathbb{Z}$ and weighted degrees δ_j such that the following conditions hold.

- For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$: all $\delta_j \geq 0$ and $\sum_j \delta_j > 0$.
- For $G_{\Omega_q} = \mathcal{K}_{\Omega_n}$: all $w_i \geq 0$ and $\sum_i w_i > 0$.
- For $G_{\Omega_q} = \mathcal{K}_{\Omega_p}$: $\sum_j \delta_j \neq 0$.

Notice that any f with some zero component function (up to the relevant G-equivalence) is w.q.h. for $\mathcal{A}_{\Omega_p}$ and $\mathcal{K}_{\Omega_p}$ (and also for $\mathcal{L}_{\Omega_p}$ and $\mathcal{C}_{\Omega_p}$), and any f such that $df(0)$ has positive rank is w.q.h. for $\mathcal{K}_{\Omega_n}$ and $\mathcal{K}_{\Omega_p}$. These "trivial forms of weak quasihomogeneity" correspond to the fact that diffeomorphisms of a proper submanifold in $(\mathbb{K}^q, 0)$ can be extended to volume preserving diffeomorphisms of $(\mathbb{K}^q, \Omega_q, 0)$. Furthermore, if f is G_{Ω_q}-w.q.h. then the statement about equality of G- and G_{Ω_q}-orbits over $\mathbb{C}$ (and the corresponding one over $\mathbb{R}$) in the previous paragraph holds in the smooth category (where smooth means complex-analytic over $\mathbb{C}$ and C^∞ or real-analytic over $\mathbb{R}$, as usual). For a G_{Ω_q}-w.q.h. map f the above (generalized) weights and weighted degrees yield a generalized Euler vector field in $(\mathbb{K}^q, 0)$ ($q = n$ or p) that allows us to integrate the (a priori formally defined) vector fields at the infinitesimal level to give the required smooth diffeomorphisms.

For f not G_{Ω_q}-w.q.h. we are interested in upper and lower bounds for the dimension of $\mathcal{M}(G_{\Omega_q}, f)$ and in the question whether the G-finiteness of f implies the finiteness of $\mathcal{M}(G_{\Omega_q}, f)$. We have several results in this direction.

(1) For any G_{Ω_q} for which there is a version of weak quasihomogeneity we have the following easy upper bound (in the formal category) for G-semiquasihomogeneous (s.q.h.) maps $f = f_0 + h$, where f_0 q.h. (and hence G_{Ω_q}-w.q.h.) and G-finite and h has positive degree (relative to the weights of f_0). The normal space $NG \cdot f_0 := \mathcal{M}_n \cdot \theta_{f_0}/LG \cdot f_0$ (where θ_{f_0} denotes the C_n-module of sections of $f_0^* T\mathbb{K}^p$) decomposes into a part of non-positive filtration and a part of positive filtration, denoted by $(NG \cdot f_0)_+$. Denoting the number of G-moduli of positive filtration of f by $m(G, f)$ we have the inequality

$$\dim \mathcal{M}(G_{\Omega_q}, f) + m(G, f) \leq \dim(NG \cdot f_0)_+.$$

(Note that the same inequality holds for the extended pseudo-groups G_e, $G_{\Omega_q,e}$.) For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$ all our examples support the following conjecture: for f as above, the upper bound is actually an equality. For $\mathcal{A}$-s.q.h. map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, \Omega_p, 0)$ with $n \geq p - 1$ and (n, p) in the nice range of dimensions or of corank one (outside the nice range) the validity of this conjecture would have an interesting consequence. Following Damon and Mond [13] we denote by $\mu_\Delta(f)$ the discriminant (for $n \geq p$) or image (for $p = n + 1$) Milnor number of f (the discriminants and images $\Delta(f)$ in these dimensions are hypersurfaces in the target, and $\Delta(f_t)$ of a stable perturbation f_t of f has the homotopy type of a wedge of $\mu_\Delta(f)$ spheres). For a q.h. map-germ f_0 we have $\operatorname{cod}(\mathcal{A}_e, f_0) = \mu_\Delta(f_0)$ for $n \geq p$ by the main result in [13] and for $p = n+1$ by Mond's conjecture (see Conjecture I in [10], for $n = 1, 2$ this conjecture has been proved by Mond and others). Now if our conjecture is true we obtain for s.q.h. maps $f = f_0 + h$ the following interesting consequence of these results:

$$\mathrm{cod}(\mathcal{A}_{\Omega_p,e}, f) = \mu_\Delta(f).$$

For $(n,p) = (1,2)$ the invariant $\mu_\Delta(f)$ is just the classical δ-invariant, hence we recover the formula $\mathrm{cod}(\mathcal{A}_{\Omega_p,e}, f) = \delta(f)$ of Ishikawa and Janeczko [29] in the special case of s.q.h. curves (their formula holds for any $\mathcal{A}$-finite curve-germ). Notice that for $f = f_0 + h$ we have $\mu_\Delta(f) = \mu_\Delta(f_0)$ (because any deformation by terms of positive filtration is topologically trivial). Our conjecture implies that the coefficients of each of the $\dim(N\mathcal{A}_e \cdot f_0)_+$ terms of h are moduli for $\mathcal{A}_{\Omega_p,e}$ (some of them may be moduli for $\mathcal{A}_e$ too), hence $\mathrm{cod}(\mathcal{A}_{\Omega_p,e}, f) = \mathrm{cod}(\mathcal{A}_e, f_0) = \mu_\Delta(f_0)$, which gives the formula above.

(2) For $G_{\Omega_q} = \mathcal{K}_{\Omega_n}$ we have more general results (in the analytic category) which, for example, imply the following. For any $\mathcal{K}$-finite map f the moduli space $\mathcal{M}(\mathcal{K}_{\Omega_n}, f)$ is finite dimensional. Furthermore, if $f^{-1}(0)$ lies in a hypersurface $h^{-1}(0)$ having (at worst) an isolated singular point at the origin then $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) \le \mu(h)$ (notice that if $f = (g_1, \dots, g_p)$ defines an ICIS then we can take a generic $\mathbb{C}$-linear combination $h = \sum_i a_i g_i$ having finite Milnor number $\mu(h)$).

(3) For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$ the moduli space $\mathcal{M}(\mathcal{A}_{\Omega_p}, f)$ is finite dimensional for maps f whose image (or discriminant) has (at worst) an isolated singularity at the origin. This applies to $\mathcal{A}$-finite maps $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $p \ge 2n$ or $p = 2$ (and any n). For the other pairs of dimensions (n,p) we only have the finiteness results for $\mathcal{A}$-s.q.h. maps (see (1) above).

(4) For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$ and $\mathcal{K}_{\Omega_n}$ we have the following criterion for $\dim \mathcal{M}(G_{\Omega_q}, f) \ge 1$: suppose f_0 is q.h. and the restriction of $\gamma_{f_0} : LG \to LG \cdot f_0$ to the filtration-0 parts of the modules in source and target has 1-dimensional kernel, then the parameter u of a deformation $f = f_0 + u \cdot M$ by some non-zero element $M \in (NG \cdot f_0)_+$ is a modulus for G_{Ω_q}. Using this criterion in combination with the existing $\mathcal{A}$- and $\mathcal{K}$-classifications in the literature we conclude the following. Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is $\mathcal{A}$-simple and $n \ge p$ or $p = 2n$ or $(n,p) = (2,3), (1,p)$ (and any corank) or $(n,p) = (3,4)$ and corank 1 then: f is w.q.h. if and only if $\dim \mathcal{M}(\mathcal{A}_{\Omega_p}, f) = 0$. Or suppose that f has $\mathcal{K}$-modality at most one, $rank(df(0)) = 0$ and $n \ge p$ then: f is q.h. if and only if $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) = 0$.

The contents of the remaining sections of this papers are as follows.

§1. *Brief summary of earlier related works*: by considering the moduli spaces $\mathcal{M}(G_{\Omega_q}, f)$ parameterizing the G_{Ω_q}-orbits inside $G \cdot f$ one can relate the seemingly unrelated earlier works on volume-preserving diffeomorphisms in singularity theory.

§2. *H-isotopic volume forms*: for a subgroup H of the group of diffeomorphisms Theorem 2.8 gives a criterion for a pair of volume forms to be H-isotopic, and Proposition 2.13 gives a sufficient condition on LH under which all pairs of volume forms are H-isotopic. The results will be applied to the subgroups $H = G_f^q$ defined above.

§3. *The moduli space $\mathcal{M}(G_{\Omega_q}, f)$*: the space parameterizing the G_{Ω_q}-orbits in a given G-orbit is isomorphic to $C_r/\mathrm{div}(LG_f^q)$ (Theorem 3.4) and it vanishes for G_{Ω_q}-w.q.h. maps f (Proposition 3.8). These results imply, for example, that (over $\mathbb{C}$) the stable orbits for $\mathcal{A}_{\Omega_p}$ and $\mathcal{A}$ and the simple orbits for $\mathcal{K}_{\Omega_n}$ and $\mathcal{K}$ agree (see Remark 3.10).

§4. *A cohomological description of $\mathcal{M}(G_{\Omega_q}, f)$ and some finiteness results*: for finitely generated ideals $\mathcal{I}$ in C_n we define a subcomplex $(\Lambda^*(\mathcal{I}), d)$ of the de Rham complex whose nth cohomology vanishes for w.q.h. ideals $\mathcal{I}$ (Theorem 4.4). For $\mathcal{I} = f^*\mathcal{M}_p$ (not necessarily w.q.h.) $H^n(\Lambda^*(\mathcal{I}))$ is isomorphic to $\mathcal{M}(\mathcal{K}_{\Omega_n}, f)$ and is finite if $\mathcal{I}$ contains the vanishing ideal of a variety W with (at worst) an isolated singular point at 0, see Theorem 4.13 (for a hypersurface germ W we have $H^n(\Lambda^*(\mathcal{I})) \leq \mu(W)$, see Theorem 4.14). These finiteness results imply for example: $\mathcal{M}(\mathcal{K}_{\Omega_n}, f)$ is finite if f defines an ICIS, and $\mathcal{M}(\mathcal{A}_{\Omega_p}, f)$ is finite for $p \geq 2n$ and $\mathcal{A}$-finite f.

§5. *The foliation of $\mathcal{A}$-orbits by $\mathcal{A}_{\Omega_p}$-orbits*: in those dimensions (n, p), for which the classification of $\mathcal{A}$-simple orbits is known, an $\mathcal{A}$-simple germ f is w.q.h. if and only if $\mathcal{M}(\mathcal{A}_{\Omega_p}, f) = 0$. The classifications of the $\mathcal{A}_{\Omega_p}$-simple orbits in dimensions $(n, 2)$ and $(n, 2n)$, $n \geq 2$, are described in Propositions 5.2, 5.3 and 5.4. In §5.3 the foliation of s.q.h. but not w.q.h. $\mathcal{A}$-orbits by $\mathcal{A}_{\Omega_p}$-orbits is investigated for $\mathcal{A}$-unimodal germs into the plane, and in §5.4 weak quasihomogeneity is defined for multigerms under $\mathcal{A}_{\Omega_p}$-equivalence.

§6. *The foliation of $\mathcal{K}$-orbits by $\mathcal{K}_{\Omega_n}$- and $\mathcal{K}_{\Omega_p}$-orbits*: a $\mathcal{K}$-unimodal germ f of rank 0 is q.h. if and only if $\mathcal{M}(\mathcal{K}_{\Omega_n}, f) = 0$, and $\mathcal{M}(\mathcal{K}_{\Omega_n}, f) = 0$ implies $\mathcal{M}(\mathcal{K}_{\Omega_p}, f) = 0$ (recall that germs f of positive rank are trivially w.q.h., hence their $\mathcal{K}$-, $\mathcal{K}_{\Omega_n}$- and $\mathcal{K}_{\Omega_p}$-orbits coincide). Examples of rank 0 germs f defining an ICIS of codimension greater than one are presented for which $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) < \mu(f) - \tau(f)$. For hypersurfaces we have $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) = \mu(f) - \tau(f)$ (by a result of Varchenko [48]), in all our higher codimensional examples we have $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) \leq \mu(f) - \tau(f)$ (and for s.q.h. germs f it is easy to see that this inequality holds in general).

§7. *The groups $G_{\Omega_q} \neq \mathcal{A}_{\Omega_p}$, $\mathcal{K}_{\Omega_n}$, $\mathcal{K}_{\Omega_p}$*: in the final section we consider the remaining groups G_{Ω_q} for which there are G-finite singular maps (as opposed to functions). Examples indicate that already G-stable, singular and not trivially w.q.h. maps f have positive modality for these groups G_{Ω_q} (for $\mathcal{A}_{\Omega_n}$ the fold map even has infinite modality).

1. Brief summary of earlier related works

Having defined the moduli space $\mathcal{M}(G_{\Omega_q}, f)$ we can now conveniently describe the known results within this framework. Most of these results are on functions (hypersurface singularities), and (as explained above) one can either fix f and classify volume forms in the presence of a hypersurface defined by f (up to $G^q_f = \mathcal{R}^n_f$, $\mathcal{A}^n_f$ or $\mathcal{K}^n_f$-equivalence) or fix a volume form and classify functions up to $G_{\Omega_q} = \mathcal{R}_{\Omega_n}$, $\mathcal{A}_{\Omega_n}$ or $\mathcal{K}_{\Omega_n}$-equivalence. Much less is known for maps (see §1.2).

1.1. **Results on functions (hypersurface singularities).** First, consider $\mathcal{R}_{\Omega_n}$-equivalence for functions $f : (\mathbb{K}^n, \Omega_n, 0) \to \mathbb{K}$, $n \geq 2$. The isochore Morse-Lemma from the late 1970s by Vey [49] and Colin de Verdière and Vey [9] gives a normal form for an A_1 singularity involving a functional modulus. More recently isochore versal deformations were studied in [8] and [22]. The following result by Francoise [19, 20] generalizes the isochore Morse-Lemma: let $b_1 = 1, b_2, \ldots, b_{\mu(f)}$ be a base for $N\mathcal{R}_e \cdot f$ then

$$\mathcal{M}(\mathcal{R}_{\Omega_n}, f) \cong \mathbb{K}\{(h_i \circ f)b_i : h_i \in C_1, i = 1, \ldots, \mu(f)\}.$$

Hence f has precisely $\mu(f)$ functional moduli (the h_i are arbitrary smooth function-germs in one variable).

Second, for $\mathcal{A}_{\Omega_n}$ it is clear that (keeping the above notation) $(h_1 \circ f)1 \in L\mathcal{L}_e \cdot f$, hence

$$\mathcal{M}(\mathcal{A}_{\Omega_n}, f) \cong \mathbb{K}\{(h_i \circ f)b_i : h_i \in C_1, i = 2, \ldots, \mu(f)\}.$$

This moduli space vanishes for an A_1 singularity, and non-Morse functions f have $\mu(f)-1$ functional moduli.

Finally, for $\mathcal{K}_{\Omega_n}$ the situation is much better. The following generalization of the corresponding $\mathcal{K}_f^n$ classification of volume forms has been studied, for example, by Arnol'd [1], Lando [32, 33], Kostov and Lando [31] and Varchenko [48]: given a hypersurface $f^{-1}(0)$ and a non-vanishing function-germ h, classify n-forms of the type $f^a h dx_1 \wedge \ldots \wedge dx_n$ up to diffeomorphisms that preserve $f^{-1}(0)$. For $a=0$ we have the special case of volume forms, and in this case the result of Varchenko gives

$$\mathcal{M}(\mathcal{K}_{\Omega_n}, f) \cong \langle f, \nabla f\rangle / \langle \nabla f\rangle,$$

which has dimension $\mu(f)-\tau(f)$. Both Francoise and Varchenko made extensive use of results of Brieskorn [5], Sebastiani [47] and Malgrange [35] on the de Rham complex of differential forms on a hypersurface with isolated singularities.

