Singularities of secant maps of immersed surfaces

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Abstract. The secant map of an immersion sends a pair of points to the direction of the line joining the images of the points under the immersion. The germ of the secant map of a generic codimension-c immersion $X : \mathbb{R}^n \to \mathbb{R}^{n+c}$ at the diagonal in the source is a $\mathbb{Z}_2$ stable map-germ $\mathbb{R}^{2n} \to \mathbb{R}^{n+c-1}$ in the following cases: (i) $c \geq 2$ and $(2n, n+c-1)$ is a pair of dimensions for which the $\mathbb{Z}_2$ stable germs of rank at least n are dense, and (ii) for generically immersed surfaces (i.e. $n = 2$ and any $c \geq 1$). In the latter surface case the $\mathcal{A}^{\mathbb{Z}_2}$-classification of germs of secant maps at the diagonal is described and it is related to the $\mathcal{A}$-classification of certain singular projections of the surfaces.

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1 Introduction

In [5] Bruce has shown that the germ of the (projectivized) secant map of a generic space-curve in $\mathbb{R}^3$ is, at a point on the diagonal in the source, $\mathcal{A}^{\mathbb{Z}_2}$-equivalent to a $\mathbb{Z}_2$ stable germ – these had been classified earlier on by Bierstone [4]. (Here and below $\mathbb{Z}_2$ stable means $\mathcal{A}^{\mathbb{Z}_2}$-stable and stable means $\mathcal{A}$-stable.) Notice that the secant map of a curve has an obvious S_2 symmetry (permuting a pair p, q of source points), which corresponds to a $\mathbb{Z}_2$ symmetry (reflection in the diagonal $p = q$). And at a point outside the diagonal it is $\mathcal{A}$-equivalent (as a mono-germ) to a stable germ (notice that the $\mathbb{Z}_2$ symmetry of the secant map implies that the off-diagonal parts of the critical sets are mapped 2:1 into the target, which makes the secant map highly unstable as a multi-germ). Conversely, all $\mathbb{Z}_2$ stable and stable germs arise as germs of secant maps of generic space-curves.

In the present paper we study the (projectivized) secant maps of higher dimensional and codimensional immersions. The secant map $\hat{S} : (\mathbb{R}^n)^2 \to \mathbb{P}^m$ of an immersion-germ $X : \mathbb{R}^n \to \mathbb{R}^{m+1}$ maps a pair of distinct points $(p, q) \in (\mathbb{R}^n)^2$ to the direction of $X(q) - X(p)$, and it maps a point (p, p) on the source-diagonal to the direction of the directional derivative $D_\omega X(p)$ (here ω can be considered as the limiting direction of a vector $q - p$ as $q \to p$). The secant map of an n-dimensional immersion is evidently S_2 symmetric (as for curves, where $n = 1$, we can permute pairs p, q of source points), but in order to obtain a $\mathbb{Z}_2$ symmetric germ $\hat{S}$ of the secant map at the diagonal for $n \geq 2$ we first have to blow up the

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diagonal $p = q$ in the source to a hyperplane $\lambda = 0$ and then consider reflections in $\lambda = 0$ (see Section 2). For immersions of codimension greater than one (i.e. for $m > n$) there is a jet-map sending pairs of jets of immersion-germs to jets of secants map-germs, whose restriction to the (blow-up of the) source-diagonal is transverse to the $\mathcal{A}^{\mathbb{Z}_2}$-stable orbits (at least in pairs of dimensions $(2n, m)$ in which the $\mathcal{A}^{\mathbb{Z}_2}$-stable germs up to a certain corank are dense). This implies that the germs of $\hat{S}$ at the diagonal are $\mathcal{A}^{\mathbb{Z}_2}$-stable for a residual set of immersions, but it turns out that not all $\mathcal{A}^{\mathbb{Z}_2}$-stable germs are germs of secant maps of generic n-dimensional immersions, $n \geq 2$ (the above jet-map can, of course, be transverse to a given orbit by not intersecting it). For generic 2-dimensional immersions in $\mathbb{R}^n$, $n \geq 3$, we will give an explicit classification of the germs of the secant map at the diagonal.

This "good behavior" (i.e. their $\mathcal{A}^{\mathbb{Z}_2}$-stability) of germs of secant maps of generic immersions at the diagonal does not extend to higher secant maps nor to parametrizations of secant varieties. The higher k-secant maps ($k \geq 3$), mapping a k-tuple of source points to the linear subspace defined by their X-images (in the appropriate Grassmannian), are S_k-symmetric germs at the main diagonal in the source – but here the corresponding jet-map fails to be transverse to the stable $\mathcal{A}^{S_k}$-orbits: for example, the trisecant plane map of a generic space curve can have highly unstable singularities (see [22]). Also, replacing the (ordinary) secant map by the map $\tilde{F}$ sending a pair of points to the line joining their images under a given immersion $X : \mathbb{R}^n \to \mathbb{R}^{n+c}$, we obtain a $\mathbb{Z}_2$-symmetric map $\mathbb{R}^{2n+1} \to \mathbb{R}^{n+c}$. But, at least for $c > 2$, the germ of this map at the source diagonal is never $\mathcal{A}^{\mathbb{Z}_2}$-stable (see Proposition 2.5). Notice that the restiction of $\tilde{F}$ to the source diagonal gives a parametrization of the tangent variety of X, and the latter generically has non-isolated singularities of infinite $\mathcal{A}$-codimension (see the survey on singularities of the tangent variety of a space-curve in [11]).

The plan of this paper is as follows. In Section 2 we define the projectivized secant map $\hat{S} : (\mathbb{R}^n)^2 \to \mathbb{P}^m$ of an immersion $X : \mathbb{R}^n \to \mathbb{R}^{m+1}$, and relate the secant map to the n-parameter family of inner projections of $X(\mathbb{R}^n)$ from centers in $X(\mathbb{R}^n)$. We show that the mono-germ of the secant map at a point (p, q) outside the diagonal is a versal n-parameter deformation of a germ $\mathbb{R}^n \to \mathbb{R}^m$ for a residual set of embeddings (Proposition 2.4). And we show (with some restrictions on the pairs of dimensions $(2m, n)$, see above) that the germ of $\hat{S}$ at the diagonal is $\mathcal{A}^{\mathbb{Z}_2}$-equivalent to some $\mathbb{Z}_2$-stable germ of $\mathcal{A}^{\mathbb{Z}_2}$-codimension at most $2n - 1$ for a residual set of immersions $X : \mathbb{R}^n \to \mathbb{R}^{m+1}$, $m > n$ (Proposition 2.2). We also show that the germ of the parametrization of the secant variety at the diagonal is never $\mathcal{A}^{\mathbb{Z}_2}$-stable for immersions of codimension greater than two (see Proposition 2.5). Section 3 describes the main result of the present paper, the classification of the germs of secant maps $\hat{S}$ at the diagonal of surfaces generically immersed in $\mathbb{R}^n$, $n \geq 3$ (Theorem 3.1). These secant map-germs $\hat{S}$ are $\mathcal{A}^{\mathbb{Z}_2}$ stable germs $\mathbb{R}^4 \to \mathbb{R}^n$, $n \geq 2$, and for $n \geq 3$ any such $\mathcal{A}^{\mathbb{Z}_2}$ stable germ is the germ of the secant map of some generically immersed surface (for $n = 2$ there are complex $\mathcal{A}^{\mathbb{Z}_2}$-orbits corresponding to several real orbits, whose representatives are distinguished by $\pm$ signs, and some of these real forms cannot be the germ of any secant map). For $n \geq 5$, the $\mathcal{A}^{\mathbb{Z}_2}$ classes in this classification of secant maps are in 1:1 correspondence with certain $\mathcal{A}$ classes of germs of projections onto hyperplanes of the immersion-germ X at the corresponding points p along a certain bad direction in T_pX. For $n = 3$ and 4 there are certain $\mathcal{A}^{\mathbb{Z}_2}$-orbits of secant germs that can be further stratified by the

$\mathcal{A}$-types of such projections. But the $\mathcal{A}$-classes of the projections distinguish the $\mathcal{A}^{\mathbb{Z}_2}$-orbits of secants germs for any $n \geq 3$. Section 4 contains the classification of $\mathbb{Z}_2$ stable germs $\mathbb{R}^4 \to \mathbb{R}^n$, for $n \geq 2$ (relevant for the secant map-germs $\hat{S}$ of generic immersions $X : \mathbb{R}^2 \to \mathbb{R}^{n+1}$).