We will see that this dimension formula for $\mathcal{M}(\mathcal{K}_{\Omega_n}, f)$ does, in general, not hold for map-germs f defining an ICIS of codimension greater than one. The obvious counter-examples are weakly quasihomogeneous maps f that are not quasihomogeneous: for such f the dimension of the moduli space is zero, but $\mu(f)-\tau(f)>0$. More subtle counter-examples (Example 6.2 below) are the members of Wall's $\mathcal{K}$-unimodal series $FW_{1,i}$ of space-curves (which are not weakly quasihomogeneous): here the dimensions of the moduli spaces are equal to one and $\mu-\tau$ is equal to two.

1.2. **Results for maps.** Motivated by Arnold's classification of A_{2k} singularities of curves in a symplectic manifold [3] Ishikawa and Janeczko [29] have (in our notation) classified all $\mathcal{A}_{\Omega_p}$-simple map-germs $f:(\mathbb{C},0)\to(\mathbb{C}^2,\Omega_p,0)$. Notice that the volume-preserving diffeomorphisms of $\mathbb{C}^2$ are also symplectomorphisms. Looking at their classification we observe that $\mathcal{M}(\mathcal{A}_{\Omega_p}, f)=0$ if f is the germ of a q.h. curve. Furthermore, it is shown in [29] that $\mathrm{cod}(\mathcal{A}_{\Omega_p,e}, f)=\delta(f)$, hence the $\mathcal{A}$-finiteness of f (which is equivalent to $\delta(f)<\infty$) implies the finiteness of the moduli space $\mathcal{M}(\mathcal{A}_{\Omega_p}, f)$.

Notice that for $p=1$ any volume-preserving diffeomorphism of $(\mathbb{K}^p,0)$ is the identity. For functions the groups G_{Ω_q}, where $q=n$, are therefore the only ones of interest, and the results in §1.1 (which could be reproved using our approach) completely settle the classification problem for function-germs in the volume-preserving case. We will therefore concentrate on maps of target dimension $p>1$ (but all general results also hold for $p=1$, of course).

2. H-isotopic volume forms

In this section we study H-isotopies joining pairs of volume forms for subgroups H of $\mathcal{D}_q := \mathrm{Diff}(\mathbb{K}^q,0)$. In the subsequent sections we will always apply these results to the subgroups $H=G_f^q$ introduced in the introduction, but it might be worth mentioning that the results of this section have some additional applications, for example to singularities of vector fields (and the proofs remain valid for subgroups H of the group of diffeomorphisms of an oriented, compact, smooth q-dimensional manifold).

Let Λ^k denote the space (of germs) of smooth differential k-forms on $(\mathbb{K}^q,0)$, and denote the subset of Λ^q of (germs of) volume forms by Vol. For a given subgroup $H\subset\mathcal{D}_q$ we consider a C_q-module M in the Lie algebra LH of H (and $M=LH$

if LH itself is a C_q-module). In the following Ω and Ω_i always denote (germs of) volume forms in $(\mathbb{K}^q, 0)$.

Definition 2.1. We say that Ω_0 and Ω_1 are H-diffeomorphic if there is a diffeomorphism $\Phi \in H$ such that $\Phi^*\Omega_1 = \Omega_0$

Definition 2.2. We say that Ω_0 and Ω_1 are H-isotopic if there is a smooth family of diffeomorphisms $\Phi_t \in H$ for $t \in [0,1]$ such that $\Phi_1^*\Omega_1 = \Omega_0$ and $\Phi_0 = \mathrm{Id}$.

Remark 2.3. Two H-isotopic volume forms Ω_0 and Ω_1 are obviously H-diffeomorphic. The converse is not true in general. For example $dx_1 \wedge dx_2$ and $-dx_1 \wedge dx_2$ are diffeomorphic but not isotopic, since any diffeomorphism mapping one to the other changes orientation.

Definition 2.4. We say that Ω_0 and Ω_1 are M-equivalent if there is a vector field $X \in M$ such that $\Omega_0 - \Omega_1 = d(X \rfloor \Omega)$ (for any volume form Ω).

Remark 2.5. Definition 2.4 does not depend on the choice of a volume form Ω. If Ω' is another volume form then $\Omega = f\Omega'$ for some non-vanishing function f. Then $\Omega_1 - \Omega_0 = d(X \rfloor \Omega) = d(fX \rfloor \Omega')$ and $fX \in M$ (M being a module).

Theorem 2.6. *If Ω_0 and Ω_1 are M-equivalent volume forms, which for $\mathbb{K} = \mathbb{R}$ define the same orientation, then Ω_0 and Ω_1 are H-isotopic.*

Proof. We use Moser's homotopy method [42]. Let $\Omega_t = \Omega_0 + t(\Omega_1 - \Omega_0)$ for $t \in [0,1]$. It is easy to see that if Ω_0 and Ω_1 define the same orientation then $\Omega_t \in \mathrm{Vol}$ for any $t \in [0,1]$. We are looking for a family of diffeomorphisms $\Phi_t \in H$, $t \in [0,1]$, such that

$$\Phi_t^*\Omega_t = \Omega_0 \tag{2.1}$$

and $\Phi_0 = \mathrm{Id}$. Differentiating (2.1) we obtain

$$\Phi_t^*(L_{Y_t}\Omega_t + \Omega_1 - \Omega_0) = 0,$$

where $Y_t \circ \Phi_t = \frac{d}{dt}\Phi_t$, which implies that

$$d(Y_t \rfloor \Omega_t) = \Omega_0 - \Omega_1. \tag{2.2}$$

But Ω_0 and Ω_1 are M-equivalent, hence there exists a vector field $X \in M$ such that $\Omega_0 - \Omega_1 = d(X \rfloor \Omega)$ for some volume form Ω. We want to find a family of vector fields Y_t satisfying the following condition:

$$Y_t \rfloor \Omega_t = X \rfloor \Omega. \tag{2.3}$$

But $\Omega_t = g_t\Omega$ for some non-vanishing smooth function g_t. Hence $Y_t = (1/g_t)X$ is a solution of (2.3) and $Y_t \in M$, because $X \in M$ and M is a module. The vector field Y_t vanishes at the origin, hence its flow exists on some neighborhood of the origin for all $t \in [0,1]$. Integrating Y_t we obtain a smooth family of diffeomorphisms $\Phi_t \in H$ for $t \in [0,1]$ such that $\Phi_0 = \mathrm{Id}$ and $\Phi_t^*\Omega_t = \Omega_0$, which implies that Ω_0 and Ω_1 are H-isotopic. □

Next, we will show that for subgroups H of $\mathcal{D}_q$ with LH a submodule of the C_q-module θ_q the existence of an H-isotopy between a pair of volume forms is equivalent to the LH-equivalence of this pair, provided that LH is closed with respect to integration in the following sense.

Definition 2.7. We say LH is closed with respect to integration if for any smooth family $X_t \in LH$, $t \in [0,1]$, the integral $\int_0^1 X_t dt$ belongs to LH.

Theorem 2.8. *Let LH be a submodule of θ_q, which is closed with respect to integration. Over $\mathbb{K}=\mathbb{R}$ we also assume that Ω_0 and Ω_1 define the same orientation. Then Ω_0 and Ω_1 are LH-equivalent if and only if Ω_0 and Ω_1 are H-isotopic.*

Proof. The "only if" part follows directly from Theorem 2.6.

For the converse, we require the following lemma

Lemma 2.9. *Let Φ_t be a smooth family of diffeomorphisms and let X_t be a family of vector fields such that $\frac{d}{dt}\Phi_t = X_t \circ \Phi_t$. Then $\frac{d}{dt}\Phi_t^{-1} = -(\Phi_t^* X_t)\circ \Phi_t^{-1}$.*

Proof of Lemma 2.9. Differentiating $\Phi_t^{-1}\circ\Phi_t = \mathrm{Id}$ we obtain

$$0 = \frac{d}{dt}(\Phi_t^{-1}\circ\Phi_t) = \frac{d}{dt}(\Phi_t^{-1})\circ\Phi_t + d(\Phi_t^{-1})\frac{d}{dt}\Phi_t,$$

which implies that $\frac{d}{dt}(\Phi_t^{-1}) = -d(\Phi_t^{-1})(X_t\circ\Phi_t)\circ\Phi_t^{-1}$. But, by definition, $\Phi_t^* X_t = d(\Phi_t^{-1})(X_t\circ\Phi_t)$. □

Returning to the proof of the theorem, we assume that Ω_0 and Ω_1 are H-isotopic. Then there exists, for all $t\in[0,1]$, a smooth family of diffeomorphisms $\Phi_t\in H$ such that $\Phi_0=\mathrm{Id}$ and $\Phi_1^*\Omega_0=\Omega_1$. Let $(\Phi_t)' = \frac{d}{dt}\Phi_t = X_t\circ\Phi_t$, then

$$\Omega_1-\Omega_0 = \Phi_1^*\Omega_0-\Omega_0 = \int_0^1 (\Phi_t^*\Omega_0)'dt = \int_0^1(\Phi_t^*\mathcal{L}_{X_t}\Omega_0)dt = \int_0^1 \Phi_t^* d(X_t\rfloor\Omega_0)dt =$$

$$d\left(\int_0^1 \Phi_t^*(X_t\rfloor\Omega_0)dt\right) = d\left(\int_0^1 (\Phi_t^*X_t)\rfloor\Phi_t^*\Omega_0)dt\right) = d\left(\int_0^1 (\Phi_t^*X_t)\rfloor h_t\Omega_0)dt\right)$$

for some smooth family of positive functions h_t. Thus

$$\Omega_1-\Omega_0 = d\left(\int_0^1 h_t\Phi_t^*X_t dt\rfloor\Omega_0\right).$$

Lemma 2.9 implies $\Phi_t^*X_t\in LH$, and using the fact that LH is a module we also have $h_t\Phi_t^*X_t\in LH$. And LH is closed with respect to integration, hence $\int_0^1 h_t\Phi_t^*X_t dt$ belongs to LH too. Therefore Ω_0 and Ω_1 are LH-equivalent, as desired. □

Definition 2.10. The divergence of a vector field $X\in\theta_q$ with respect to a given volume form Ω is, by definition, the smooth function $\mathrm{div}_\Omega(X) = d(X\rfloor\Omega)/\Omega$. When the volume form Ω is understood from the context then we simply write $\mathrm{div}(X)$. And we have a map $\mathrm{div}:\theta_q\to C_q$ defined by $X\mapsto\mathrm{div}(X)$.

Corollary 2.11. *Under the assumption of Theorem 2.8 the number of H-moduli of volume forms is equal to*

$$\dim_{\mathbb{K}}\frac{C_q}{\mathrm{div}(LH)}.$$

Proof. It is easy to see that spaces $C_q/\mathrm{div}(LH)$ and $\Lambda^q/\{d(X\rfloor\Omega): X\in LH\}$ are isomorphic. By Theorem 2.8 the number of H-moduli of volume forms is equal to the dimension of $\mathrm{Vol}/\sim_{LH}$. But it is easy to see that the spaces $\Lambda^q/\{d(X\rfloor\Omega): X\in LH\}$ and $\mathrm{Vol}/\sim_{LH}$ are equal if there exists a $X\in LH$ such that $d(X\rfloor\Omega)$ is a volume form. Otherwise $\Lambda^q/\{d(X\rfloor\Omega): X\in LH\}\setminus\mathrm{Vol}/\sim_{LH}$ is a linear subspace of positive codimension in $\Lambda^q/\{d(X\rfloor\Omega): X\in LH\}$. This implies that

$$\dim_{\mathbb{K}}\frac{\Lambda^q}{\{d(X\rfloor\Omega): X\in LH\}} = \dim_{\mathbb{K}}\mathrm{Vol}/\sim_{LH}.$$

□

Next, we describe two sufficient conditions for the existence of a single M-equivalence class of volume forms in $(\mathbb{K}^q,0)$ (recall M is a C_q-module in LH). For the first sufficient condition we require the following

Definition 2.12. A linear vector field

$$E_w=\sum_{i=1}^{q} w_i x_i \frac{\partial}{\partial x_i}.$$

with integer coefficients w_i is called a generalized Euler vector field (for coordinates $(x_1,\dots,x_q)\in\mathbb{K}^q$ and weights $w=(w_1,\dots,w_q)$).

We first consider generalized Euler vector fields with non-negative weights w_i (for positive weights we obtain the usual Euler vector fields). For $\mathcal{K}_{\Omega_p}$-equivalence we also require linear vector fields with negative coefficients (see Theorem 3.9 below).

Proposition 2.13. *Let X be the germ of a smooth vector field on $(\mathbb{K}^q,0)$ which is locally diffeomorphic to a generalized Euler vector field with non-negative weights and positive total weight. If X generates a C_q-module in LH then any two germs of volume forms (which over $\mathbb{K}=\mathbb{R}$ define the same orientation) are H-isotopic.*

Proof. Let E_w be (the germ of) the Euler vector field for a coordinate system $(x,y)=(x_1,\dots,x_k,y_1,\dots,y_{q-k})$ with weights $w=(w_1,\dots,w_k,0,\cdots,0)$, where $w_1,\cdots,w_k$ are positive and let Ω_0 be the germ of the volume-form $dx_1\wedge\ldots\wedge dx_k\wedge dy_1\wedge\ldots\wedge dy_{q-k}$. By Theorem 2.6, it is enough to show that for any smooth q-form ω on $(\mathbb{K}^q,0)$ there exists a smooth function-germ g on $(\mathbb{K}^q,0)$ such that $\omega=d(gE_w\rfloor\Omega_0)$.

Let $G_t(x,y)=(e^{w_1t}x_1,\dots,e^{w_kt}x_k,y_1,\dots,y_{q-k})$ for $t\le 0$. It is easy to see that

$$(G_t)':=\frac{d}{dt}G_t=E_w\circ G_t,\ G_0=\mathrm{Id},\ \lim_{t\to-\infty}G_t(x,y)=(0,y)$$

for any $(x,y)\in\mathbb{K}^q$. Thus

$$\omega=G_0^*\omega-\lim_{t\to-\infty}G_t^*\omega=\int_{-\infty}^{0}(G_t^*\omega)'dt. \tag{2.4}$$

But $\omega=f\Omega_0$ for some smooth function-germ f and

$$(G_t^*\omega)'=G_t^*L_{E_w}\omega=G_t^*d(E_w\rfloor\omega)=d(G_t^*(E_w\rfloor\omega)),$$

hence

$$(G_t^*\omega)'=d(G_t^*(E_w\rfloor f\Omega_0))=d((f\circ G_t)G_t^*(E_w\rfloor\Omega_0)).$$

One then checks by a direct calculation that $G_t^*(E_w\rfloor\Omega_0)=e^{t\sum_{i=1}^k w_i}(E_w\rfloor\Omega_0)$. Therefore $(G_t^*\omega)'=d((f\circ G_t)e^{t\sum_{i=1}^k w_i}(E_w\rfloor\Omega_0))$. Combining this with (2.4) we obtain

$$\omega=d(\int_{-\infty}^{0}((f\circ G_t)e^{t\sum_{i=1}^k w_i})dt(E_w\rfloor\Omega_0))=d(g(E_w\rfloor\Omega_0)),$$

where g is a function-germ on $(\mathbb{K}^q,0)$ defined as follows:

$$g(x,y)=\int_{-\infty}^{0}(e^{t\sum_{i=1}^k w_i}(f(G_t(x,y)))dt.$$

The function-germ g is smooth, because

$$\int_{-\infty}^{0}(e^{t\sum_{i=1}^k w_i}(f(G_t(x,y)))dt=\int_0^1(s^{\alpha}f(F_s(x,y))ds,$$

where $\alpha = (\sum_{i=1}^{k} w_i) - 1$ and

$$F_s(x_1, \ldots, x_k, y_1, \ldots, y_{q-k}) = (s^{w_1}x_1, \ldots, s^{w_k}x_k, y_1, \ldots, y_{q-k})$$

for any $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_{q-k})$ and $s \in [0, 1]$. Multiplying the weights by a sufficiently large constant we may assume that $\alpha > 1$. □

We conclude this section by stating a second sufficient condition for the existence of a single M-orbit of volume forms. Here we assume that LH contains a module $\mathcal{M}_q X$, where X is the germ of a non-vanishing vector field and $\mathcal{M}_q$ is the maximal ideal of C_q.