2 Secants and inner projections of immersions $X : \mathbb{R}^n \to \mathbb{R}^{m+1}$, $m \geq n$

Let $X : \mathbb{R}^n \to \mathbb{R}^{m+1}$, $p = (p_1, \dots, p_n) \mapsto X(p)$ be an immersion, and represent the $\mathbb{P}^{n-1}$ of lines through $0 \in \mathbb{R}^n$ by one unit vector ω on each line. The map $\beta(p, \lambda, \omega) = (p, p + \lambda \cdot \omega) =: (p, q)$ blows up the diagonal in $(\mathbb{R}^n)^2$, which has codimension n, to the hyperplane $\beta^{-1}(\{p = q\}) = \{\lambda = 0\}$ and is one-to-one outside $\lambda = 0$. Let $[v]$ denote homogeneous coordinates of the vector v. We want to define a projectivized secant map $\mathbb{R}^n \times \mathbb{R} \times \mathbb{P}^{n-1} \to \mathbb{P}^m$ given by $(p, q) \mapsto [X(q) - X(p)]$, for $p \neq q$, and on the diagonal by $(p, p) \mapsto [D_\omega X(p)]$ (with D_ω the directional derivative). The desired map is given by

$$\tilde{S} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{P}^{n-1} \to \mathbb{P}^m, \quad (p, \lambda, \omega) \mapsto [\lambda^{-1}(X(p + \lambda \cdot \omega) - X(p))].$$

On the diagonal we have $\tilde{S}(p, 0, \omega) = [D_\omega X(p)]$, and outside the diagonal we obtain $\tilde{S}(p, \lambda, \omega) = [X(q) - X(p)]$, for $\lambda \neq 0$. (For an immersion X and some neighborhood of $\lambda = 0$, the vector $\lambda^{-1}(X(p+\lambda\cdot\omega) - X(p))$ is always non-zero. In Proposition 2.4, which describes the off-diagonal behaviour of the secant map, we assume that X is an embedding, so that $\tilde{S}$ is defined for all λ.)

Next, we want to consider the germ of this secant map at a point of the diagonal. Writing an immersion germ as

$$X(p) = (p, g_{n+1}(p), \dots, g_{m+1}(p)), \quad g_i \in \mathcal{M}_n^2,$$

and by taking an affine chart such that $(1, v) = (1, v_2, \dots, v_n)$, $v_i = \omega_i/\omega_1$ we get

$$\tilde{S} = [1, v_2, \dots, v_n, \lambda^{-1}(g_{n+1}(q) - g_{n+1}(p)), \dots, \lambda^{-1}(g_{m+1}(q) - g_{m+1}(p))].$$

Composing with the ("symmetry restoring") linear right coordinate change

$$L(\bar{p}_1, \bar{p}_2, \dots, \bar{p}_n, \lambda, v_2, \dots, v_n) =$$

$$\big((\bar{p}_1 - \lambda)/2, (\bar{p}_2 - \lambda v_2)/2, \dots, (\bar{p}_n - \lambda v_n)/2, \lambda, v_2, \dots, v_n\big)$$

and omitting the first (constant) component of $\tilde{S}$, we get a map

$$\hat{S} : \mathbb{R}^{2n} \to \mathbb{R}^m, \quad (\bar{p}, \lambda, v) \mapsto \hat{S}(\bar{p}, \lambda, v).$$

The germ of $\hat{S}$ at 0 is the germ of the projectivized secant map at the point $(p, q) = (0, 0) \in (\mathbb{R}^n)^2$ on the source-diagonal and the direction $\omega = [1 : 0 \dots : 0]$. We can (and will) always consider a neighborhood of this direction (in which $\omega_1 \neq 0$, where $\omega = [\omega_1 : \dots : \omega_n]$) by applying an element of $SO(n-1)$ to the tangent space of the immersion. We have the following easy

LEMMA 2.1. *The germ of the projectivized secant map $\hat{S}: \mathbb{R}^{2n} \to \mathbb{P}^m$ at a point on the source-diagonal is even.*

Proof. We claim that $\hat{S}(\bar{p}, \lambda, v) = \hat{S}(\bar{p}, -\lambda, v)$, so that, by Whitney's lemma on even map-germs [24], $\hat{S} = f(\bar{p}, \lambda^2, v)$ for some smooth germ f. The term $c_\alpha p^\alpha$ in g_l (where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index) corresponds in $\tilde{S}$ to

$$\lambda^{-1} c_\alpha \big((p_1 + \lambda)^{\alpha_1} \prod_{i>1} (p_i + \lambda v_i)^{\alpha_i} - p^\alpha\big) =: \lambda^{-1} c_\alpha Q,$$

where Q is divisible by λ. Now

$$Q \circ L = 2^{-|\alpha|} \big((\bar{p}_1 + \lambda)^{\alpha_1} \prod_{i>1} (\bar{p}_i + \lambda v_i)^{\alpha_i} - (\bar{p}_1 - \lambda)^{\alpha_1} \prod_{i>1} (\bar{p}_i - \lambda v_i)^{\alpha_i}\big)$$

is odd in λ, hence $\lambda^{-1} c_\alpha Q \circ L$ and therefore $\hat{S}$ are even as claimed. □

Hence the $\mathbb{Z}_2$ symmetric secant map $\hat{S}$ is the composition of a map-germ $f: \mathbb{R}^{2n}, 0 \to \mathbb{R}^m, 0$, $(\bar{p}, u, v) \mapsto f(\bar{p}, u, v)$ with a folding map $(\bar{p}, \lambda, v) \mapsto (\bar{p}, \lambda^2, v)$. It is well-known that the $\mathcal{A}^{\mathbb{Z}_2}$-classification of $\mathbb{Z}_2$ symmetric germs corresponds to the $\mathcal{A}(H)$-classification of such germs f (see [1, 6]), where $\mathcal{A}(H)$ denotes the geometric subgroup of $\mathcal{A}$ in which the diffeomorphisms in the source preserve the hyperplane $H := \{u = 0\}$ in the source. If $\mathcal{R}(H)$ is the subgroup of $\mathcal{R}$ of elements preserving H, we set $\mathcal{A}(H) = \mathcal{L} \times \mathcal{R}(H)$ and $\mathcal{K}(H) = \mathcal{C} \cdot \mathcal{R}(H)$ (semi-direct product). For $\hat{S}$ and f as above, we set

$$\operatorname{cod}(\mathcal{A}^{\mathbb{Z}_2}, \hat{S}) := \operatorname{cod}(\mathcal{A}(H), f),$$

and similarly for the corresponding extended groups $\mathcal{A}_e^{\mathbb{Z}_2}$ and $\mathcal{A}(H)_e$ of non-origin preserving diffeomorphisms. The germ $\hat{S}$ is (infinitesimally) $\mathbb{Z}_2$ stable if $\operatorname{cod}(\mathcal{A}_e^{\mathbb{Z}_2}, \hat{S}) := \operatorname{cod}(\mathcal{A}(H)_e, f) = 0$, and the $\mathcal{A}^{\mathbb{Z}_2}$-codimension of a $\mathbb{Z}_2$ stable germ from $\mathbb{R}^{2n}$ to $\mathbb{R}^m$ is at most $2n - 1$ (see Lemma 4.1).

Working with the group of $\mathcal{A}(H)$ equivalences of map-germs f — rather than with $\mathcal{A}^{\mathbb{Z}_2}$ equivalence of equivariant secant-germs $\hat{S}$ — not only has technical advantages in the classification but also in transversality arguments relating submanifolds of jet spaces $J^k(2n, m)$ of k-jets of maps f to submanifolds in multi-jet spaces of jets of pairs of immersions X (and the results obtained for f then give the desired results for secants maps $\hat{S}$ by composing f with the above folding map).

The next result shows that, for generic codimension ≥ 2 immersions, the germ of $\hat{S}$ at a point of the diagonal is a $\mathbb{Z}_2$ stable germ (provided the $\mathbb{Z}_2$ stable germs are dense for the relevant pairs of dimensions $(2n, p)$; for other pairs of dimension this statement holds with C^0-$\mathbb{Z}_2$ stable in place of $\mathbb{Z}_2$ stable).

PROPOSITION 2.2. *Let $(2n, m)$ be a pair of dimensions for which the $\mathcal{K}(H)$-orbits of germs $f: \mathbb{R}^{2n}, 0 \to \mathbb{R}^m, 0$ of rank at least n and $\mathcal{K}(H)$-codimension at most $2n - 1$ are $\mathcal{K}(H)$-simple. Then, for a residual subset of* $\operatorname{Imm}(\mathbb{R}^n, \mathbb{R}^{m+1})$, *where $m > n$, the germ of the secant map $\hat{S}: \mathbb{R}^{2n}, (p, p) \to \mathbb{R}^m, \hat{S}(p, p)$ at the diagonal in the source is a $\mathbb{Z}_2$ stable germ of $\mathcal{A}^{\mathbb{Z}_2}$-codimension at most $2n - 1$ and rank at least $n - 1$.*

Proof. We will show that, for $m > n$, the restriction to the blow-up of the diagonal, Γ_k, of the map

$$(J^{2k+1}(n, m+1))^2 \to J^k(2n, m)$$

sending $(2k+1)$-jets of pairs of immersions germs $j^{2k+1}(X(p), X(q))$ to the k-jets of the associated maps f (whose composition with the folding map gives $\hat{S}$) is transverse to the closure of $j^k\mathcal{A}(H)$-orbits of codimension no greater than $2n-1$. Taking coordinates p and $q = p + \lambda \cdot v$ and restricting to the diagonal $\lambda = 0$, we have local coordinates $(p, v) \in \mathbb{R}^{2n-1}$ on the (blow-up of the) diagonal, and $\Gamma_k : J^{2k+1}(n, m+1) \times \mathbb{R}^{n-1} \to J^k(2n, m)$ maps pairs $(c_\alpha p^\alpha, v)$, with $c_\alpha p^\alpha$ a monomial of some component function g_l of X, to terms $\lambda^{-1} c_\alpha Q \circ L(\bar{p}, u^{1/2}, v)$ of the corresponding component function of $f(\bar{p}, u, v)$ (recall that $\lambda^{-1} Q \circ L$, defined in the proof of Lemma 2.1, is even in λ). The Γ_k-preimages of the closures of these $j^k\mathcal{A}(H)$-orbits are therefore Whitney stratified subsets of the same codimension or are empty. The image of $(j^{2k+1}X, v)(\mathbb{R}^{2n-1})$ will therefore, for a residual set of immersions X, miss the Γ_k-preimages of orbits of $\mathcal{A}(H)$-codimension $2n$ or greater.

Claim 1. Suppose f $\mathcal{A}(H)$-stable, then $\mathcal{A}(H) \cdot f \subset \mathcal{K}(H) \cdot f$ is open.

This follows from the following analogue of a result of Mather [16]: suppose f and g are $\mathcal{A}(H)$ stable, then they are $\mathcal{A}(H)$-equivalent if and only if they are $\mathcal{K}(H)$-equivalent. In order to prove this, one can adapt a result of Martinet [15] on the relation between $\mathcal{K}_e$-versality and $\mathcal{A}$-stability to the subgroups $\mathcal{K}(H)_e$ and $\mathcal{A}(H)$. (A brief summary of Martinet's result is given on p. 502 of Wall's survey [23]: one considers a "regular unfolding" $F : \mathbb{R}^d \times \mathbb{R}^{2n} \to \mathbb{R}^d \times \mathbb{R}^m$ of f, i.e. one which is transverse to $\mathbb{R}^d \times \{0\}$, then $V_F := F^{-1}(\mathbb{R}^d \times \{0\})$ is a smooth submanifold of $\mathbb{R}^d \times \mathbb{R}^{2n}$. Let $\pi_F : V_F \to \mathbb{R}^d$ denote the restriction of the obvious projection onto the first factor. Checking that one can replace the group $\mathcal{R}$ by $\mathcal{R}(H)$ we get the following variant of Martinet's result, (3.5) in [23]: F is $\mathcal{K}(H)_e$-versal if and only if π_F is $\mathcal{A}(H)$-stable. And one deduces the above variant of Mather's result by taking regular unfoldings $F(y, x) = (y, f(x) - y)$ of $x \mapsto f(x) = y$ and (similarly) G of g – notice that $\pi_F = f$, because $x \mapsto (f(x), x)$ is a parametrization of V_F.)

For $\mathcal{A}(H)$-stable germs f, transversality to the $\mathcal{K}(H)$-orbit of f therefore implies transversality to the $\mathcal{A}(H)$-orbit.

Claim 2. For $m > n$ the maps Γ_k are transverse to the $\mathcal{K}^k(H)$-orbits of codimension at most $2n-1$ and rank at least n.

The map-germ f is given by

$$f = (v_2, \ldots, v_n, G_{n+1}(\bar{p}, u, v), \ldots, G_{m+1}(\bar{p}, u, v)),$$

where the k-jets of the G_j are the k-jets of functions

$$\sum_{|\alpha| \le 2k+1} \lambda^{-1} c_\alpha Q \circ L(\bar{p}, u^{1/2}, v)$$

(recall the definition of the map Γ_k). The map Γ_1 is a submersion and the maps $\mathbb{R}^{2n} \to \mathbb{R}^m$ of rank less than n form a submanifold B of $J^1(2n, m)$ of codimension greater than or equal to $(m - n + 1)(n + 1) > 2n - 1$, for $m > n$. The set $\Gamma_1^{-1}(B)$ therefore has empty intersection with j^3X for a residual set of immersions X. Now (avoiding the bad set B of germs of rank less than n)

we can choose coordinates $p_1, \dots, p_n$ in the source of X and an affine chart $(1, v_2, \dots, v_n)$ for $\mathbb{P}^n$ such that either

$$f = (v_2, \dots, v_n, \bar{p}_1, G_{n+2}(\bar{p}, u, v), \dots, G_{m+1}(\bar{p}, u, v)),$$

or

$$f = (v_2, \dots, v_n, u + G_{n+1}(\bar{p}, 0, 0), G_{n+2}(\bar{p}, u, v), \dots, G_{m+1}(\bar{p}, u, v))$$

(in the former case $\hat{S}$ has rank at least n and in the latter case at least $n-1$). Now consider in both cases the restriction f' of f to the subspace of the source given by the vanishing of the first n variables, and let Γ'_k denote the corresponding restriction of Γ_k. Again using the functions $\lambda^{-1} c_\alpha Q \circ L$ above we see that Γ'_k is a submersion, and hence transverse to $\mathcal{K}(H) \cdot f'$, and therefore to $\mathcal{K}(H) \cdot f$ (f being the suspension of f'). □