Proposition 2.14. *If $X \in \theta_q$, $X(0) \neq 0$, and the C_q-module $\mathcal{M}_q X$ is contained in LH then any two germs of volume forms (which over $\mathbb{K} = \mathbb{R}$ define the same orientation) are H-isotopic.*

Proof. $X(0) \neq 0$ implies that X is diffeomorphic to $\partial/\partial x_1$. Any germ of a q-form has in such a coordinate system, for some $f \in C_q$, the following form

$$f(x)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_q = d(\int_0^{x_1} f(t, x_2, \cdots, x_q)dt \frac{\partial}{\partial x_1} \rfloor dx_1 \wedge dx_2 \wedge \cdots \wedge dx_q).$$

And $\int_0^{x_1} f(t, x_2, \cdots, x_q)dt\partial/\partial x_1$ belongs to $\mathcal{M}_q \partial/\partial x_1$. Thus any two germs of volume forms (which over $\mathbb{R}$ define the same orientation) are H-isotopic, by Theorem 2.6. □

3. The moduli space $\mathcal{M}(G_{\Omega_q}, f)$

In this section we study smooth map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ (for $\mathbb{K} = \mathbb{C}$ smooth means complex-analytic, for $\mathbb{K} = \mathbb{R}$ smooth means either C^∞ or real-analytic). We set $\mathcal{R} := \mathcal{D}_n$ and $\mathcal{L} := \mathcal{D}_p$ (one can compose f with elements of $\mathcal{D}_n$ on the right and with elements of $\mathcal{D}_p$ on the left, which explains this notation).

Let G be one of the Mather groups $\mathcal{A}, \mathcal{K}, \mathcal{R}, \mathcal{L}$ or $\mathcal{C}$ (all of which can be considered as subgroups of $\mathcal{A}$ or $\mathcal{K}$, e.g. $\mathcal{R} \times 1 \subset \mathcal{A}$) acting on the space of smooth map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. And let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_p)$ be coordinates on $\mathbb{K}^n$ and $\mathbb{K}^p$, respectively. The differential of the orbit map $g \mapsto g \cdot f$ ($g \in G$ and the action on f depends on the definition of G)

$$\gamma_f : LG \longrightarrow LG \cdot f$$

has kernel LG_f (where G_f is the stabilizer of f in G). Recall that for $G = \mathcal{A}$ the map γ_f is given by

$$L\mathcal{A} = \mathcal{M}_n\theta_n \oplus \mathcal{M}_p\theta_p \to \mathcal{M}_n\theta_f, \ (a, b) \mapsto tf(a) - \omega f(b),$$

where $tf(a) = df(a)$ and $wf(b) = b \circ f$, and for $G = \mathcal{K}$ it is given by

$$L\mathcal{K} = \mathcal{M}_n\theta_n \oplus gl_p(C_n) \to \mathcal{M}_n\theta_f, \ (a, B) \mapsto tf(a) - B \cdot f.$$

The kernel of γ_f inherits a C_r module structure from LG, where $r = p$ (or $r = n$) for G a subgroup of $\mathcal{A}$ (or $\mathcal{K}$). Projecting onto source or target factors

$$LG_f^n \longleftarrow LG_f \longrightarrow LG_f^p$$

preserves this C_r module structure. Denoting the factors of G_f by G_f^n and G_f^p their Lie algebras are the above projections. We also denote the factors of G by G^n and G^p (hence e.g. for $G = \mathcal{A}$ we have $G^n = \mathcal{R}$). Superscripts always denote projections onto one of the factors.

Consider subgroups G_{Ω_n} and G_{Ω_p} of G in which the diffeomorphisms (or families of diffeomorphisms for $G = \mathcal{C}$, see below) preserve a given volume form Ω_n or Ω_p in the source or target, respectively. For $r = n$ or p and a given volume form Ω_r on $\mathbb{K}^r$ let $\operatorname{div} : \mathcal{M}_q\theta_q \to C_r$ be the map that sends a vector field (vanishing at 0 in $\mathbb{K}^n$ or $\mathbb{K}^p$) to its divergence.

For $\mathcal{K}$-equivalence in combination with a volume form in the target there are two ways to define the $\mathcal{C}_{\Omega_p}$ component. But both version yield identical $\mathcal{K}_{\Omega_p}$-orbits (just as the alternative definitions of $\mathcal{K}$ yield the same $\mathcal{K}$-orbits).

(1) In the original definition of $\mathcal{K}$ by Mather, $\mathcal{C}$ consists of diffeomorphisms $H = (\phi(x), \varphi(x, y)) \in \mathcal{D}_{n+p}$, with $\varphi(x, 0) = 0$ for all $x \in (\mathbb{K}^n, 0)$, and the action on f is given by $H \cdot f := \varphi(x, f \circ \phi(x))$. We can think of H as a n-parameter family of diffeomorphisms $\{\varphi_x\}$, $x \in \mathbb{K}^n$, acting on f by sending x to $\varphi_x \circ f(x)$. If Ω_p is a volume form on $(\mathbb{K}^p, 0)$ we require that each φ_x preserves Ω_p (i.e. $\varphi_x^*\Omega_p = \Omega_p$ for all $x \in (\mathbb{K}^n, 0)$). In this way we obtain a subgroup $\mathcal{C}_{\Omega_p}$ of $\mathcal{C}$, and $\mathcal{K}_{\Omega_p} := \mathcal{R} \cdot \mathcal{C}_{\Omega_p}$ (semi-direct product).

(2) In the linearized version of $\mathcal{K}$ we set $\mathcal{C} := GL_p(C_n)$ and restrict to $\mathcal{C}_{\Omega_p} = SL_p(C_n)$, then $L\mathcal{C}_{\Omega_p} = sl_p(C_n)$ consists of $p \times p$ matrices over C_n with zero trace. And, again, $\mathcal{K}_{\Omega_p} := \mathcal{R} \cdot \mathcal{C}_{\Omega_p}$. Then div can be considered as a map $B \mapsto \operatorname{trace} B$ as follows: the map $gl_p(C_n) \to \mathcal{M}_n\theta_f$, sending B to $B \cdot f$ (multiplication of f as a column vector of its component functions by a matrix $B = (b_{ij})$), can also be written $B \cdot f = X_B \circ f$, where $X_B = \sum_{i=1}^{p}(b_{i1}(x)y_1 + \ldots + b_{ip}(x)y_p)\partial/\partial y_i$ is a linear vector field in $\mathbb{K}^p$ with coefficients $b_{ij} \in C_n$. Hence $\operatorname{div} X_B = \operatorname{trace} B \in C_n$.

For any of the above volume preserving subgroups G_{Ω_q} of G we have the following

Proposition 3.1. *For $q = n$ or p, and* $\operatorname{div} : \mathcal{M}_q\theta_q \to C_r$ *(where $r = n$ for $G_f^q = \mathcal{K}_f^p$ and $r = q$ in all other cases), we have an isomorphism*

$$\mathcal{M}(G_{\Omega_q}, f) := \frac{LG \cdot f}{LG_{\Omega_q} \cdot f} \cong \frac{C_r}{\operatorname{div}(LG_f^q)}.$$

Proof. Let $\pi : LG \to LG^q$ be the projection onto one of the factors, so that for $u = (a, b)$ we have $v := \pi(u)$ is equal to $a \in \mathcal{M}_n\theta_n$ or b, where either $b \in \mathcal{M}_p\theta_p$ (for $G = \mathcal{A}$) or $b = X_B$ for some $B \in gl_p(C_n)$ (for $G = \mathcal{K}$). (Recall that in the latter case $\operatorname{div}(X_B) = \operatorname{trace} B$.) Then consider the epimorphism

$$\beta : LG \longrightarrow C_r, \ u \mapsto \operatorname{div}(v).$$

Factoring out the kernel we obtain an isomorphism

$$\bar{\beta} : \frac{LG}{LG_{\Omega_q}} \longrightarrow C_r.$$

We also have a well-defined map

$$\gamma : \frac{LG}{LG_{\Omega_q}} \longrightarrow \frac{\mathcal{M}_n \cdot \theta_f}{LG_{\Omega_q} \cdot f}$$

sending $[(a, b)]$ to $[tf(a) - \omega f(b)]$ (for G a subgroup of $\mathcal{A}$) and $[(a, B)]$ to $[tf(a) - B \cdot f]$ or, equivalently, $[(a, X_B)]$ to $[tf(a) - X_B \circ f]$ (for G a subgroup of $\mathcal{K}$). We see that

$$\operatorname{im}\gamma = \frac{LG \cdot f}{LG_{\Omega_q} \cdot f}$$

and that $\bar{\beta}(\ker \gamma) = \operatorname{div}(LG_f^q)$. Factoring out the kernel of γ yields an isomorphism $\bar{\gamma}$ onto $\operatorname{im}\gamma$ so that $\bar{\beta} \circ \bar{\gamma}^{-1}$ is the desired isomorphism. $\square$

Remark 3.2. For $G = \mathcal{A}$ the vector fields $(a, b) \in L\mathcal{A}_f$, $b \in L\mathcal{A}_f^p$ and $a \in L\mathcal{A}_f^n$ are also said to be f-related, liftable and lowerable, respectively.

Notice that LG_f^q inherits a C_r module structure, where $r = n$ or p, from LG_f and LG. In fact, we have

Lemma 3.3. *LG_f is a C_r-submodule of LG ($r = p$ or n for G a subgroup of $G = \mathcal{A}$ or $\mathcal{K}$, respectively), which is closed under integration. The same is true for the factors LG_f^q of LG_f.*

Proof. The statements about the module structure are obvious. And for 1-parameter families of vector fields $v_t = (a_t, b_t)$ (for $G = \mathcal{A}$) or (a_t, X_{B_t}) (for $G = \mathcal{K}$), $t \in [0, 1]$, in the kernel of γ_f we have $0 = \int_0^1 \gamma_f(v_t)dt = \gamma_f(\int_0^1 v_t dt)$, hence $\int_0^1 v_t dt \in LG_f$. And it is clear that the q-component of $\int_0^1 v_t dt$ belongs to LG_f^q. $\square$

We can now deduce from Proposition 3.1 and Corollary 2.11 the following

Theorem 3.4. *For all volume preserving subgroups G_{Ω_q} of G, except for $\mathcal{A}_{\Omega_n}$, the dimension of*

$$\mathcal{M}(G_{\Omega_q}, f) := \frac{LG \cdot f}{LG_{\Omega_q} \cdot f} \cong \frac{C_r}{\operatorname{div}(LG_f^q)}$$

is equal to the number of G_{Ω_q}-moduli of f and also to the number of G_f^q-moduli of volume forms in $(\mathbb{K}^q, 0)$. (For $\mathcal{A}_{\Omega_n}$ the above statement holds in the formal category, in the smooth category the number of moduli is at least $\dim \mathcal{M}(\mathcal{A}_{\Omega_n}, f)$*.)*

Proof. In all cases, except $L\mathcal{A}_f^n$, the component LG_f^q of LG_f is a module over the ring C_r appearing as the target of the map $\operatorname{div} : \mathcal{M}_q\theta_q \to C_r$. And LG_f^q is closed under integration, by the above lemma, hence Corollary 2.11 applies. For $L\mathcal{A}_f^n$ we notice that Proposition 3.1 is a statement about vector spaces (a C_r module structure is not required). $\square$

Remark 3.5. At this point it is perhaps useful to briefly recall the following. The G-modality of a map-germ f is, roughly speaking, the least m such that a small neighborhood of f can be covered by a finite number of m-parameter families of G-orbits. (More precisely, we consider the $j^k(G)$-orbits in some neighborhood of $j^k f$ in a finite-dimensional jet-space $J^k(n, p)$ for some k for which all these $j^k(G)$-orbits are G-sufficient – recall that the G-determinacy degree of f in general fails to be upper semicontinuous under deformations of f, see [50] for a survey of results on G-determinacy.) Map-germs f of G-modality $0, 1, 2, \ldots$ are said to be G-simple, G-unimodal, G-bimodal and so on. An m-G-modal family depends on no more than m parameters (moduli), for $G = \mathcal{R}$ and function-germs it depends on exactly m moduli [21]. For a subgroup G_{Ω_q} of a Mather group G and an m-parameter family of map-germs f^λ the dimension of $\mathcal{M}(G_{\Omega_q}, f^\lambda)$ is equal to the number of G_{Ω_q}-moduli of f^λ, and also to the number of $G_{f^\lambda}^q$-moduli of volume forms in $(\mathbb{K}^q, 0)$, for each fixed vector $\lambda \in \mathbb{K}^m$ of G-moduli of f^λ.

We are now interested in classes of map-germs f for which the moduli spaces $\mathcal{M}(G_{\Omega_q}, f)$ vanish. For the groups $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$, $\mathcal{K}_{\Omega_n}$ and $\mathcal{K}_{\Omega_p}$ such classes of maps are given by the following weak forms of quasihomogeneity.

Definition 3.6. A map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, which is q.h. for weights $w_i \in \mathbb{Z}$ ($1 \le i \le n$) and weighted degrees δ_j ($1 \le j \le p$), is said to be *weakly quasihomogeneous* (w.q.h.) for the group G_{Ω_q} if the following conditions hold.

- For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$: all $\delta_j \geq 0$ and $\sum_j \delta_j > 0$.
- For $G_{\Omega_q} = \mathcal{K}_{\Omega_n}$: all $w_i \geq 0$ and $\sum_i w_i > 0$.
- For $G_{\Omega_q} = \mathcal{K}_{\Omega_p}$: $\sum_j \delta_j \neq 0$.

Remark 3.7. (i) The condition w.q.h. depends on the group G_{Ω_q}, when the group is clear from the context we will simply say that f is w.q.h.

(ii) For any subgroup $G_{\Omega_q} \neq \mathcal{A}_{\Omega_n}$ of G we have the following *"trivial versions of w.q.h"* for f: (1) for $q = p$ and f G-equivalent to some map-germ having a zero component function, and (2) for $q = n$ and $df(0)$ of positive rank. For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$, $\mathcal{K}_{\Omega_n}$ and $\mathcal{K}_{\Omega_p}$ it is easy to see that "trivially w.q.h." is a special case of w.q.h.: for (1) we give the zero component function positive weighted degree (and set all weights w_i or all other degrees δ_j to zero), and for (2) we have (up to G-equivalence) $f = (x_1, g(x_2, \ldots, x_n))$, so we take $w_1 = 1$ and $w_i = 0$, $i > 1$.

We then have the following

Proposition 3.8. *Let f be w.q.h. for one of the groups $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$, $\mathcal{K}_{\Omega_n}$ or $\mathcal{K}_{\Omega_p}$ (or "trivially w.q.h." for any group). Then $\mathcal{M}(G_{\Omega_q}, f) = 0$.*

Proof. We will show that $LG^q \cdot f \subset LG_{\Omega_q} \cdot f$ (here $LG^q \cdot f$ is one of the factors of $LG \cdot f$), so that $LG_{\Omega_q} \cdot f = LG \cdot f$.

For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$ we have to show that $L\mathcal{L} \cdot f \subset L\mathcal{A}_{\Omega_p} \cdot f$. Clearly it is enough to check this inclusion for the elements of $L\mathcal{L} \cdot f$ that do not belong to $L\mathcal{L}_{\Omega_p} \cdot f$. Let $y^\alpha = \prod_l y_l^{\alpha_l}$ and $|\alpha| \geq 0$. The following elements of $L\mathcal{A}_{\Omega_p} \cdot f$ yield $\omega f(y^\alpha y_i \cdot \partial/\partial y_i) \in L\mathcal{L} \cdot f$, $i = 1, \ldots, p$:

$$\omega f(-(1+\alpha_j) y_1 y^\alpha \cdot \partial/\partial y_1 + (1+\alpha_1) y_j y^\alpha \cdot \partial/\partial y_j), \; j = 2, \ldots, p$$

together with

$$tf(f^*(y^\alpha) \sum_{i=1}^n w_i x_i \cdot \partial/\partial x_i) - \sum_{j=2}^p \delta_j \cdot \omega f\Big(-\frac{1+\alpha_j}{1+\alpha_1} y^\alpha y_1 \cdot \partial/\partial y_1 + y^\alpha y_j \cdot \partial/\partial y_j \Big)$$

$$= (1+\alpha_1)^{-1} \sum_{j=1}^p (1+\alpha_j)\delta_j \cdot \omega f(y^\alpha y_1 \cdot \partial/\partial y_1).$$

Notice that $\sum_j (1+\alpha_j)\delta_j \neq 0$, for any exponent vector α, is equivalent to f being w.q.h. for the group $\mathcal{A}_{\Omega_p}$.

For $G_{\Omega_q} = \mathcal{K}_{\Omega_n}$ we have to show that $L\mathcal{R} \cdot f \subset L\mathcal{K}_{\Omega_n} \cdot f$. Exchanging the roles of the source and target vector fields, we see that the following elements of $L\mathcal{K}_{\Omega_n} \cdot f$ yield $tf(x^\alpha x_i \cdot \partial/\partial x_i) \in L\mathcal{R} \cdot f$, $i = 1, \ldots, n$:

$$tf(-(1+\alpha_j) x_1 x^\alpha \cdot \partial/\partial x_1 + (1+\alpha_1) x_j x^\alpha \cdot \partial/\partial x_j), \; j = 2, \ldots, n$$

together with

$$x^\alpha \sum_{i=1}^n \delta_i f_i \cdot \partial/\partial y_i - \sum_{j=2}^n tf\Big(w_j \Big(-\frac{1+\alpha_j}{1+\alpha_1} x_1 x^\alpha \cdot \partial/\partial x_1 + x_j x^\alpha \cdot \partial/\partial x_j \Big) \Big)$$

$$= (1+\alpha_1)^{-1} \sum_{j=1}^n (1+\alpha_j) w_j \cdot tf(x_1 x^\alpha \cdot \partial/\partial x_1).$$

Notice that $\sum_j (1+\alpha_j) w_j \neq 0$, for any exponent vector α, is equivalent to f being w.q.h. for the group $\mathcal{K}_{\Omega_n}$.

For $G_{\Omega_q} = \mathcal{K}_{\Omega_p}$ we have to show that $L\mathcal{C} \cdot f \subset L\mathcal{K}_{\Omega_p} \cdot f$. Notice that $L\mathcal{C}_{\Omega_p} = sl_p(C_n)$ consists of elements B of $gl_p(C_n)$ with trace 0, hence we have a C_n-module structure. Therefore, if E_{ij} denotes a $p \times p$ matrix with entry (i,j) equal to 1 and all other entries 0 then it is enough to show that $E_{ii} \cdot f \in L\mathcal{K}_{\Omega_p} \cdot f$, for $i = 1, \ldots, p$. (Notice that this implies that $L\mathcal{C} \cdot f \subset L\mathcal{K}_{\Omega_p} \cdot f$, both for the linearized version $GL_p(C_n)$ of $\mathcal{C}$ and for Mather's original $\mathcal{C}$, because of the C_n-module structure.) Taking for $j = 2, \ldots, p$

$$-f_1 \cdot \partial/\partial y_1 + f_j \cdot \partial/\partial y_j$$

(corresponding to $(E_{jj} - E_{11}) \cdot f$ with $(E_{jj} - E_{11}) \in L\mathcal{C}_{\Omega_p}$) and

$$tf\Big(\sum_{i=1}^{n} w_i x_i \cdot \partial/\partial x_i\Big) - \big((-\delta_2 - \ldots - \delta_p)E_{11} + \delta_2 E_{22} + \ldots + \delta_p E_{pp}\big) \cdot f$$

$$= \sum_{j=1}^{p} \delta_j f_1 \cdot \partial/\partial y_1$$

we see that $E_{ii} \cdot f \in L\mathcal{K}_{\Omega_p} \cdot f$ $(i = 1, \ldots, p)$ provided that $\sum_j \delta_j \neq 0$.

Finally, by the remark in the introduction, there is nothing to prove in the "trivially w.q.h. cases" (arbitrary diffeomorphisms in a proper subspace can be extended to volume preserving diffeomorphisms of the total space $(\mathbb{K}^q, 0)$). □

The proposition says that, at the infinitesimal level, the tangent spaces of the G-orbit and of the G_{Ω_q}-orbit of f coincide. For $\mathbb{K} = \mathbb{R}$ let G^+ be the subgroup of G for which the diffeomorphisms of the G^q factor of G are orientation preserving. We then have at the level of orbits the following

Theorem 3.9. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be w.q.h. for one of the groups $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$, $\mathcal{K}_{\Omega_n}$ or $\mathcal{K}_{\Omega_p}$ (or "trivially w.q.h." for any group) then:*

(i) any two volume forms Ω, Ω' on $\mathbb{K}^q$ (so that, in the case of $\mathbb{K} = \mathbb{R}$, $\Omega|_0$ and $\Omega'|_0$ define the same orientation in $T_0\mathbb{R}^q$) are G^q_f-isotopic.

(ii) $f' \sim_G f$ (for $\mathbb{K} = \mathbb{C}$) and $f' \sim_{G^+} f$ (for $\mathbb{K} = \mathbb{R}$) imply $f' \sim_{G_{\Omega_q}} f$ (for some given volume form Ω_q on $\mathbb{K}^q$).

Proof. Using the weights w_i (for $q = n$) or weighted degrees δ_j (for $q = p$) in the definition of a G_{Ω_q}-w.q.h. map f we can define generalized Euler vector fields in $\mathbb{C}^q$. For $G_{\Omega_q} = \mathcal{A}_{\Omega_p}$ and $\mathcal{K}_{\Omega_n}$ the vector fields have non-negative coefficients, hence Proposition 2.13 implies statement (i). For $\mathcal{K}_{\Omega_p}$ we can have negative coefficients and we deduce statement (i) by a slightly modified argument (see below). The equivalence of (i) and (ii) is clear (over $\mathbb{C}$ the G-orbits are connected).

For $\mathcal{K}_{\Omega_p}$-equivalence the weighted degrees δ_i of f yield a generalized Euler vector field $E_\delta = \sum_{i=1}^{p} \delta_i y_i \partial/\partial y_i$ in $(\mathbb{K}^p, 0)$. We first claim that any volume form Ω_p is $\mathcal{K}^p_f$-equivalent to some linear volume form $g(x) dy_1 \wedge \cdots dy_p$ parameterized by $g \in C_n$ with $g(0) \neq 0$. Let Ψ be an origin-preserving diffeomorphism of $(\mathbb{K}^p, 0)$ such that, for $\Omega_p = h(y) dy_1 \wedge \cdots \wedge dy_p$, we have $\Psi^* \Omega_p = dy_1 \wedge \cdots \wedge dy_p$. Its inverse has the form

$$\Psi^{-1}(y) = \Big(\sum_{i=1}^{p} \phi_{1i}(y) y_i, \cdots, \sum_{i=1}^{p} \phi_{pi}(y) y_i\Big).$$

We have $\Psi^{-1} \circ f(x) = \Phi_x \circ f(x)$ for the following family Φ_x of diffeomorphisms of $(\mathbb{K}^p, 0)$ parameterized by $x \in (\mathbb{K}^n, 0)$

$$\Phi_x(y) = (\sum_{i=1}^{p} \phi_{1i}(f(x))y_i, \cdots, \sum_{i=1}^{p} \phi_{pi}(f(x))y_i).$$

Hence $\Psi \circ \Phi_x \circ f = f$ (i.e., $\Psi \circ \Phi_x \in \mathcal{K}_f^p$) and $\Phi_x^* \Psi^* \Omega_p = g(x) dy_1 \wedge \cdots \wedge dy_p$, where $g(x) = \det(d\Phi_x)$. Clearly $g(0) \neq 0$, which implies the above claim.

It is therefore sufficient to consider the equivalence of parameterized linear volume forms. Notice that E_δ generates a C_n-submodule of $L\mathcal{K}_f^p$ and

$$g(x) dy_1 \wedge \cdots \wedge dy_p = d \left(\frac{g(x)}{\sum_{i=1}^{p} \delta_i} E_\delta \rfloor dy_1 \wedge \cdots \wedge dy_p \right)$$

(recall that $\sum_{i=1}^{p} \delta_i \neq 0$), hence any pair of such volume forms is $L\mathcal{K}_f^p$-equivalent. Furthermore, by the argument in the proof of Theorem 2.6, such a pair of volume forms (which, for $\mathbb{K} = \mathbb{R}$, is required to define the same orientation) is $\mathcal{K}_f^p$-isotopic. □

"Non-trivial applications" of the above result – namely to weakly quasihomogeneous map-germs f that are neither quasihomogeneous nor trivially weakly quasihomogeneous – will be considered later. For quasihomogeneous and trivially quasihomogeneous germs f we have the following immediate applications.

Remark 3.10. (1) Quasihomogeneous case: all $\mathcal{A}$-stable and all $\mathcal{K}$-simple map-germs f are quasihomogeneous. Hence the classifications, over $\mathbb{C}$, of stable germs for $\mathcal{A}$ and $\mathcal{A}_{\Omega_p}$ and of simple germs for $\mathcal{K}$, $\mathcal{K}_{\Omega_n}$ and $\mathcal{K}_{\Omega_p}$ agree – over $\mathbb{R}$, each $\mathcal{A}$-stable or $\mathcal{K}$-simple orbit corresponds to one or two stable or simple orbits for the volume preserving subgroups.

(2) Trivially weakly quasihomogeneous case: (i) the classifications of map-germs f, with $df(0)$ of positive rank, for the groups $\mathcal{K}$, $\mathcal{K}_{\Omega_n}$ and $\mathcal{K}_{\Omega_p}$ agree. (ii) For map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ whose image lies in a proper submanifold of $(\mathbb{K}^p, 0)$ (such f have, up to a target coordinate change, a zero component function) the $\mathcal{A}$- and $\mathcal{A}_{\Omega_p}$-orbits, the $\mathcal{L}$- and $\mathcal{L}_{\Omega_p}$-orbits, and the $\mathcal{C}$- and $\mathcal{C}_{\Omega_p}$-orbits agree. Notice, for example, that the $\mathcal{A}$ and $\mathcal{A}_{\Omega_p}$ classifications of simple curve-germs agree for $p \geq 7$ (Arnol'd [2] has shown that all stably simple curves can be realized in 6-space, hence all $\mathcal{A}$-simple curves in higher dimensions have zero component functions).

4. A cohomological description of $\mathcal{M}(G_{\Omega_q}, f)$ and some finiteness results

The results on $\mathcal{M}(\mathcal{K}_{\Omega_n}, f)$ can be reformulated for ideals, and for this reformulation we obtain a further isomorphism in terms of cohomology. This cohomological description yields some finiteness results in the non-w.q.h. case. Let $\mathcal{I} \subset C_n$ be a finitely generated ideal (recall: for $C_n = \mathcal{O}_n$ all ideals are f.g., for $C_n = \mathcal{E}_n$, the ring of C^∞ function germs, there are non-f.g. ideals like $\mathcal{M}_n^\infty$).

We say that $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{D}_n$-equivalent if and only if there is a diffeomorphism germ $\phi \in \mathcal{D}_n$ such that $\phi^* \mathcal{I} = \mathcal{J}$. The stabilizer of $\mathcal{I}$ is $(\mathcal{D}_n)_{\mathcal{I}} = \{\phi : \phi^* \mathcal{I} = \mathcal{I}\}$, and

$$L(\mathcal{D}_n)_{\mathcal{I}} = \text{Derlog}(\mathcal{I}) = \{Y \in \theta_n : Y\mathcal{I} \subset \mathcal{I}\},$$

where, for $h \in \mathcal{I}$, we set $Yh := dh \cdot Y$. For $\mathcal{I} = \langle g_1, \ldots, g_p \rangle$ and $f := (g_1, \ldots, g_p) : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ we have the following:

$$\phi^* \langle g_1, \ldots, g_p \rangle := \langle g_1 \circ \phi, \ldots, g_p \circ \phi \rangle = \langle g_1, \ldots, g_p \rangle$$

if and only if $f = B \cdot (f \circ \phi)$ for some $B \in \mathrm{GL}_p(C_n)$. Hence $\mathrm{Derlog}(\mathcal{I}) = L\mathcal{K}_f^n$, and setting $\mathcal{D}_{\Omega_n} := \{h \in \mathcal{D}_n : h^* \Omega_n = \Omega_n\}$ (for a given volume form Ω_n in $(\mathbb{K}^n, 0)$) we have the following isomorphisms for the (infinitesimal) $\mathcal{D}_{\Omega_n}$-moduli space of $\mathcal{I}$:

$$\mathcal{M}(\mathcal{D}_{\Omega_n}, \mathcal{I}) := \frac{C_n}{\mathrm{div}(\mathrm{Derlog}(\mathcal{I}))} \cong \frac{C_n}{\mathrm{div}(L\mathcal{K}_f^n)} \cong \frac{L\mathcal{K} \cdot f}{L\mathcal{K}_{\Omega_n} \cdot f}.$$

This moduli space is also isomorphic to the nth cohomology group of the following complex $(\Lambda^*(\mathcal{I}), d)$. Defining for $k = 0, \ldots, n$ the vector spaces

$$\Lambda^k(\mathcal{I}) := \{\alpha + d\beta \in \Lambda^k : d\mathcal{I} \wedge \alpha \subset \mathcal{I}\Lambda^{k+1},\ d\mathcal{I} \wedge \beta \subset \mathcal{I}\Lambda^k\}$$

we obtain a subcomplex $(\Lambda^*(\mathcal{I}), d)$ of the de Rham complex (Λ^*, d). Sometimes we shall simply write $\Lambda^*(\mathcal{I}) = (\Lambda^*(\mathcal{I}), d)$ and similarly for the other complexes defined below (the differential is always the same d).