Next, the following relation between secant maps $\hat{S}$ and n-parameter families of projections of $X(\mathbb{R}^n) \subset \mathbb{R}^{m+1}$ from centers in $X(\mathbb{R}^n)$ – so called inner projections in the terminology of [3] – will be useful. Recall that the map $\beta(p, \lambda, \omega) = (p, p + \lambda \cdot \omega)$ blows up the diagonal in $(\mathbb{R}^n)^2$, which has codimension n, to the hyperplane $\lambda = 0$. Its composition $\tilde{\beta} := \beta \circ L$ with the symmetry restoring linear right coordinate change L is given by

$$(\bar{p}, \lambda, \omega) \mapsto ((\bar{p} - \lambda \cdot \omega)/2, (\bar{p} + \lambda \cdot \omega)/2) =: (p, q).$$

If π_c denotes the projection from a center $c \in \mathbb{R}^{m+1}$, then the projectivized secant map factors as follows:

$$\begin{array}{ccccc} & & \mathbb{R}^n \times \mathbb{R} \times \mathbb{P}^{n-1} & & \\ & \tilde{\beta} \swarrow & & \searrow \hat{S} & \\ \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} & \longrightarrow & \mathbb{P}^m \\ (p, q) & \longmapsto & (X(p), X(q)) & \longmapsto & \pi_{X(p)}(X(q)) \end{array}$$

Outside the diagonal, where $p \neq q$, one can use this observation to show that the germ of $\hat{S}$ is locally a versal n-parameter deformation of a germ $\mathbb{R}^n \to \mathbb{R}^m$ for a residual set of immersions X, see Proposition 2.4 below.

On the diagonal, the family inner projections is only defined after blowing-up the source diagonal and dividing by λ. The following remark implies that the germ of the secant-map $\hat{S}$ at $(p, \lambda, v) = (p, 0, 0)$ determines the projection $\pi_1(X)$ of the germ of the immersion X at p into a hyperplane in $\mathbb{R}^{m+1}$ orthogonal to $e_1 = dX_p(e'_1) \in \mathbb{R}^{m+1}$, with $e'_1 = (1, 0, \dots, 0) \in T_p\mathbb{R}^n$ (notice that $e'_1 = (1, v)$ for $v = (v_2, \dots, v_n) = 0$). In Section 3 the $\mathcal{A}^{\mathbb{Z}_2}$-classification of secant-maps $\hat{S}$ and the related $\mathcal{A}$-classification of the associated orthogonal projections $\pi_1(X)$ will be described, the remark is also useful in relating $\mathcal{A}^{\mathbb{Z}_2}$-orbit membership conditions to transversality conditions on the immersion. Setting

$$\gamma(p_1, \dots, p_n) = (p_1, \dots, p_n, p_1, p_2/p_1, \dots, p_n/p_1),$$

we have the following.

REMARK 2.3. Let $\hat{S}$ be a germ of the secant map of an immersion $p \mapsto X(p)$ at $(\bar{p}, \lambda, v) = (0,0,0)$, then $\pi_1 \circ X(p) = p_1 \cdot \hat{S}(\gamma(p)))$.

Proof. Let $X(p) = (p, g_{n+1}(p), \dots, g_{m+1}(p))$ be in Monge form. From the diagram above we have

$$\hat{S} = [1, v_2, \dots, v_n, h_{n+1}, \dots, h_{m+1}],$$

where

$$h_i := \lambda^{-1}(g_i((\bar{p} + \lambda \cdot \omega)/2) - g_i((\bar{p} - \lambda \cdot \omega)/2))$$

and $\omega = (1, v_2, \dots, v_n)$. Composing with γ and multiplying through with p_1 gives the desired formula (notice that $g_i(0) = 0$). □

Next, consider the off-diagonal behaviour of the secant map. Notice that we can exchange the roles of $X(p)$ and $X(q)$ in the projection $\pi_{X(p)}(X(q))$, this global $\mathbb{Z}_2$-symmetry makes the off-diagonal part of the secant map highly unstable as a bi-germ. But considering $\hat{S}$ as a mono-germ we have the following.

PROPOSITION 2.4. *The germ of the secant map*

$$\hat{S} : \mathbb{R}^{2n}, (p_0, q_0) \to \mathbb{R}^m, \hat{S}(p_0, q_0)$$

at any pair of points $p_0 \neq q_0$ is a versal n-parameter deformation of a germ $\mathbb{R}^n$ to $\mathbb{R}^m$ for a residual subset of $\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^{m+1})$.

Proof. Consider disjoint neighborhoods U and V of p_0 and q_0, respectively. The n-parameter family of projections $\pi_{X(U)} : U \times \mathbb{R}^{m+1} \to U \times \mathbb{R}^m$, $(p, r) \to \pi_{X(p)}(r)$ is a family of submersions, and hence $\mathcal{A}$-versal. By a transversality theorem for composite maps [19, 10] and a partition of unity argument, we can approximate the embedding germ $X(V)$ in a neighborhood of $X(q_0)$ by an embedding germ $Y(V)$, without changing the embedding germ $X(U)$ near $X(p_0)$, such that the composite family $\pi_{X(U)} : U \times V \to U \times \mathbb{R}^m$, $(p, q) \to \pi_{X(p)}(Y(q))$ is an $\mathcal{A}$-versal n-parameter deformation of central projections of the embedding germ $Y(V) \subset \mathbb{R}^{m+1}$. □

We conclude this section by considering parametrizations of the secant variety of an immersion $X : \mathbb{R}^n \to \mathbb{R}^{n+c}$, $X(p) = (p, g_{n+1}(p), \dots, g_{n+c}(p))$, $g_i \in \mathcal{M}_n^2$. The map

$$F : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+c}, \quad (p, q, t) \mapsto 1/2(X(q) + X(p)) + t(X(q) - X(p))$$

has the symmetry $F(q, p, -t) = F(p, q, t)$. We can again extend F to the "diagonal" $p = q$ by taking $q = p + \lambda(1, v_2, \dots, v_n)$ and by replacing $X(q) - X(p)$ by $\lambda^{-1}(X(q) - X(p))$. By composing again on the right with the symmetry restoring linear coordinate change L we obtain a map $\tilde{F}(\bar{p}, \lambda, v, t)$. And setting $w = (1, v) = (1, v_2, \dots, v_n)$, we see that the first term $1/2(X((\bar{p} + \lambda w)/2) + X((\bar{p} - \lambda w)/2)$ in $\tilde{F}$ is even in λ, and the second term is even by Lemma 2.1. Setting $u = \lambda^2$, the least degenerate $j^1\tilde{F}$ is $j^1\mathcal{A}(H)$-equivalent to $h := (t, \bar{p}_2, \dots, \bar{p}_n, u, 0, \dots, 0)$ (if the x^2 coefficients of all the g_i components of X vanish then we obtain a more degenerate 1-jet). The $j^1\mathcal{A}(H)$-codimension of h is cn, and Lemma 4.1 below implies that the $\mathcal{A}(H)$-stable germs $\mathbb{R}^{2n+1} \to \mathbb{R}^{n+c}$ have $\mathcal{A}(H)$-codimension at most $2n$. Hence $\tilde{F}$ is $\mathcal{A}(H)$-unstable, and hence $\mathcal{A}^{\mathbb{Z}_2}$-unstable, for any $c > 2$, and we obtain

PROPOSITION 2.5. *The germ at the diagonal, $\tilde{F} : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+c}$, of the parametrization of the secant variety of any immersion $X : \mathbb{R}^n \to \mathbb{R}^{n+c}$ of codimension $c > 2$ is $\mathcal{A}^{\mathbb{Z}_2}$-unstable.*

REMARK. The restriction of the secant variety to the diagonal is the tangent variety, and the latter has already for the simplest case of space-curves in $\mathbb{R}^3$ non-isolated singularities of infinite $\mathcal{A}$-codimension. The map $\tilde{F}$ might therefore be more degenerate than just $\mathcal{A}^{\mathbb{Z}_2}$-unstable. (In fact, one checks, that for curves in 3-space the map $\tilde{F}$ has infinite $\mathcal{A}^{\mathbb{Z}_2}$-codimension.)