The nth cohomology group of the complex $(\Lambda^*(\mathcal{I}), d)$ is

$$H^n((\Lambda^*(\mathcal{I}), d) = \Lambda^n / d\Lambda^{n-1}(\mathcal{I}) = \Lambda^n / \{d\alpha \in \Lambda^n : d\mathcal{I} \wedge \alpha \subset \mathcal{I}\Lambda^n\}.$$

For a given volume form Ω_n the map

$$\mathrm{Derlog}(\mathcal{I}) \ni X \mapsto X \rfloor \Omega_n \in \{\alpha \in \Lambda^{n-1} : d\mathcal{I} \wedge \alpha \subset \mathcal{I}\Lambda^n\}$$

is an isomorphism. Notice that the tangent space to Λ^n can be identified with C_n, and recall that $\mathrm{div}_{\Omega_n}(X) = d(X \rfloor \Omega_n)/\Omega_n$. Hence we see that

$$H^n((\Lambda^*(\mathcal{I}), d)) \cong C_n / \mathrm{div}(\mathrm{Derlog}(\mathcal{I})) \cong \mathcal{M}(\mathcal{D}_{\Omega_n}, \mathcal{I}).$$

Furthermore, Theorem 2.8 implies the following

Proposition 4.1. *Two volume forms (defining the same orientation) are $(\mathcal{D}_n)_{\mathcal{I}}$-isotopic if and only if they define the same cohomology class in $H^n((\Lambda^*(\mathcal{I}), d))$.*

Definition 4.2. We say that an ideal $\mathcal{I}$ in C_n is w.q.h. if it has a set of generators $g_1, \ldots, g_p$ such that the corresponding map $f = (g_1, \ldots, g_p)$ is $\mathcal{K}_{\Omega_n}$-w.q.h. (notice that this is a natural generalization of homogeneous ideals).

Remark 4.3. If the ideal $\mathcal{I}$ is w.q.h. then the variety defined by $\mathcal{I}$ is "quasihomogeneous with respect to a smooth submanifold" in the sense of [17].

We can now reformulate Theorem 3.9 as follows

Theorem 4.4. *Let $\mathcal{I}$ be a w.q.h. ideal in $C_n = \mathcal{O}_n$ or $\mathcal{E}_n$. For $C_n = \mathcal{E}_n$ we assume that $\mathcal{I}$ is finitely generated, and (over $\mathbb{R}$) $\mathcal{D}_n^+$ denotes the group of orientation preserving diffeomorphisms. Then we have the following:*

(i) any two volume forms on $\mathbb{K}^n$ (which, in the case $\mathbb{K} = \mathbb{R}$, define the same orientation in $T_0\mathbb{R}^n$) can be joined (via pullback) by a 1-parameter family of diffeomorphisms ϕ_t such that $\phi_t^* \mathcal{I} = \mathcal{I}$ (i.e., by a $(\mathcal{D}_n)_{\mathcal{I}}$-isotopy).

(ii) For a given volume form Ω_n, let $\mathcal{D}_{\Omega_n}$ be the subgroup of $\mathcal{D}_n$ whose elements preserve Ω_n. Then $\phi^* \mathcal{I} = \mathcal{J}$ for some $\phi \in \mathcal{D}_n$ (for $\mathbb{K} = \mathbb{C}$) or some $\phi \in \mathcal{D}_n^+$ (for $\mathbb{K} = \mathbb{R}$) implies $h^* \mathcal{I} = \mathcal{J}$ for some $h \in \mathcal{D}_{\Omega_n}$.

Remark 4.5. For $\mathcal{A}$-equivalence we have the following cohomological description. Given a map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, let Δ_f be the discriminant (for $n \geq p$) or the image (for $n < p$) of f. If f satisfies the necessary and sufficient condition (namely, GTQ for $n \geq p$ or NHS for $n < p$) for the equality $\mathrm{Derlog}(\mathcal{I}(\Delta_f)) = \mathrm{Lift}(f)$ of Theorem 2 in [7] then we have the following isomorphism:

$$\mathcal{M}(\mathcal{A}_{\Omega_p}, f) \cong H^p((\Lambda^*(\mathcal{I}(\Delta_f), d)),$$

here $\mathcal{I}(\Delta_f)$ is the vanishing ideal of $\Delta_f \subset (\mathbb{C}^p, 0)$. For the precise definitions of GTQ (generically a trivial unfolding of a q.h. germ) and NHS (no hidden singularities) we refer to [7].

Notice that f w.q.h. (for $\mathcal{A}_{\Omega_p}$) implies that the ideal $\mathcal{I}(\Delta_f)$ is weakly quasi-homogeneous. But there are w.q.h. maps f that fail to be GTQ. We give two examples illustrating these facts.

Example 4.6. The map $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ given by $f(x, y, x) = (x, xy+y^5+y^7z)$ is w.q.h. with weights $(4, 1, -2)$ and weighted degrees $(4, 5)$. The discriminant of f is the origin in $(\mathbb{C}^2, 0)$. The critical set is the z-axis, which consists of $\mathcal{A}$-unstable points, hence f fails to be $\mathcal{A}$-finite.

Example 4.7. In [7]

$$f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0),\ f(u, x, y) = (u, x^4 + y^4 + ux^2y^2)$$

is presented as an example of a non-GTQ map-germ. But f is weakly quasi-homogeneous (for the weights $(0, 1, 1)$). Notice that, again, f fails to be $\mathcal{A}$-finite.

One may not care much about such degenerate examples of infinite $\mathcal{A}$-codimension. In Section 5 we describe more subtle examples of weakly quasi-homogeneous map-germs that are $\mathcal{A}$-finite and even $\mathcal{A}$-simple.

Next, we will derive some finiteness results for $H^n(\Lambda^*(\mathcal{I}))$ when $\mathcal{I}$ is not necessarily w.q.h., and apply these to deduce G_{Ω_q}-finiteness from G-finiteness of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ (for certain G and (n, p)). We assume here that $\mathbb{K} = \mathbb{C}$ and that all germs (at 0) are $\mathbb{C}$-analytic. For $\mathcal{I} = \langle g_1, \ldots, g_s \rangle$ we denote the ideal of maximal minors of the Jacobian of $g = (g_1, \ldots, g_s)$ (viewed as a map-germ) by $J(g)$, and we set $\nabla g_i := J(g_i)$. Recall that $\langle g_1, \ldots, g_s, J(g) \rangle$ is the vanishing ideal of the set of $\mathcal{K}$-unstable points of g, so that (by the Nullstellensatz) g is $\mathcal{K}$-finite if and only if $\mathcal{M}_n^r \subset \langle g_1, \ldots, g_s, J(g) \rangle$, for some $r < \infty$, or iff g has (at worst) an isolated singular point at 0. Also notice that

$$\langle g_1, \ldots, g_s, J(g) \rangle \subset \langle g_1, \ldots, g_s, \nabla g_1, \cdots, \nabla g_s \rangle$$

implies that, for $\mathcal{K}$-finite g, the ideal on the RHS of this inclusion has finite colength.

We will relate the complex $\Lambda^*(\mathcal{I})$ to the following subcomplex of the de Rham complex: $(\mathcal{A}_0^*(\mathcal{I}), d)$, where $\mathcal{A}_0^k(\mathcal{I}) = \{\alpha + d\beta \in \Lambda^k : \alpha \in \mathcal{I}\Lambda^k,\ \beta \in \mathcal{I}\Lambda^{k-1}\}$. If $\mathcal{I}$ is the vanishing ideal of a variety V then this complex is called the complex of zero algebraic restrictions to V (see [18], [17], [16]). The cohomology of the quotient complex $(\Lambda^*/\mathcal{A}_0^*(\mathcal{I}(V)), d)$ has been studied in detail in earlier works (see [43],[4],[5], [47],[26],[27]). Notice that the k-th cohomology $H^k(\Lambda^*/\mathcal{A}_0^*(\mathcal{I}))$ of this quotient complex and the $(k+1)$-th cohomology $H^{k+1}(\mathcal{A}_0^*(\mathcal{I}))$ of the above subcomplex are related by the map

$$d : \frac{\{\omega \in \Lambda^k : d\omega \in \mathcal{A}_0^{k+1}(\mathcal{I})\}}{d\Lambda^{k-1} + \mathcal{A}_0^k(\mathcal{I})} \longrightarrow \frac{\{\gamma \in \mathcal{A}_0^{k+1}(\mathcal{I}) : d\gamma = 0\}}{d\mathcal{A}_0^k(\mathcal{I})},$$

which is an isomorphism by the exactness of the de Rham complex of germs of differential forms on $\mathbb{C}^n$.

We are interested in $H^n(\mathcal{A}_0^*(\mathcal{I}))$. First notice the following fact.

Proposition 4.8. *If an ideal $\mathcal{I}$ in $\mathcal{O}_n$ has generators $g_1, \ldots, g_s$, where each g_i is $\mathcal{K}$-equivalent to a $\mathcal{K}_{\Omega_n}$-w.q.h. function-germ, then $H^n(\mathcal{A}_0^*(\mathcal{I})) = 0$.*

Remark 4.9. The hypothesis that each g_i is $\mathcal{K}$-equivalent to some function-germ that is q.h. for non-negative weights and total positive weight (and hence $\mathcal{K}_{\Omega_n}$-w.q.h.) does not require that the map $g = (g_1, \ldots, g_s)$ is $\mathcal{K}_{\Omega_n}$-w.q.h. (the source diffeomorphisms in the $\mathcal{K}$-equivalences can be different for each g_i).

Proof. It is enough to show that any n-form in $\mathcal{I}\Lambda^n$ is the differential of a $(n-1)$-form in $\mathcal{I}\Lambda^{n-1}$. Let $\omega = \sum_{i=1}^s g_i\omega_i$, where the ω_i are n-forms. Any n-form on $\mathbb{C}^n$ is closed and each $g_i = k_i\Phi^* h_i$, where k_i is a non-vanishing function-germ, Φ_i is a diffeomorphism-germ and h_i is w.q.h. with non-negative weights, at least one of which is positive. We then apply the following lemma to each $h_i(\Phi_i^{-1})^*(k_i\omega_i)$ separately.

Lemma 4.10. *If h is w.q.h. then for any n-form ω there exists an $(n-1)$-form β such that $h\omega = d(h\beta)$.*

Proof of Lemma 4.10. If h generates the vanishing ideal of $\{h = 0\}$ then this is a corollary of the relative Poincare lemma for varieties that are quasi-homogeneous with respect to a smooth submanifold [17]. More generally (for $\langle h\rangle$ not necessarily radical) we use the same method as in the proof of Proposition 2.13.

Let E_w be (the germ of) the Euler vector field for h and let G_t be the flow of E_w. Then $G_t^* h = e^{\delta t} h$, where δ is quasi-degree of h. By direct calculation we obtain

$$\omega = \int_{-\infty}^{0} (G_t^*\omega)' dt = d(h\beta), \tag{4.1}$$

where $\beta = \int_{-\infty}^0 e^{\delta t} G_t^*(E_w \rfloor \omega) dt$ is a smooth $(n-1)$-form. □

To conclude the proof of the proposition, we have from Lemma 4.10

$$g_i\omega_i = \Phi_i^*(h_i(\Phi_i^{-1})^*(k_i\omega_i)) = \Phi_i^*(d(h_i\beta_i)) = d(g_i\alpha_i),$$

where $\alpha_i = \frac{1}{k_i}\Phi_i^*\beta_i$. Hence $\omega = \sum_{i=1}^s g_i\omega_i = d(\sum_{i=1}^s g_i\alpha_i)$, as desired. □

We can now relate the dimensions of nth cohomology groups of the two complexes in question.

Theorem 4.11. *For $g_1, \cdots, g_s \in \mathcal{I}$ we have*

$$\dim H^n(\Lambda^*(\mathcal{I})) \le \dim \frac{\mathcal{O}_n}{\langle g_1, \cdots, g_s, \nabla g_1, \cdots, \nabla g_s\rangle} + \dim H^n(\mathcal{A}_0^*(\langle g_1, \cdots, g_s\rangle)).$$

Proof. For $\mathcal{J} := \langle g_1, \cdots, g_s\rangle \subset \mathcal{I}$, clearly $\mathcal{J}\Lambda^{n-1} \subset \Lambda^{n-1}(\mathcal{I})$, which implies that

$$\dim H^n(\Lambda^*(\mathcal{I})) = \dim \Lambda^n/d(\Lambda^{n-1}(\mathcal{I})) \le \dim \Lambda^n/d(\mathcal{J}\Lambda^{n-1}),$$

where $\dim \Lambda^n/d(\mathcal{J}\Lambda^{n-1}) = \dim \Lambda^n/\mathcal{A}_0^n(\mathcal{J}) + \dim \mathcal{A}_0^n(\mathcal{J})/d(\mathcal{J}\Lambda^{n-1})$. Furthermore, from

$$\mathcal{A}_0^n(\mathcal{J}) = \{\sum_{i=1}^s g_i\omega_i + dg_i \wedge \sigma_i : \omega_i \in \Lambda^n,\ \sigma_i \in \Lambda^{n-1},\ i = 1, \cdots, s\}$$

we see that $\Lambda^n/\mathcal{A}_0^n(\mathcal{J})$ is isomorphic to $\mathcal{O}_n/\langle g_1,\cdots,g_s,\nabla g_1,\cdots,\nabla g_s\rangle$. Finally, $d(\mathcal{J}\Lambda^{n-1}) = d(\mathcal{A}_0^{n-1}(\mathcal{J}))$ implies that $\mathcal{A}_0^n(\mathcal{J})/d(\mathcal{J}\Lambda^{n-1})$ and $H^n(\mathcal{A}_0^*(\mathcal{J}))$ are equal. □

Theorem 4.11 and Proposition 4.8 imply the following corollary

Corollary 4.12. *If $g_1,\cdots,g_s \in \mathcal{I}$ satisfy the conditions of Proposition 4.8 then*

$$\dim H^n(\Lambda^*(\mathcal{I})) \leq \dim \frac{\mathcal{O}_n}{\langle \nabla g_1,\cdots,\nabla g_s\rangle}.$$

Proof. Proposition 4.8 implies that $\dim H^n(\mathcal{A}_0^*(\langle g_1,\cdots,g_s\rangle)) = 0$, and $g_i \in \langle \nabla g_i\rangle$ (because g_i is w.q.h.). □

We can now deduce the following finiteness results.

Theorem 4.13. *Let W be a variety-germ with an isolated singularity at 0. If the vanishing ideal of W is contained in $\mathcal{I}$ then $\dim H^n(\Lambda^*(\mathcal{I})) < \infty$.*

Proof. Let $\mathcal{I}(W)$ be generated by $g_1,\cdots,g_s$. Clearly $g_1,\cdots,g_s \in \mathcal{I}$ and from Theorem 4.11 we have

$$\dim H^n(\Lambda^*(\mathcal{I})) \leq \dim \frac{\mathcal{O}_n}{\langle g_1,\cdots,g_s,\nabla g_1,\cdots,\nabla g_s\rangle} + \dim H^n(\mathcal{A}_0^*(\mathcal{I}(W))).$$

From the hypothesis on W we then obtain the finiteness of the dimensions on the right: $H^n(\mathcal{A}_0^*(\mathcal{I}(W)))$ is finite by a result of Bloom and Herrera [4] and the colength of $\langle g_1,\cdots,g_s,\nabla g_1,\cdots,\nabla g_s\rangle$ in $\mathcal{O}_n$ is also finite for such W (see our earlier remark). □

Theorem 4.14. *Let $\langle g\rangle$ be the vanishing ideal of a hypersurface having an isolated singularity at 0. If g is contained in $\mathcal{I}$ then $\dim H^n(\Lambda^*(\mathcal{I})) \leq \mu(g)$, where $\mu(g)$ is the Milnor number of g.*

Proof. For $\langle g\rangle \subset \mathcal{I}$ we obtain from Theorem 4.11

$$\dim H^n(\Lambda^*(\mathcal{I})) \leq \dim \frac{\mathcal{O}_n}{\langle g,\nabla g\rangle} + \dim H^n(\mathcal{A}_0^*(\langle g\rangle)).$$

The desired bound then follows from the following formula of Brieskorn [5] and Sebastiani [47]: $\dim H^n(\mathcal{A}_0^*(\langle g\rangle)) = \mu(g) - \tau(g)$, where $\tau(g) := \dim \mathcal{O}_n/\langle g,\nabla g\rangle$ is the Tjurina number of g. □

Remark 4.15. Theorem 4.13 implies that for a finitely generated ideal $\mathcal{I} = \langle g_1,\ldots,g_p\rangle$ corresponding to a $\mathcal{K}$-finite map $f = (g_1,\ldots,g_p)$ the dimension of $H^n(\Lambda^*(\mathcal{I}))$ is finite dimensional. For the ideal of an ICIS we have a more precise bound. For a $\mathbb{C}$-linear combination $h = \sum_{i=1}^p a_i g_i$ we have $\langle h\rangle \subset \mathcal{I}$, hence $\dim H^n(\Lambda^*(\mathcal{I})) \leq \mu(h)$ (for $\mu(h) < \infty$ we apply Theorem 4.14, and otherwise the upper bound is trivial). Furthermore, for a generic projection $\pi : \mathbb{C}^p \to \mathbb{C}$, $(y_1,\ldots,y_p) \mapsto \sum_{i=1}^p a_i y_i$ the Milnor number of $h = \pi\circ g$, where $g = (g_1,\ldots,g_p)$, is finite (recall the usual method for calculating the Milnor number of an ICIS).

The above finiteness results can be generalized to the case of subgroups H of the group of germs of $\mathbb{C}$-analytic diffeomorphisms of $\mathbb{C}^q$. Using the isomorphism $\theta_q \ni X \mapsto X\rfloor\Omega \in \Lambda^q$ we can prove in the same way the following result.

Theorem 4.16. *Let $\mathcal{J}$ be an ideal in $\mathcal{O}_q$ generated by $g_1, \cdots, g_s$. If $\mathcal{J}\theta_q$ is contained in LH then*

$$\dim \frac{\mathcal{O}_q}{\mathrm{div}(LH)} \leq \dim \frac{\mathcal{O}_q}{\langle g_1, \cdots, g_s, \nabla g_1, \cdots, \nabla g_s\rangle} + \dim H^q(\mathcal{A}_0^*(\langle g_1, \cdots, g_s\rangle)).$$

In particular we obtain the following

Corollary 4.17. *Consider the image* $\mathrm{im}f$ *of a complex-analytic map-germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$*, and recall that* $L\mathcal{A}_f^p = Lift(f)$*.*

(a) If $\mathrm{im}f \subset W$*, for some variety-germ* W *with an isolated singularity at* 0*, then* $\dim \mathcal{O}_p/\mathrm{div}(L\mathcal{A}_f^p)$ *is finite.*

(b) If $\mathrm{im}f \subset g^{-1}(0)$*, for some hypersurface germ* $g^{-1}(0)$ *with an isolated singularity at* 0*, then* $\dim \mathcal{O}_p/\mathrm{div}(L\mathcal{A}_f^p) \leq \mu(g)$*.*

Remark 4.18. Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an $\mathcal{A}$-finite map-germ with target dimension $p \geq 2n$. Then $\mathrm{im}f$ is a variety-germ with an isolated singularity at 0, hence $\mathcal{A}$-finiteness implies $\mathcal{A}_{\Omega_p}$-finiteness (in the sense that the moduli-space $\mathcal{M}(\mathcal{A}_{\Omega_p}, f)$ is finite dimensional, by the above result). This generalizes the corresponding result in [29] for plane curves.

Also, for map-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$, $n \geq 2$, for which $L\mathcal{A}_f^p = Lift(f)$ is equal to Derlog of the discriminant we have that the $\mathcal{A}$-finiteness of f implies the $\mathcal{A}_{\Omega_p}$-finiteness (notice, the discriminant is a curve with isolated singularities).

For $p < 2n$ the image (for $n < p$) or the discriminant (for $n \geq p \geq 3$) of an $\mathcal{A}$-finite singular map-germ f in general has non-isolated singularities (except perhaps for a generalized fold map f). Hence the above finiteness result cannot be applied.

5. The foliation of $\mathcal{A}$-orbits by $\mathcal{A}_{\Omega_p}$-orbits

In this section we study the foliation of $\mathcal{A}$-orbits of map germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, \Omega_p, 0)$ by $\mathcal{A}_{\Omega_p}$-orbits. Our main objective here is the classification of $\mathcal{A}_{\Omega_p}$-simple orbits inside the $\mathcal{A}$-simple orbits, and in dimensions $(n, 2)$ and $(n, 2n)$, $n \geq 2$, we give explicit lists (see §5.1 and §5.2). We also consider $\mathcal{A}_{\Omega_p}$-orbits of positive modality that are s.q.h. but not w.q.h (see §5.3) and w.q.h. multigerms (see §5.4).

For the pairs (n, p) for which the $\mathcal{A}$-simple orbits are known – i.e., for $n \geq p$, $(1, p)$ any p, $p = 2n$, $(2, 3)$ (any corank) and for $(3, 4)$ (of corank 1), see the references below – we find that:

(1) an $\mathcal{A}$-simple germ is $\mathcal{A}_{\Omega_p}$-simple if and only if it does not lie in the closure of the orbit of any non-weakly quasihomogeneous germ,
(2) for $n < 2p$ and for $p = 2n$, with $n \leq 3$, an $\mathcal{A}$-simple germ is $\mathcal{A}_{\Omega_p}$-simple if and only if it does not lie in the closure of the orbit of any non-quasihomogeneous germ.

(The classifications of $\mathcal{A}$-simple orbits can be found in the following papers: $(n, p) = (1, 2)$ [6], $(1, 3)$ [23], $(1, p)$ $(p \geq 3)$ [2], $(n, 2n)$ $(n \geq 2)$ [30], $(2, 3)$ [41], $(3, 4)$ [28], $(n, 2)$ $(n \geq 2)$ [44, 46] and $(3, 3)$ [36]. The survey in [25] describes the simple singularities of projections of complete intersections, this *a priori* finer classification corresponds to the $\mathcal{A}$-classification for $n \geq p$.)

After explaining the techniques for verifying the above claim, we will describe two particular cases in detail. First, the classification of $\mathcal{A}_{\Omega_p}$-simple orbits in dimensions $(n, 2)$, $n > 1$, because for $p = 2$ the volume preserving and the symplectic classifications agree. Combining this classification with the one by Ishikawa and

Janeczko [29] for curves (i.e., for $(1,2)$) yields all simple map-germs into the symplectic plane. And second, the classification of $\mathcal{A}_{\Omega_p}$-simple orbits in dimensions $(n,2n)$, where "non-trivial" weakly quasihomogeneous germs (that are not quasihomogeneous nor "trivially w.q.h.") start appearing.

Notice that the condition w.q.h. (for $\mathcal{A}_{\Omega_p}$) in Proposition 3.8 and Theorem 3.9 is a sufficient condition for the absence of $\mathcal{A}_{\Omega_p}$-moduli, we do not know whether it is necessary. However, for all $\mathcal{A}$-simple germs in the dimension ranges (n,p) in which the $\mathcal{A}$-simple classification is known (see above) the condition w.q.h. is necessary and sufficient for the absence of $\mathcal{A}_{\Omega_p}$-moduli. This obviously implies the criterion above: an $\mathcal{A}$-simple germ f is $\mathcal{A}_{\Omega_p}$-simple if and only if f is only adjacent to w.q.h. germs. All known examples of $\mathcal{A}$-simple map-germs f that fail to be w.q.h. are of the form $f = f_0 + h$, where f_0 is quasihomogeneous, h is a monomial vector of positive filtration (weighted degree) and the restriction of $\gamma_{f_0} : L\mathcal{A} \to L\mathcal{A} \cdot f$ to the filtration-0 parts (of the filtered modules in source and target) has 1-dimensional kernel. In this situation the coefficient of h is a modulus for $\mathcal{A}_{\Omega_p}$ (see Lemma 5.1 below).

Consider $L\mathcal{A}_{\Omega_p} \cdot f \subset L\mathcal{A} \cdot f = tf(\mathcal{M}_n \cdot \theta_n) + wf(\mathcal{M}_p \cdot \theta_p)$. For the subgroup $\mathcal{A}_{\Omega_p} = \mathcal{R} \times \mathcal{L}_{\Omega_p}$ of $\mathcal{A}$ we have to restrict the homomorphism $wf : \theta_p \to \theta_f$, $wf(b) = b \circ f$ to divergence free vector fields b, hence $L\mathcal{L}_{\Omega_p} \cdot f$ is no longer a C_p-module. Let Λ_d denote the $\mathbb{K}$-vector space of homogeneous divergence free vector fields in $\mathbb{K}^p$ of degree d. Notice that Λ_d is the kernel of the epimorphism

$$\operatorname{div} : (\theta_p)_{(d)} := \frac{\mathcal{M}_p^d \cdot \theta_p}{\mathcal{M}_p^{d+1} \cdot \theta_p} \to H_{(d-1)} := \frac{\mathcal{M}_p^{d-1}}{\mathcal{M}_p^d},$$

which maps a vector field on $\mathbb{K}^p$ of degree d to its divergence. Hence

$$\dim \Lambda_d = \dim(\theta_p)_{(d)} - \dim H_{(d-1)} = (p-1)\binom{p+d-1}{d} + \binom{p+d-2}{d}.$$

The $\dim \Lambda_d$ vector fields

$$\prod_{l \neq i} y_l^{\alpha_l} \partial/\partial y_i, \ \sum_l \alpha_l = d, \ i = 1, \ldots, p$$

and (setting $h_{y_i} := \partial h/\partial y_i$)

$$-h_{y_j} \partial/\partial y_1 + h_{y_1} \partial/\partial y_j, \ h = \prod_l y_l^{\alpha_l}, \ \alpha_1, \alpha_j \geq 1, \sum_l \alpha_l = d+1, \ j = 2, \ldots, p$$

are clearly linearly independent and hence form a basis for Λ_d. The tangent space to the $\mathcal{L}_{\Omega_p}$-orbit at f is then given by $L\mathcal{L}_{\Omega_p} \cdot f = f^* \oplus_{d \geq 1} \Lambda_d$.

The criterion in the next easy lemma is sufficient for detecting in the existing classifications of $\mathcal{A}$-simple orbits those which are foliated by an r-parameter family, $r \geq 1$, of $\mathcal{A}_{\Omega_p}$-orbits.

Lemma 5.1. *Consider a map-germ $f_u : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ of the form $f_u = f + u \cdot M$, where f is a quasi-homogeneous germ, $u \in \mathbb{K}$ and $M = X^\alpha \cdot \partial/\partial y_j \notin L\mathcal{A} \cdot f = L\mathcal{A}_{\Omega_p} \cdot f$ is a monomial vector of positive weighted degree (with respect to the weights of f). Then we have the following:*

(i) The coefficient u is not a modulus for $\mathcal{A}$-equivalence.

(ii) For a set of weights for which f is weighted homogeneous, let $(\theta_n)_0$, $(\theta_p)_0$ and $(\theta_f)_0$ denote the filtration-0 parts of the modules of source-, target-vector fields

and vector fields along f, respectively. If the kernel of the linear map

$$\gamma_f : (\theta_n)_0 \oplus (\theta_p)_0 \to (\theta_f)_0,\ (a,b) \mapsto tf(a) - wf(b),$$

of $\mathbb{K}$-vector spaces is 1-dimensional then u is an $\mathcal{A}_{\Omega_p}$-modulus of f_u.

Proof. Let f be weighted-homogeneous for the weights $w_1, \dots, w_n$, and associate to the target variables the weights $\delta_1, \dots, \delta_p$. Then the weighted degree of $\partial/\partial y_i$ is $-\delta_i$ so that f has filtration 0 and M has filtration $r > 0$.

For $\mathcal{A}$-equivalence we consider the following element of $L\mathcal{A} \cdot f_u$:

$$tf_u(\sum_{i=1}^{n} w_i x_i \cdot \partial/\partial x_i) - wf_u(\sum_{j=1}^{p} \delta_j y_j \cdot \partial/\partial y_j) = ruM.$$

From Mather's lemma (Lemma 3.1 in [38]) we conclude that the connected components of $\mathbb{K} \setminus \{0\}$ of the parameter axis lie in a single $\mathcal{A}$-orbit, hence u is not a modulus for $\mathcal{A}$.

For the second statement we observe that $\dim \ker \gamma_f = 1$ implies that this kernel is spanned by the pair of Euler vector fields (E_w, E_δ) in source and target (which is unique up to a multiplication by an element of $\mathbb{K}^*$). And $M \notin L\mathcal{A} \cdot f$ implies that the only generator of M in $L\mathcal{A} \cdot f_u$ must be of the form $tf_u(a) - wf_u(b)$ with (a,b) a non-zero multiple of (E_w, E_δ). But E_δ has non-zero divergence, hence this generator does not belong to $L\mathcal{A}_{\Omega_p} \cdot f_u$. Now Mather's lemma implies that u is a modulus for $\mathcal{A}_{\Omega_p}$. □

5.1. **$\mathcal{A}_{\Omega_p}$-simple, hence symplectically simple, maps from n-space to the plane.** The following classification, in combination with Ishikawa and Janeczko's classification of plane curves [29], provides a complete list of simple map-germs into the plane $\mathbb{C}^2$, up to source diffeomorphisms and target symplectomorphisms (volume preserving diffeomorphisms of $\mathbb{C}^2$ are symplectomorphisms).

Proposition 5.2. *Any $\mathcal{A}_{\Omega_p}$-simple map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$, $n \geq 2$, is equivalent to one of the following normal forms (here $Q = \sum_{i=1}^{n-2} z_i^2$ for $n > 2$ and $Q = 0$ for $n = 2$): (x, y); $(x, y^2 + Q)$; $(x, xy + y^3 + Q)$; $(x, y^3 + x^k y + Q)$, $k > 1$; $(x, xy + y^4 + Q)$.*

Proof. Any $\mathcal{A}$-simple germ in dimensions $(n, 2)$, $n \geq 2$, which does not appear in the above list, is adjacent to one of the following germs (for $n > 2$, up to a suspension by Q defined above): $(x, xy + y^5 + y^7)$, $(x, xy^2 + y^4 + y^5)$ or $(x^2 + y^3, y^2 + x^3)$ (see the adjacency diagrams in [44] and [46]). These three germs fail to be weakly quasihomogeneous and they satisfy the hypotheses of Lemma 5.1, hence they have at least one modulus for $\mathcal{A}_{\Omega_p}$. In fact, the parameter a in $(x, xy + y^5 + ay^7)$, $(x, xy^2 + y^4 + ay^5 + \dots)$ and $(x^2 + ay^3, y^2 + x^3)$ is a modulus for $\mathcal{A}_{\Omega_p}$. □

5.2. **$\mathcal{A}_{\Omega_p}$-simple maps from n-space to $2n$-space.** In the same way we obtain the $\mathcal{A}_{\Omega_p}$-simple germs in dimensions $(n, 2n)$, $n \geq 2$ (notice that $n = 1$ again corresponds to the classification in [29]). Except for the appearance of a series of w.q.h. germs (see the last two normal forms in Proposition 5.4 below, corresponding to type 22_k and 23 in [30]), which are not q.h. nor trivially w.q.h., this classification follows from the classification of $\mathcal{A}$-orbits (and some information about adjacencies between these orbits) in [30], using the same arguments as in dimensions $(n, 2)$. The classifications in dimensions $(2, 4)$ and $(n, 2n)$, $n \geq 3$, are as follows.