3 Secant maps of generically immersed surfaces in $\mathbb{R}^n$, $n \geq 3$

In this section the germs of the secant map at a point in the source diagonal will be classified for generically immersed surfaces in $\mathbb{R}^n$, $n \geq 3$. Locally such an immersion is given by an embedding-germ

$$X(x,y) = (x, y, \sum_{i+j>1} a_{ij}^{(3)} x^i y^j, \dots, \sum_{i+j>1} a_{ij}^{(n)} x^i y^j)$$

at $(x,y) = (0,0)$. The projectivized secant map (in the affine chart $(1,v)$ for ω, see Section 2) is given by $\tilde{S} = [1 : v : S_3 : \dots : S_n]$, hence we have a map-germ into $(n-1)$-space $\hat{S}(\bar{x}, \bar{y}, \lambda, v) := (v, S_3, \dots, S_n)$, where $\bar{x}$, $\bar{y}$ correspond to the coordinates $\bar{p}_1$, $\bar{p}_2$ in Section 2 (after applying the coordinate change L that restores the $\mathbb{Z}_2$-symmetry of $\hat{S}$). Using the map Γ_k (defined in the proof of Proposition 2.2), one obtains $S_i = a_{20}^{(i)}\bar{x} + a_{11}^{(i)}(\bar{y} + v\bar{x})/2 + a_{02}^{(i)} v\bar{y} + a_{30}^{(i)}(u + 3\bar{x}^2)/4 + \dots$, with $u = \lambda^2$.

Let $\pi_x \circ X$ be the projection-germ at $x = y = 0$ of the immersed surface $X(\mathbb{R}^2)$ along the x-direction. By Remark 2.3, the projection-germ $\pi_x \circ X$ is determined by the secant map-germ $\hat{S}$ at $\bar{x} = \bar{y} = \lambda = v = 0$ (the direction $(1,v) = (1,0)$ in $T_{X(0,0)}X(\mathbb{R}^2)$ corresponds to the x-direction), but $\mathcal{A}^{\mathbb{Z}_2}$-equivalence of $\hat{S}$ does in general not preserve the $\mathcal{A}$-class of $\pi_x \circ X$. In the classification of secant-germs of generically immersed surfaces we therefore decompose certain $\mathcal{A}^{\mathbb{Z}_2}$-orbits of secant-maps into strata on which the $\mathcal{A}$-type of the corresponding projection $\pi_x \circ X$ is invariant. In the normal forms $\hat{S}$ certain monomials are therefore present which affect the $\mathcal{A}$-type of $\pi_x \circ X$ but which could be removed by an $\mathcal{A}^{\mathbb{Z}_2}$-change. On the other hand, redundant terms in the corresponding projection-germs $\pi_x \circ X$ have been eliminated (by suitable $\mathcal{A}$-changes). (Also recall that if $\mathrm{cod}(\mathcal{A}^{\mathbb{Z}_2}, \hat{S})$ denotes the $\mathcal{A}^{\mathbb{Z}_2}$-codimension of $\hat{S}$ in the space of $\mathbb{Z}_2$-symmetric germs and $\hat{S} = f \circ (\bar{x}, \bar{y}, \lambda^2, v)$ then $\mathrm{cod}(\mathcal{A}^{\mathbb{Z}_2}, \hat{S}) = \mathrm{cod}(\mathcal{A}(H), f)$.) The main result of the present section is then the following.

THEOREM 3.1. *For a residual subset of* $\mathrm{Imm}(\mathbb{R}^2, \mathbb{R}^n)$, $n \geq 3$ *the germ of the secant map $\hat{S} : \mathbb{R}^4 \to \mathbb{R}^{n-1}$ at a point on the diagonal is equivalent to one of the*

$\mathbb{Z}_2$ stable germs in Table 1. The projection-germs $\pi_x \circ X$ associated with the $\hat{S}$ are also listed (up to $\mathcal{A}$-equivalence).

Table 1: Generic secant germs for immersions in $\mathbb{R}^n$

n	$\hat{S}(\bar{x},\bar{y},\lambda,v) \sim_{\mathcal{A}^{\mathbb{Z}_2}}$	$\mathcal{A}^{\mathbb{Z}_2}$-cod	$\pi_x \circ X \sim_{\mathcal{A}}$	$\mathcal{A}$-cod
3	$(v,\bar{x})$	0	(y,x^2)	1
	$\sim_{\mathcal{A}^{\mathbb{Z}_2}} (v,\bar{y}+\bar{x}^2)$		$(y,xy+x^3)$	2
	$\sim_{\mathcal{A}^{\mathbb{Z}_2}} (v,\bar{y}+\bar{x}^3)$		$(y,xy+x^4)$	3
	$\sim_{\mathcal{A}^{\mathbb{Z}_2}} (v,\bar{y}+\bar{x}^4 \pm \bar{x}^6)$		$(y,xy+x^5 \pm x^7)$	4
	$(v,\lambda^2+\bar{x}^2 \pm \bar{y}^2)$	2	$(y,x^3 \pm xy^2)$	3
	$(v,\lambda^2+v\bar{y}+\bar{x}^2+\bar{y}^3)$	3	(y,x^3+xy^3)	4
	$(v,\lambda^4+v\lambda^2+\bar{x}^3+\bar{x}\bar{y})$	3	$(y,x^2y+x^4+x^5)$	4
4	$(v,\bar{x},\bar{y})$	0	(y,x^2,xy)	2
	$s^{\pm}=(v,\bar{x},\pm\lambda^2+\bar{y}^2)$	1	$(y,x^2,x^3 \pm xy^2)$	3
	$s^{+} \sim_{\mathcal{A}^{\mathbb{Z}_2}} (v,\bar{y}+\bar{x}^4,\lambda^2+\bar{x}^2)$		$(y,xy+x^5,x^3)$	4
	$\sim_{\mathcal{A}^{\mathbb{Z}_2}} (v,\bar{y}+\bar{x}^7,\lambda^2+\bar{x}^2)$		$(y,xy+x^8,x^3)$	5
	$(v,\bar{x},\lambda^2+v\bar{y}+\bar{y}^3)$	2	(y,x^2,x^3+xy^3)	4
	$(v,\bar{x},\lambda^4+v\lambda^2 \pm \bar{y}^2)$	2	$(y,x^2,x^5 \pm xy^2)$	4
	$(v,\bar{x},\lambda^2+v\bar{y}+\bar{x}\bar{y}^2 \pm \bar{y}^4)$	3	$(y,x^2,x^3 \pm xy^4)$	5
	$(v,\bar{x},v\bar{y}+\bar{x}\bar{y}^2+\bar{y}^3 \pm \bar{y}\lambda^2)$	3	$(y,x^2,xy^3 \pm x^3y)$	5
	$(v,\bar{x},\bar{y}^2+v\lambda^2 \pm \lambda^6+\bar{x}\lambda^4)$	3	$(y,x^2,xy^2 \pm x^7)$	5
5	$(v,\bar{x},\bar{y},\lambda^2)$	0	(y,x^2,xy,x^3)	3
	$(v,\bar{x},\bar{y},\lambda^4+v\lambda^2)$	1	(y,x^2,xy,x^5)	4
	$(v,\bar{x},\bar{y},\lambda^6+v\lambda^4+\bar{x}\lambda^2)$	2	(y,x^2,xy,x^7)	5
	$(v,\bar{x},\lambda^2+v\bar{y},\bar{y}^2)$	2	(y,x^2,x^3,xy^2)	5
	$(v,\bar{x},\bar{y},\lambda^8+v\lambda^6+\bar{x}\lambda^4+\bar{y}\lambda^2)$	3	(y,x^2,xy,x^9)	6
	$(v,\bar{x},\lambda^2 \pm \bar{y}^2,v\bar{y}+\bar{y}^3+\bar{x}\bar{y}^2)$	3	$(y,x^2,x^3 \pm xy^2,xy^3)$	6
6	$(v,\bar{x},\bar{y},\lambda^2,0)$	0	$(y,x^2,xy,x^3,0)$	4
	$(v,\bar{x},\bar{y},\lambda^4+\bar{x}\lambda^2,v\lambda^2)$	2	$(y,x^2,xy,x^5,0)$	6
	$(v,\bar{x},\bar{y}^2,\lambda^2+v\bar{y},\bar{x}\bar{y})$	3	$(y,x^2,xy^2,x^3,0)$	7
7	$(v,\bar{x},\bar{y},\lambda^2,0,0)$	0	$(y,x^2,xy,x^3,0,0)$	5
	$(v,\bar{x},\bar{y},\lambda^4+v\lambda^2,\bar{x}\lambda^2,\bar{y}\lambda^2)$	3	$(y,x^2,xy,x^5,0,0)$	8
≥ 8	$(v,\bar{x},\bar{y},\lambda^2,0,\dots,0)$	0	$(y,x^2,xy,x^3,0,\dots,0)$	$n-2$

REMARK 3.2. (i) Projecting an immersion-germ $X:\mathbb{R}^2,p \to \mathbb{R}^n, X(p)$ along a direction $w \in T_{X(p)}X(\mathbb{R}^2)$ always gives rise to a singular projection-germ of $\mathcal{A}$-codimension at least $n-2$. Furthermore, in a 3-parameter family of such projection-germs, varying with $(p,w) \in \mathbb{R}^2 \times \mathbb{P}^1$, we generically expect that $n-2 \leq \mathrm{cod}(\mathcal{A},\pi_x \circ X) \leq n+1$. Comparing the $\mathcal{A}$-classes of projections $\pi_x \circ X$ in the theorem (which correspond to secant-germs $\hat{S}$ of generic immersions) with the existing classifications of $\mathcal{A}$-orbits of maps $\mathbb{R}^2 \to \mathbb{R}^{n-1}$ we find that for $n \neq 5$ all $\mathcal{A}$-classes $[f]$ with $n-2 \leq \mathrm{cod}(\mathcal{A},f) \leq n+1$ arise as the $\mathcal{A}$-class of a projection $\pi_x \circ X$ associated with some $\hat{S}$: for $n=3$ we get the $\mathcal{A}$-classes 2, 3, 4_k $(k=2,3)$, 5, 6 and 11_5 from [21] and for $n=4$ we get S_k $(k=0,\dots,4)$, B_k $(k=2,3)$, C_3 and H_k $(k=2,3)$ from [18]. Finally, one checks that in higher dimensions $n \geq 6$ this is also true (the normal forms for $\pi_x \circ X$ in the table represent the only $\mathcal{A}$-orbits of $\mathcal{A}$-codimension between $n-2$ and $n+1$). For $n=5$ we get I_k $(k=1,\dots,4)$, II_2, and $\mathrm{III}_{2,3}$ from [13], but there is no $\mathbb{Z}_2$-stable

secant germ $\hat{S}$ whose associated projection $\pi_x \circ X$ is $\mathcal{A}$-equivalent to VII_1 (and $\mathrm{cod}(\mathcal{A}, \mathrm{VII}_1) = 6$, see [13]).

(ii) For $n = 3$ there are $\mathcal{A}^{\mathbb{Z}_2}$-stable germs that cannot be equivalent to any germ of a secant-map $\hat{S}$. The $\mathcal{A}^{\mathbb{Z}_2}$-stable germs $(v, \lambda^2 - \bar{x}^2 - \bar{y}^2)$, $(v, \lambda^2 + v\bar{y} - \bar{x}^2 + \bar{y}^3)$ and $(v, \lambda^4 + v\lambda^2 + \epsilon\bar{x}^2 + \epsilon\bar{y}^2)$ $(\epsilon = \pm 1)$ which, over $\mathbb{C}$, are $\mathcal{A}^{\mathbb{Z}_2}$-equivalent to the representatives $\hat{S}$ of the third, fourth and fifth $\mathcal{A}^{\mathbb{Z}_2}$-orbit in the table, respectively, are not equivalent to the germ of any $\hat{S}$. The reason in all three cases is that an ax^3 term in the last component of the immersion-germ X yields an $a(\lambda^2 + 3\bar{x}^2)/4$ term in the last component of $\hat{S}$.

Proof of Theorem 3.1. We consider map-germs $f(\bar{x}, \bar{y}, u, v)$ (up to $\mathcal{A}(H)$-equivalence) whose composition with $(\bar{x}, \bar{y}, u, v) \mapsto (\bar{x}, \bar{y}, \lambda^2, v)$ yield $\mathbb{Z}_2$-symmetric map-germs. Theorem 4.2 in Section 4 contains the classification of $\mathcal{A}(H)$-stable germs $f : \mathbb{R}^4, 0 \to \mathbb{R}^{n-1}, 0$, $n \geq 3$. Proposition 2.2 implies that for a residual set of immersion-germs $X : \mathbb{R}^2 \to \mathbb{R}^n$, $n \geq 4$, the associated germs f are $\mathcal{A}(H)$-stable. Furthermore, for $n = 3$ (codimension-1 immersions) the proof of Proposition 2.2 implies that for a residual set of immersions f is an $\mathcal{A}(H)$-stable germ of rank 2 or a germ of lower rank. From the classification in Theorem 4.2 we have that for $n - 1 = 2$ there are three real stable $\mathcal{A}(H)$-orbits of rank 1 with representatives $f_i = (v, u^2 + vu + \epsilon_1\bar{x}^2 + \epsilon_2\bar{y}^2)$, where $(\epsilon_1, \epsilon_2) = (+1, +1)$ for $i = 1$, $(-1, -1)$ for $i = 2$ and $(+1, -1)$ for $i = 3$, and all germs of rank less than two lie in the closure of these orbits. Now one checks that the jet-map Γ_2 (defined in the proof of Proposition 2.2) has empty intersection with the closures of $j^2\mathcal{A}(H) \cdot f_i$, $i = 1, 2$ and that $\Gamma_2^{-1}(A)$, where A is the closure of $j^2\mathcal{A}(H) \cdot f_3$, is a submanifold of $J^3(2,3) \times \mathbb{R}$ defined by $a_{20}^{(3)} = a_{11}^{(3)} = a_{30}^{(3)} = 0$ (notice that $f_3 \sim_{\mathcal{A}(H)} (v, u^2 + vu + \bar{x}\bar{y})$). Furthermore, the Γ_2-preimages of the orbits of $j^2\mathcal{A}(H)$-codimension greater than three in the closure of $j^2\mathcal{A}(H) \cdot f_3$ have codimension greater than three (or are empty). Hence we conclude that in all cases (i.e. the ones covered by Proposition 2.2 – or its proof – and also for $n = 3$ and rank ≤ 1) we obtain $\mathcal{A}(H)$-stable map-germs f (and hence $\mathcal{A}^{\mathbb{Z}_2}$-stable secant maps $\hat{S}$) for a residual set of immersion-germs.

Finally, we have to consider the following two points: (i) we want to see which $\mathcal{A}(H)$-stable germ $\mathbb{R}^4 \to \mathbb{R}^{n-1}$ can be realized as a germ f (or $\hat{S}$) of an immersion and (ii) we want to decompose the $\mathcal{A}(H)$-orbits of f into $\mathcal{A}$-invariant strata of $\pi_x \circ X$. For (i) we consider the list of $\mathcal{A}(H)$-stable map-germs f in Theorem 4.2 and check whether the Γ_k-preimage of a k-sufficient orbit $\mathcal{A}(H) \cdot f$ is non-empty. In the normal forms of Theorem 4.2 the variables x_1 and u always correspond to v and u in f (or to v and λ^2 in $\hat{S}$). One finds that for $n \geq 4$ all these Γ_k-preimages are non-empty, and for $n = 3$ the Γ_k-preimages are empty in the cases mentioned in Remark 3.2 (ii).