Proposition 5.3. *Any $\mathcal{A}_{\Omega_p}$-simple map-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^4, 0)$ is equivalent to one of the following normal forms:* $(x, y, 0, 0)$; (x, xy, y^2, y^{2k+1}), $k \geq 1$; $(x, y^2, y^3, x^k y)$, $k \geq 2$; $(x, y^2, y^3 + x^k y, x^l y)$, $l > k \geq 2$; $(x, y^2, x^2 y + y^{2k+1}, xy^3)$, $k \geq 2$; $(x, y^2, x^2 y, y^5)$; $(x, y^2, x^3 y + y^5, xy^3)$; $(x, xy, xy^2 + y^{3k+1}, y^3)$, $k \geq 1$; $(x, xy, xy^2 + y^{3k+2}, y^3)$, $k \geq 1$; $(x, xy + y^{3k+2}, xy^2, y^3)$, $k \geq 1$; (x, xy, y^3, y^4); (x, xy, y^3, y^5).

Proposition 5.4. *Any $\mathcal{A}_{\Omega_p}$-simple map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n}, 0)$, $n \geq 3$, is equivalent to one of the following normal forms (here $\mathbf{x}$ denotes $x_1, \ldots, x_{n-1}$, and notice that the last two normal forms are only $\mathcal{A}_{\Omega_p}$-simple for $n \geq 4$):*

$(\mathbf{x}, y, 0, \ldots, 0)$
$(\mathbf{x}, x_1 y, \ldots, x_{n-1} y, y^2, y^{2k+1})$, $k \geq 1$
$(\mathbf{x}, x_2 y, \ldots, x_{n-1} y, y^2, y^3, x_1^k y)$, $k \geq 2$
$(\mathbf{x}, x_2 y, \ldots, x_{n-1} y, y^2, y^3 + x_1^k y, x_1^l y)$, $l > k \geq 2$
$(\mathbf{x}, x_2 y, \ldots, x_{n-1} y, y^2, x_1^2 y + y^{2k+1}, x_1 y^3)$, $k \geq 2$
$(\mathbf{x}, x_2 y, \ldots, x_{n-1} y, y^2, x_1^2 y, y^5)$
$(\mathbf{x}, x_2 y, \ldots, x_{n-1} y, y^2, x_1^3 y + y^5, x_1 y^3)$
$(\mathbf{x}, x_3 y, \ldots, x_{n-1} y, y^2, x_1^2 y, x_2^2 y, y^3 + x_1 x_2 y)$
$(\mathbf{x}, x_3 y, \ldots, x_{n-1} y, y^2, x_1^2 y, x_2^2 y, y^3)$
$(\mathbf{x}, x_3 y, \ldots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3 + x_2^2 y)$
$(\mathbf{x}, x_3 y, \ldots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3 + x_2^3 y)$
$(\mathbf{x}, x_3 y, \ldots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3)$
$(\mathbf{x}, \mathbf{x} y, x_1 y^2 + y^{3k+1}, y^3), k \geq 1$
$(\mathbf{x}, \mathbf{x} y, x_1 y^2 + y^{3k+2}, y^3), k \geq 1$
$(\mathbf{x}, x_1 y + y^{3k+2}, x_2 y, \ldots, x_{n-1} y, x_1 y^2, y^3), k \geq 1$
$(\mathbf{x}, x_1 y, x_2 y + y^{3k+2}, x_3 y, \ldots, x_{n-1} y, x_1 y^2 + y^{3l+1}, y^3), l > k \geq 1$
$(\mathbf{x}, x_1 y, x_2 y + y^{3k+2}, x_3 y, \ldots, x_{n-1} y, x_1 y^2 + y^{3l+2}, y^3), l > k \geq 1$
$(\mathbf{x}, x_1 y + y^{3l+2}, x_2 y + y^{3k+2}, x_3 y, \ldots, x_{n-1} y, x_1 y^2, y^3), l > k \geq 1$
$(\mathbf{x}, \mathbf{x} y, y^3, y^4)$
$(\mathbf{x}, \mathbf{x} y, y^3, y^5)$
$(\mathbf{x}, x_1 y + y^3, x_2 y, \ldots, x_{n-1} y, x_1 y^2 + y^{2k+1}, x_2 y^2 + y^4)$, *for* $k = 2$ *and* $n \geq 4$
$(\mathbf{x}, x_1 y + y^3, x_2 y, \ldots, x_{n-1} y, x_1 y^2 + y^5, y^4)$, *for* $n \geq 4$.

Proof. Except for the germs of type 22_k and 23 in dimensions $(n, 2n)$, $n \geq 4$ (these are the last two germs in the second list above), all $\mathcal{A}$-simple germs in [30] are either quasihomogeneous or they satisfy the hypotheses of Lemma 5.1 and hence have at least one $\mathcal{A}_{\Omega_p}$-modulus.

Consider, then, the series 22_k of map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{2n}, 0)$, $n \geq 3$ given by:

$$g_k = (x_1, \ldots, x_{n-1}, x_1 y + y^3, x_2 y, \ldots, x_{n-1} y, x_1 y^2 + y^{2k+1}, x_2 y^2 + y^4), k \geq 2.$$

The germs 22_k are not semi-quasihomogeneous: if we write $g_k = f + y^{2k+1} \cdot e_{2n-1}$ then the weighted homogeneous initial part f is not $\mathcal{A}$-finite. For $n = 3$ all the germs 22_k are $\mathcal{A}$-simple, for $n \geq 4$ only 22_2 is $\mathcal{A}$-simple (the germs $22_{\geq 3}$ do not have an $\mathcal{A}$-modulus, but they lie in the closure of non-simple $\mathcal{A}$-orbits), see [30].

Now consider $\mathcal{A}_{\Omega_p}$-equivalence. Writing $f_u = f + u \cdot y^{2k+1} \cdot e_{2n-1}$ we see that $\dim \ker \gamma_f = n - 2$. For $n = 3$ part (ii) of Lemma 5.1 therefore implies that the coefficient u is an $\mathcal{A}_{\Omega_p}$-modulus. For $n \geq 4$ the germs f_u are weakly quasihomogeneous (take weights $w(x_1) = w(x_2) = w(y) = 0$ and $w(x_i) = 1$, $i \geq 3$) and $\mathcal{A}_{\Omega_p}$-equivalent to g_k (for $u \neq 0$).

For the germ of type 23 the argument is the same. □

5.3. **Semi-quasihomogeneous, but not weakly quasihomogeneous, singularities.** Non-w.q.h. maps have a decomposition $f = f_0 + h$ with f_0 q.h. and h of positive degree (relative to the weights of f_0). The normal space $N\mathcal{A} \cdot f_0 := \mathcal{M}_n \cdot \theta_{f_0}/L\mathcal{A} \cdot f_0$ decomposes into a part of non-positive filtration and a part of positive filtration, denoted by $(N\mathcal{A} \cdot f_0)_+$. Using the fact that $L\mathcal{A}_{\Omega_p} \cdot f_0 = L\mathcal{A} \cdot f_0$ and Mather's lemma we obtain the following formal pre-normal form for an element of an $\mathcal{A}_{\Omega_p}$-orbit inside $\mathcal{A} \cdot f$:

$$f' = f_0 + \sum_{h_i \in B(f_0)_+} a_i h_i,$$

where $B(f_0)_+$ denotes a base for $(N\mathcal{A} \cdot f_0)_+$ as a $\mathbb{K}$-vector space. Notice that for semi-quasihomogeneous maps f the above sum is finite (because f_0 is $\mathcal{A}$-finite), otherwise it is infinite.

Preliminary empirical examples indicate that in the s.q.h. case (where f_0 is $\mathcal{A}$-finite) the above pre-normal for f' is in fact a (formal) normal form for $\mathcal{A}_{\Omega_p}$. In this case the coefficients a_i are independent moduli for $\mathcal{A}_{\Omega_p}$ (some a_i might also be moduli for $\mathcal{A}$). If this observation holds in general for s.q.h. maps in dimensions (n, p), $n \geq p - 1$, (and Conjecture I in [10] is true) then such maps f satisfy the formula

$$\operatorname{cod}(\mathcal{A}_{\Omega_p,e}, f) = \mu_\Delta(f)$$

as pointed out in the introduction (here μ_Δ denotes the discriminant Milnor number (for $n \geq p$) or the image Milnor number (for $p = n + 1$)). Also notice that for $n \geq p$ we have $\operatorname{cod}(\mathcal{A}_{\Omega_p,e}, f) \leq \mu_\Delta(f)$, independent of the correctness of the above conjectures.

Let us consider some examples in dimensions $(n, 2)$, $n \geq 2$.

Example 5.5. The $\mathcal{A}$-simple non-w.q.h. germs in dimensions $(n, 2)$ have the following formal normal forms for $\mathcal{A}_{\Omega_p}$ (the normal forms $(*)$ are not s.q.h. and Q denotes a sum of squares in additional variables): $(x, xy + y^5 + ay^7 + Q)$; $(x, xy^2 + y^5 + ay^6 + by^9 + Q)$; $(x, x^2y + y^4 + ay^5 + Q)$; $(*)$ $(x, xy^2 + y^4 + \sum_{k\geq 2} a_k y^{2k+1} + Q)$; $(*)$ $(x^2 + ay^{2l+1}, y^2 + x^{2m+1})$, $l \geq m \geq 1$.

The first three normal forms f are s.q.h. and their $\mathcal{A}_{\Omega_p,e}$-codimensions are equal to the $\mathcal{A}_e$-codimensions of their initial parts f_0, and these are given by 3, 5 and 4, respectively. And from [13] we have the formula $\mu_\Delta(f) = \mu(\Sigma_f) + d(f)$ (relating the discriminant Milnor to the Milnor number of the critical set and the double-fold number), which gives for the three normal forms $3 = 0 + 3$, $5 = 1 + 4$ and $4 = 2 + 2$, respectively.

The two series of non-s.q.h. maps f (marked by $(*)$) are GTQ in the sense of [7] and the Milnor numbers of their discriminant curves Δ_f (not to be confused with the discriminant Milnor numbers of f) are $2k+7$ and $2(l+m)+3$, respectively. These Milnor numbers are upper bounds for the $\mathcal{A}_{\Omega_p}$-moduli space of f (by Remark 4.18). Formal calculations (at the infinitesimal level using Mather's lemma) actually show that $\dim \mathcal{M}(\mathcal{A}_{\Omega_p}, (x^2 + y^{2l+1}, y^2 + x^{2m+1})) = 1$, modulo $\mathcal{M}_n^\infty \theta_f$, and that we can take the above (formal) normal form for $\mathcal{A}_{\Omega_p}$-equivalence with the parameter a as the modulus. For the other non-s.q.h. map we only know that a_2 is a modulus and that we can take $a_3 = a_4 = 0$ (provided $a_2 \neq 0$), for a_k, $k > 4$, the corresponding calculations of $L\mathcal{A}_{\Omega_p} \cdot f + \mathcal{M}_n^{k+1}\theta_f$ seem very tedious.

Finally, a brief remark on our computation of $\mu(\Delta_f)$ for the above two series. We use the formulas $2\delta = \mu + r - 1$ (relating the δ-invariant, the number of branches r

and μ of a planar curve-germ) and $\delta(\Delta_f) = c(f)+d(f)+\delta(\Sigma_f)$ (where $c(f)$ and $d(f)$ are the numbers of cusps and double folds, respectively, in a stable perturbation f_t of f, hence $\delta(\Delta_{f_t}) = c(f) + d(f)$). For $f = (x, xy^2 + y^4 + y^{2k+1})$ we obtain $\delta(\Delta_f) = 3 + k + 1$ (see Table 1 in [44]), hence $\mu(\Delta_f) = 2k + 7$ (notice that the discriminant has $r = 2$ branches). This contradicts the claim in part (c) of example 1 in [7] that Δ_f has an E_{6k+1} singularity.

Example 5.6. The $\mathcal{A}$-unimodal germs in dimensions $(n, 2)$ lie in the closure of the orbits of one of the following $\mathcal{A}$-unimodal s.q.h. germs (see [45], and Q is again a sum of squares in additional variables):

$$(x, y^4 + x^3y + ax^2y^2 + x^3y^2 + Q),\ a \neq -3/2$$
$$(x, xy + y^6 + y^8 + ay^9 + Q)$$
$$(x, xy + y^3 + ay^2z + z^3 + z^5 + Q).$$

For $\mathcal{A}_{\Omega_p}$-equivalence the corresponding normal forms f are:

$$(x, y^4 + x^3y + ax^2y^2 + bx^3y^2 + Q)$$
$$(x, xy + y^6 + ay^8 + by^9 + cy^{14} + Q)$$
$$(x, xy + y^3 + ay^2z + z^3 + bz^5 + Q).$$

All $\mathcal{A}$-unimodal germs therefore have $\mathcal{A}_{\Omega_p}$-modality at least two. Also, for the above $f = f_0 + h$ we again have $\mathrm{cod}(\mathcal{A}_{\Omega_p,e}, f) = \mathrm{cod}(\mathcal{A}_e, f_0) = \mu_\Delta(f)$.

5.4. **Weakly quasihomogeneous multigerms.** Before leaving the subject of $\mathcal{A}_{\Omega_p}$-classification we make a final remark. All the results on $\mathcal{A}_{\Omega_p}$-equivalence can be easily extended to multigerms $f = (f^1, \ldots, f^s) : (\mathbb{K}^n, S) \to (\mathbb{K}^p, \Omega_p, 0)$ at an s-tuple $S = \{q^1, \ldots, q^s\} \subset \mathbb{K}^n$ of points in the source. Such an f is $\mathcal{A}_{\Omega_p}$-w.q.h. if each component $f^i = (f^i_1, \ldots, f^i_p)$ is $\mathcal{A}_{\Omega_p}$-w.q.h. as a monogerm for (possibly different) sets of weights $\{w^i_1, \ldots, w^i_n\}$ but of the same weighted degrees $\deg f^1_j = \ldots = \deg f^s_j = \delta_j$, $j = 1, \ldots, p$. Also, if the above weights w^i_j are positive integers then we say that f is q.h. as a multigerm.

Using Mather's [39] characterization of $\mathcal{A}$-stability of multigerms in terms of multitransversality to $\mathcal{K}$-orbits of multigerms, it is not hard to see that all $\mathcal{A}$-stable multigerms are q.h. and hence $\mathcal{A}_{\Omega_p}$-w.q.h., which implies that the classifications of $\mathcal{A}$-stable and $\mathcal{A}_{\Omega_p}$-stable orbits (over $\mathbb{C}$) also agree for multigerms.