For (ii) we simultaneously consider two jet-maps: the map $\Gamma_k : J^{2k+1}(2,n) \times \mathbb{R}^1 \to J^k(4, n-1)$, sending $(j^{2k+1}X, v)$ to $j^k f$, and a second map $\tilde{\Gamma}_{2k+1} : J^{2k+1}(2,n) \times \mathbb{R}^1 \to J^{2k+1}(2, n-1)$, sending $(j^{2k+1}X, v)$ to $j^{2k+1}(\pi_{(1,v)} \circ X)$. Here $\pi_{(1,v)}$ denotes the projection onto a hyperplane in $\mathbb{R}^n$ orthogonal to the $(1, v)$ in the tangent plane $T_{X(p)}X(\mathbb{R}^2)$. (Recall that we actually fix the direction $(1, v) = (1, 0)$, the x-direction, in the tangent plane of $X(x, y) = (x, y, g_3, \ldots, g_n)$ at $(0, 0)$ and instead rotate the surface about an axis orthogonal to the (x, y)-plane, but the transversality arguments are perhaps clearer if we consider varying directions $(1, v)$.) Using the correspondence of pairs consisting of germs of secant maps $\hat{S} = f(\bar{p}, \lambda^2, v)$ at $(\bar{p}, v)$ and projection germs $\pi_{(1,v)} \circ X$ from Remark 2.3, we obtain submanifolds

$$\Gamma_k^{-1}(j^k\mathcal{A}(H) \cdot f) \cap \tilde{\Gamma}_{2k+1}^{-1}(j^{2k+1}\mathcal{A} \cdot (\pi_{(1,v)} \circ X)) \subset J^{2k+1}(2, n) \times \mathbb{R}^1,$$

and such submanifolds of codimension greater than three will generically have empty intersection with $(j^{2k+1}X, v)(\mathbb{R}^3)$. For the pairs $\hat{S}, \pi_x \circ X$ in Table 1 we find that the codimensions of these submanifolds are given by the sum of the $\mathcal{A}^{\mathbb{Z}_2}$-codimension of $\hat{S}$ and the difference of the $\mathcal{A}$-codimensions of $\pi_x \circ X$ and of the least degenerate projection-germ corresponding to some secant map-germ in the $\mathcal{A}^{\mathbb{Z}_2}$-orbit of $\hat{S}$. □

The projection-germs $\pi_x \circ X$ distinguish all the $\mathcal{A}^{\mathbb{Z}_2}$-orbits of secant map-germs in the classification above. In some cases there is also a relation between $\hat{S}$ and curvature properties of $X(\mathbb{R}^2)$.

REMARK 3.3. For $n = 3$ the last three $\mathcal{A}^{\mathbb{Z}_2}$-orbits of secant map-germs $\hat{S}$ occur at parabolic points of the surface $X(\mathbb{R}^2)$ and the direction $(1, v) = (1, 0)$ is an asymptotic direction. The last orbit occurs, in fact, at points of tangency between the parabolic and the flecnodal curve (such points of tangency are called godron- or gutter-points and correspond to cusps of the Gauss map, see e.g. [12]). For the first $\mathcal{A}^{\mathbb{Z}_2}$-orbit there is no restriction on the Gaussian curvature (here the degenerations of the corresponding projection-germs occur for asymptotic directions $(1, v) = (1, 0)$ at a hyperbolic point).

For $n = 4$ the $\mathcal{A}^{\mathbb{Z}_2}$-type of $\hat{S}$ imposes no restriction on the second fundamental form of $X(\mathbb{R}^2)$. Only for the degenerations of the projection-germs corresponding to the secant map-germ s^+ we obtain such a restriction. These occur at parabolic points p that are not inflection points (the invariant Δ introduced by Little [14] vanishes at p, but the 3×2 matrix α_X of the second fundamental form has rank 2) and such p have negative Gaussian curvature.

For $n = 5$, consider the 3×3 matrix α_X of the second fundamental form of a surface $X(\mathbb{R}^2)$ and the sets $M_i := \{p : \text{rank}\alpha_X(p) = i\}$. Mochida *et al.* [17] have shown that generic surfaces consist of regions of M_3 points and curves of M_2 points, but the $\mathcal{A}^{\mathbb{Z}_2}$-types of $\hat{S}$ in our classification can occur at M_3 as well as at M_2 points.

4 Classification of $\mathbb{Z}_2$ stable map-germs $\mathbb{R}^4 \to \mathbb{R}^n$

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth $\mathbb{Z}_2$-equivariant germ. We wish to classify the $\mathcal{A}^{\mathbb{Z}_2}$-stable germs, where $\mathcal{A}^{\mathbb{Z}_2} = \mathcal{L} \times \mathcal{R}^{\mathbb{Z}_2}$ (i.e. $\mathbb{Z}_2$-equivariant diffeomorphisms in the source and arbitrary diffeomorphisms in the target). By an observation of Arnol'd [1], the $\mathcal{A}^{\mathbb{Z}_2}$ classification of $\mathbb{Z}_2$-symmetric map-germs $f(x_1, \ldots, x_{n-1}, y) = f(x_1, \ldots, x_{n-1}, -y)$ over $\mathbb{C}$ coincides with the $\mathcal{A}(H)$-classification of germs $f(x_1, \ldots, x_{n-1}, u)$, where $\mathcal{A}(H) = \mathcal{L} \times \mathcal{R}(H)$ and where $\mathcal{R}(H)$ is the group of diffeomorphisms preserving the hyperplane $H = \{u = 0\}$ in the source (which are of the form $k = (k_1(x, u), \ldots, k_{n-1}(x, u), u \cdot k_n(x, u))$. Substituting $u = y^2$ into the normal forms of the latter classification gives the desired $\mathbb{Z}_2$-symmetric germs. Over $\mathbb{R}$ the last component of the source diffeomorphism k has to satisfy the extra condition $k_n(0, 0) > 0$ (so that k preserves the set $\{(x_1, \ldots, x_{n-1}, u) : u > 0\}$).

We denote the local rings of smooth source and target functions by C_n and C_p, respectively, and $\mathcal{M}_n$ and $\mathcal{M}_p$ are the corresponding maximal ideals. Let θ_f denote the C_n-module of vector fields over f (i.e. sections of $f^*T\mathbb{R}^p$). Set $\theta_n = \theta(1_{\mathbb{R}^n})$ and $\theta_p = \theta(1_{\mathbb{R}^p})$, and define homomorphisms tf and wf:

$$tf : \theta_n \to \theta_f, \quad tf(\xi) = df \cdot \xi,$$

(where df is the differential of f), and

$$wf : \theta_p \to \theta_f, \quad wf(\phi) = \phi \circ f.$$

The tangent spaces to the groups $\mathcal{G} = \mathcal{A}, \mathcal{K}, \mathcal{R}$ and $\mathcal{C}$ are then defined in the usual way (see [23]). Hence we shall only indicate the required modifications for the subgroup $\mathcal{A}(H)$ of $\mathcal{A}$ (where we have to restrict tf to $\theta_n(H)$, see below).