6. The foliation of $\mathcal{K}$-orbits by $\mathcal{K}_{\Omega_n}$- and $\mathcal{K}_{\Omega_p}$-orbits

In this section we consider the volume-preserving versions of the classification of ICIS or, in other words, of $\mathcal{K}$-finite maps $f : \mathbb{C}^n \to \mathbb{C}^p$, $n \geq p$. Recall that all $\mathcal{K}$-simple f and all f whose differential has non-zero rank are w.q.h. for both $\mathcal{K}_{\Omega_n}$ and $\mathcal{K}_{\Omega_p}$. Hence we will consider $\mathcal{K}$-unimodal germs f of rank 0 and concentrate on the more interesting group $\mathcal{K}_{\Omega_n}$ (the condition w.q.h. for $\mathcal{K}_{\Omega_p}$ is weaker than that for $\mathcal{K}_{\Omega_n}$, hence $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) = 0$ implies $\dim \mathcal{M}(\mathcal{K}_{\Omega_p}, f) = 0$). The relevant $\mathcal{K}$-classifications are therefore those in dimensions $(n, p) = (3, 2)$ and $(4, 2)$ (see [51]) and $(2, 2)$ (see [14]) and $(3, 3)$ (see [15]). Recall that the $\mathcal{K}_{\Omega_n}$-classification of hypersurfaces $f^{-1}(0)$ has been settled by the result of Varchenko [48], which gives $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) = \mu(f) - \tau(f)$.

Looking at the lists in [51, 14, 15] we see (using our results) that a $\mathcal{K}$-unimodal map-germ f is w.q.h. for $\mathcal{K}_{\Omega_n}$ if and only if it is quasihomogeneous. We can therefore state:

(1) A $\mathcal{K}$-unimodal map-germ f of rank 0 is $\mathcal{K}_{\Omega_n}$-unimodal if and only if it is q.h. and does not lie in the closure of a non-q.h. $\mathcal{K}$-orbit.
(2) For a $\mathcal{K}$-unimodal map-germ f of rank 0 such that $f^{-1}(0)$ defines a ICIS of positive dimension and of codimension greater than one we have the following: (i) f is q.h. if and only if $\mathcal{M}(\mathcal{K}_{\Omega_n}, f) = 0$, and (ii) for f non-w.q.h. the dimension of $\mathcal{M}(\mathcal{K}_{\Omega_n}, f)$ is one or two.
(3) For map-germs f of positive rank we recall that the $\mathcal{K}$- and $\mathcal{K}_{\Omega_n}$-classifications agree.

We will now apply our finiteness results for $\mathcal{M}(\mathcal{K}_{\Omega_n}, f) \cong H^n(\Lambda^*(f^*\mathcal{M}_p))$ to some examples of non-q.h. (and non-w.q.h.) map-germs f from the classifications in [51, 14, 15]. These results give upper bounds for $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f)$, and for certain f some of these upper bound will coincide with the following lower bound (which is analogous to Lemma 5.1 in the $\mathcal{A}_{\Omega_p}$ case).

Lemma 6.1. *Consider a map-germ $f_u : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of the form $f_u = f + u \cdot M$, where f is a quasi-homogeneous germ, $u \in \mathbb{C}$ and $M = X^\alpha \cdot \partial/\partial y_j \notin L\mathcal{K} \cdot f = L\mathcal{K}_{\Omega_n} \cdot f$ is a monomial vector of positive weighted degree (with respect to the weights of f). For a set of weights for which f is weighted homogeneous, let $(\theta_n)_0$, $(gl_p(\mathcal{O}_n))_0$ and $(\theta_f)_0$ denote the filtration-0 parts of the relevant modules. If the kernel of the linear map*

$$\gamma_f : (\theta_n)_0 \oplus (gl_p(\mathcal{O}_n))_0 \to (\theta_f)_0,\ (a, B) \mapsto tf(a) - B \cdot f,$$

of $\mathbb{K}$-vector spaces is 1-dimensional then u is an $\mathcal{K}_{\Omega_n}$-modulus of f_u. Hence the dimension of $\mathcal{M}(\mathcal{K}_{\Omega_n}, f_u)$ is positive.

In our first example we consider positive dimensional complete intersections, defined by $\mathcal{K}$-finite maps f, that are not hypersurfaces. In all our examples we have (for positive dimensional) ICIS that $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) \leq \mu(f) - \tau(f)$, and for a s.q.h. germ $f = f_0 + h$ this inequality holds in general. (For such $f = f_0 + h$ we have $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) = \dim \mathcal{M}(\mathcal{K}_{\Omega_n,e}, f) \leq \tau(f_0) - \tau(f)$ and $\tau(f_0) = \mu(f_0) = \mu(f)$.) The germs f in the example (which are not s.q.h.) show that this inequality can be strict.

Example 6.2. Consider the $\mathcal{K}$-unimodal space-curves $FW_{1,i}$ from [51], given by

$$f = (g_1, g_2) = (xy + z^3, xz + y^2z^2 + y^{5+i}), i > 0.$$

Writing $f = f_0 + (0, y^{5+i})$, where f_0 is q.h. for $w = (7, 2, 3)$, $\delta = (9, 10)$ and where $(0, y^{5+i})$ has filtration $2i > 0$, and applying Lemma 6.1 we have $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) \geq 1$. The component functions of f are both q.h. (for weights $w_1 = (1, 2, 1)$ and $w_2 = (7+i, 2, 3+i)$, respectively) and $\mathcal{O}_3/\langle \nabla g_1, \nabla g_2 \rangle \cong \mathbb{C}$, hence $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) \leq 1$ by Corollary 4.12. Therefore $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) = 1$ and the family $f_a = (xy + z^3, xz + y^2z^2 + ay^{5+i})$ parameterizes the $\mathcal{K}_{\Omega_n}$-orbits inside $\mathcal{K} \cdot f$.

A weaker upper bound for $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f)$ follows from Remark 4.15 (which does not require that the component functions are w.q.h.): take a projection π onto the first target coordinate, then $\pi \circ f = g_1$ and $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) \leq \mu(g_1) = 2$.

Finally, notice that $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, f) = 1$ is smaller than the difference of $\mu(f) = 16 + i$ and $\tau(f) = 14 + i$, where $\tau(f)$ denotes the dimension of $T^1_f = N\mathcal{K}_e \cdot f$. Recall that for hypersurfaces $h^{-1}(0)$ we have $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, h) = \mu(h) - \tau(h)$ (by [48]), in all our examples of higher codimensional ICIS $g^{-1}(0)$ we have $\dim \mathcal{M}(\mathcal{K}_{\Omega_n}, g) \leq \mu(g) - \tau(g)$. Notice that for a "suspension" $G = (z, g)$ of g (z an extra variable)

$\mu(G)-\tau(G)=\mu(g)-\tau(g)$, but $dG(0)$ has positive rank hence $\dim\mathcal{M}(\mathcal{K}_{\Omega_n},G)=0$, the difference between both sides of the inequality above can therefore be arbitrarily large. But the examples $f=FW_{1,i}$ show that even in the rank 0 case the Varchenko formula does not hold for ICIS of codimension greater than 1.

Also notice that the series $FW_{1,i}$, $i>0$, lies in the closure of the $\mathcal{K}$-orbit of the s.q.h. germ

$$g_\lambda=(xy+z^3,xz+y^2z^2+\lambda y^5+y^6),\lambda\neq 0,-1/4,$$

where $\mu(g_\lambda)=16$ and $\tau(g_\lambda)=15$. Omitting the higher filtration y^6-term we obtain type $FW_{1,0}$ in Wall's list [51], which is q.h. and $\mu(FW_{1,0})=\tau(FW_{1,0})=16$. Notice that $FW_{1,1}$ (with $\mu(F_{1,1})=17$ and $\tau(F_{1,1})=15$) corresponds to the exceptional parameter $\lambda=0$ in the modular stratum $\bigcup_{\lambda\in\mathbb{C}\setminus\{0,-1/4\}}\mathcal{K}\cdot g_\lambda$ (which seems to be missing in Wall's list) and does not lie in the closure of the orbit of $FW_{1,0}$.

Example 6.3. Consider the $\mathcal{K}$-unimodal equidimensional maps of type $h_{\lambda,q}$ from [15], given by

$$f=f^\lambda:=(xz+xy^2+y^3,yz,x^2+y^3+\lambda z^q)=f_0+(0,0,y^3+\lambda z^q),q>2.$$

The initial part f_0 is q.h. of type $w=(1,1,2)$, $\delta=(3,3,2)$ and $fil(0,0,y^3)=1$, $fil(0,0,z^q)=2(q-1)>1$. Applying Lemma 6.1 to $f'=f_0+(0,0,ay^3)$ we see that a is a $\mathcal{K}_{\Omega_n}$-modulus of f' and hence of f, hence $\dim\mathcal{M}(\mathcal{K}_{\Omega_n},f)\geq 1$. The component functions of $f=(g_1,g_2,g_3)$ are q.h. for distinct sets of weights, namely for $w_1=(1,1,2)$, any w_2 and $w_3=(3q,2q,6)$. Now $\mathcal{O}_3/\langle\nabla g_1,\nabla g_2,\nabla g_3\rangle\cong\mathbb{C}$, so that $\dim\mathcal{M}(\mathcal{K}_{\Omega_n},f)=1$ (by Corollary 4.12 and the above lower bound – the upper bound also follows from $\mu(g_1+g_2+g_3)=1$, by Remark 4.15). And for each $f^\lambda=(xz+xy^2+y^3,yz,x^2+y^3+\lambda z^q)$, $\lambda\in\mathbb{C}$, the family $f^\lambda_a=(xz+xy^2+y^3,yz,x^2+ay^3+\lambda z^q)$ parameterizes the $\mathcal{K}_{\Omega_n}$-orbits inside $\mathcal{K}\cdot f^\lambda$.

Example 6.4. Finally, consider the $\mathcal{K}$-unimodal equidimensional maps of type $G_{k,l,m}$ from [14], given by

$$f=(g_1,g_2)=(x^2+y^k,xy^l+y^m)=f_0+(0,y^m),$$

where $k\neq 2(m-l)$ and either $k\leq l$, $l+1<m<l+k-1$ (case (a)) or $l<k<2l-1$, $k<m<2l$ (case (b)). As above we check that the coefficient of $(0,y^m)$ is a $\mathcal{K}_{\Omega_n}$-modulus, hence $\dim\mathcal{M}(\mathcal{K}_{\Omega_n},f)\geq 1$. And again the g_i are q.h. for distinct sets of weights, but now $\mathcal{O}_2/\langle\nabla g_1,\nabla g_2\rangle\cong\mathbb{C}\{1,y,\ldots,y^r\}$, where $r=k-1$ in case (a) and $r=l$ in case (b). Hence $1\leq\dim\mathcal{M}(\mathcal{K}_{\Omega_n},f)\leq r$.

We can also obtain an upper bound using Remark 4.15: take the generic projection π onto the first target coordinate, then $g_1=\pi\circ f$ and $\dim\mathcal{M}(\mathcal{K}_{\Omega_n},f)\leq\mu(g_1)=k-1$. This gives the same upper bound in case (a), but in case (b) we have $l\leq k-1$.

7. The groups $G_{\Omega_q}\neq\mathcal{A}_{\Omega_p}$, $\mathcal{K}_{\Omega_n}$, $\mathcal{K}_{\Omega_p}$: examples of G-stable maps f of positive and infinite G_{Ω_q}-modality

In this final section we make some remarks on the remaining volume preserving subgroups G_{Ω_q} of $\mathcal{A}$ or $\mathcal{K}$. First of all we remark that placing volume forms both in the source and the target of a map f leads to moduli even for invertible linear maps $f:\mathbb{C}^n\to\mathbb{C}^n$ (the modulus being the determinant of f).

For function-germs the only relevant groups are those with a volume form to be preserved in the source, and what is known for these had been described in Section 1.

For map-germs $\mathcal{R}$-equivalence is too fine already in the absence of a volume form, hence the remaining cases of interest (not considered in the previous sections) are the groups $\mathcal{A}_{\Omega_n}$, $\mathcal{L}_{\Omega_p}$ and $\mathcal{C}_{\Omega_p}$ for pairs of dimensions (n,p), $p>1$, for which singular G-finite ($G=\mathcal{A}$, $\mathcal{L}$ or $\mathcal{C}$) map-germs f exist. And we can also discard those map-germs f that are trivially w.q.h. for the relevant group.

The following (in some sense "simplest" singular but non-w.q.h.) examples indicate that for the above three groups we immediately obtain moduli.

Example 7.1. For $\mathcal{A}_{\Omega_n}$ the fold map $f=(x,y^2)$ has infinite modality. We have

$$L\mathcal{A}^n_f=\mathbb{K}\{(x^ly^{2k},0),(0,x^ly^{2k+1});l,k\geq 0,l+k\geq 1\},$$

where the elements of $L\mathcal{A}^n_f$ are also known as lowerable vector fields (we write these source vector fields as vectors). It follows that dimension of $C_n/\mathrm{div}(L\mathcal{A}^n_f)$, which is a lower bound for the number of $\mathcal{A}_{\Omega_n}$-moduli, is infinite for the fold f.

Example 7.2. For $\mathcal{L}_{\Omega_p}$ perhaps the first interesting example of a singular germ that fails to be trivially w.q.h. is the planar cusp $f=(x^2,x^3)$. We claim that in this case $C_p/\mathrm{div}(L\mathcal{L}^p_f)\cong\mathbb{K}\{1,y_1\}$, hence the $\mathcal{L}_{\Omega_p}$-modality of f is two (f is $\mathcal{L}$-simple and the dimension of the $\mathcal{L}_{\Omega_p}$-moduli space is two).

Taking coordinates (y_1,y_2) in the target, we see that the kernel of

$$L\mathcal{L}\longrightarrow\mathcal{M}_n\theta_f,\ u\mapsto u\circ f$$

is (as a $\mathbb{K}$-vector space) generated by elements $u^i_{rsl}:=y_1^ry_2^s(y_2^{2l}-y_1^{3l})\partial/\partial y_i$, where $i=1,2$, $r,s\geq 0$ and $l\geq 1$. Set $G^i_{rsl}:=\mathrm{div}(u^i_{rsl})$, then

$$(2l+s+1)G^1_{r+1,s,l}-(r+1)G^2_{r,s+1,l}=cy_1^{3l+r}y_2^s$$

and

$$(s+1)G^1_{r+1,s,l}-(3l+r+1)G^2_{r,s+1,l}=cy_1^ry_2^{2l+s}$$

where $c=-6l^2-2l(r+1)-3l(s+1)\neq 0$. Finally, we have $G^1_{001}=-3y_1^2$, $G^2_{001}=2y_2$, $G^1_{011}=-3y_1^2y_2$ and $G^2_{101}=2y_1y_2$, and the claim follows.

Example 7.3. For $\mathcal{C}_{\Omega_p}$ we first remark that $\mathcal{C}$-finite germs f can only appear for $n\leq p$. As an example for a singular germ f, which fails to be trivially w.q.h., we can consider the fold $f=(x,y^2)$. A quick calculation yields $C_n/\mathrm{div}(L\mathcal{C}^p_f)\cong\mathbb{K}\{1,y\}$. Hence f has two $\mathcal{C}_{\Omega_p}$-moduli, which can also be checked by comparing the normal spaces for $\mathcal{C}$ and $\mathcal{C}_{\Omega_p}$. Notice that $N\mathcal{C}\cdot f$ is spanned by $(0,y)$ and $(y,0)$, whereas $N\mathcal{C}_{\Omega_p}\cdot f$ is spanned by these two elements together with $(x,0)$ and $(xy,0)$.

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