Apart from $\mathcal{A}(H)$ we need the the group $\mathcal{K}(H) = \mathcal{C} \cdot \mathcal{R}(H)$ (semi-direct product), both groups are so-called geometric subgroups of $\mathcal{A}$ and $\mathcal{K}$ so that the usual unfolding and determinacy results hold (see Damon [9]). For any such group $\mathcal{G}$ we denote, as usual, by $\mathcal{G}_e$ and $\mathcal{G}_1$ the extended pseudo-group (of non-origin preserving diffeomorphisms) and the subgroup of diffeomorphisms with 1-jet the identity. For calculations of complete transversals (see [8]) and of determinacy degrees we use the notation $\mathcal{H}$ for a unipotent subgroup of $\mathcal{A}(H)$ ($\mathcal{H}$ can contain certain elements of $\mathcal{A}(H) \setminus \mathcal{A}(H)_1$). Since $\mathcal{A}(H)$-equivalence is much finer than $\mathcal{A}$-equivalence, we often have to work with bigger unipotent groups $\mathcal{H}$ than $\mathcal{A}(H)_1$, even for stable germs f – frequently one also has to use the whole group $\mathcal{A}(H)$ (which is not unipotent) and apply Mather's lemma [16]. Combining the determinacy results in [7, 20] we get the following useful criterion: if $\mathcal{M}_n^l\theta_f \subset T\mathcal{K}(H)_e \cdot f + \mathcal{M}_n^{l+1}\theta_f$ and $\mathcal{M}_n^{k+1}\theta_f \subset T\mathcal{H} \cdot f + \mathcal{M}_n^{k+l+1}\theta_f$ then f is k-$\mathcal{H}$- and hence k-$\mathcal{A}(H)$-determined. Recall that $T\mathcal{R}(H)_e \cdot f = tf(\theta_n(H))$, where $\theta_n(H)$ is the C_n-module of source vectorfields tangent to H (i.e. $\theta_n(H) \ni \xi = \sum_{i=1}^{n-1} a_i(x,u)\partial/\partial x_i + u \cdot b(x,u)\partial/\partial u$, where $a_i, b \in C_n$). The following relation is also useful in the classifications below.

LEMMA 4.1. *For an $\mathcal{A}(H)$-finite map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we have the following relation between codimensions of orbits:*

$$\mathrm{cod}(\mathcal{A}(H)_e, f) = \max[0, \mathrm{cod}(\mathcal{A}(H), f) - n + 1].$$

Proof (Sketch). The argument is almost the same as that for the analogous formula for ordinary $\mathcal{A}$-equivalence, see for example Proposition 4.5.2 (ii) of [23]. For $\mathcal{A}(H)$-stable germs f the formula holds trivially, hence suppose f unstable. In that case the formula is equivalent to

$$\dim \left(T\mathcal{A}(H)_e \cdot f / T\mathcal{A}(H) \cdot f\right) = n + p - 1,$$

which in turn is equivalent to: if $\alpha \in \theta_n(H)$ and $\beta \in \theta_p$ are such that $tf(\alpha) + wf(\beta) \in T\mathcal{A}(H) \cdot f$ then $\beta \in \mathcal{M}_p \cdot \theta_p$ and $\pi(\alpha) \in \mathcal{M}_n \cdot \theta_{n-1}$, where π is the projection onto the first $n-1$ components (this makes the difference to the usual formula for $\mathcal{A}$). The proof now concludes as in the $\mathcal{A}$-case. □

The classification of $\mathcal{A}(H)$-stable map-germs with source dimension four is given by the following

THEOREM 4.2. *Any $\mathcal{A}(H)$-stable map-germ $f : \mathbb{R}^4, 0 \to \mathbb{R}^n, 0$, where $n \geq 2$, is equivalent to one of the germs in Table 2. (Here $\epsilon_i := \pm 1$ and the cases $(\epsilon_1, \epsilon_2) = (+1, -1), (-1, +1)$ are equivalent.).*

Table 2: $\mathcal{A}(H)$-stable germs $\mathbb{R}^4 \to \mathbb{R}^n$

$n =$	$f(x_1, x_2, x_3, u) =$	$\mathrm{cod}(\mathcal{A}(H), f) =$
2	(x_1, x_2)	0
	$(x_1, u + \epsilon_1 x_2^2 + \epsilon_2 x_3^2)$	2
	$(x_1, u + x_1 x_2 \pm x_3^2 + x_2^3)$	3
	$(x_1, u^2 + x_1 u + \epsilon_1 x_2^2 + \epsilon_2 x_3^2)$	3
3	(x_1, x_2, x_3)	0
	$(x_1, x_2, u \pm x_3^2)$	1
	$(x_1, x_2, u + x_1 x_3 + x_3^3)$	2
	$(x_1, x_2, u^2 + x_1 u \pm x_3^2)$	2
	$(x_1, x_2, u + x_1 x_3 + x_2 x_3^2 \pm x_3^4)$	3
	$(x_1, x_2, x_1 x_3 + x_2 x_3^2 + x_3^3 + x_3 u)$	3
	$(x_1, x_2, x_3^2 + x_1 u \pm u^2)$	3
4	(x_1, x_2, x_3, u)	0
	$(x_1, x_2, x_3, u^2 + x_1 u)$	1
	$(x_1, x_2, x_3, u^3 + x_1 u + x_2 u^2)$	2
	$(x_1, x_2, u + x_1 x_3, x_3^2)$	2
	$(x_1, x_2, x_3, u^4 + x_1 u + x_2 u^2 + x_3 u^3)$	3
	$(x_1, x_2, u \pm x_3^2, x_1 x_3 + x_3^3 + x_2 x_3^2)$	3
5	$(x_1, x_2, x_3, u, 0)$	0
	$(x_1, x_2, x_3, u^2 + x_2 u, x_1 u)$	2
	$(x_1, x_2, x_3^2, u + x_1 x_3, x_2 x_3)$	3
6	$(x_1, x_2, x_3, u, 0, 0)$	0
	$(x_1, x_2, x_3, u^2 + x_1 u, x_2 u, x_3 u)$	3
≥ 7	$(x_1, x_2, x_3, u, 0, \ldots, 0)$	0

Remark on Proof. By Lemma 4.1 one obtains the $\mathcal{A}(H)$-stable germs (of $\mathcal{A}(H)_e$-codimension 0) by classifying the $\mathcal{A}(H)$-orbits of germs $\mathbb{R}^4, 0 \to \mathbb{R}^n, 0$ of $\mathcal{A}(H)$-codimension at most three. The group $\mathcal{A}(H)$ is a geometric subgroup of the group $\mathcal{A}$ (as defined by Damon [9]), the classification techniques are therefore the same as for $\mathcal{A}$. Below we give an outline of this classification for the case $n = 4$, $p = 3$ (indicating the structure of the classification pattern of the $\mathcal{A}(H)$-stable orbits, but omitting all calculations). For the other $p \geq 2$ the classifications are similar (and often less extensive). □

So for $n = 4$, $p = 3$ we have the following classification.

PROPOSITION 4.3. *Any $\mathcal{A}(H)$-stable map-germ $f : \mathbb{R}^4, 0 \to \mathbb{R}^3, 0$ is equivalent to one of the following germs: (x_1, x_2, x_3), $(x_1, x_2, u \pm x_3^2)$, $(x_1, x_2, u + x_1 x_3 + x_3^3)$, $(x_1, x_2, u + x_1 x_3 + x_2 x_3^2 \pm x_3^4)$, $(x_1, x_2, x_3^2 + x_1 u \pm u^2)$, $(x_1, x_2, x_3^2 + x_1 u + u^3 + x_2 u_2)$ and $(x_1, x_2, x_3 u + x_1 x_3 + x_3^3 + x_2 x_3^2)$.*

Outline of Proof. The elements of $\mathcal{R}(H)$ are germs of diffeomorphisms k of $\mathbb{R}^4$ preserving the hyperplane $H = \{u = 0\}$, which — by Hadamard's lemma — have the form $k = (k_1, k_2, k_3, uk_4)$ with $k_i = k_i(x_1, x_2, x_3, u)$ and $k_4(0,0,0,0) > 0$. By Lemma 4.1 the $\mathcal{A}(H)$-stable germs $f : \mathbb{R}^4, 0 \to \mathbb{R}^n, 0$ are those with $\mathcal{A}(H)$-codimension at most three. Let $\mathcal{A}(H)^k := j^k\mathcal{A}(H)$ denote the Lie group of k-jets of elements of $\mathcal{A}(H)$. For $n = 3$, one finds (by direct $\mathcal{A}(H)$ coordinate changes) $\mathcal{A}(H)^1$-orbits of codimension 0, 1 and 2, respectively, with the following representatives:

$$(x_1, x_2, x_3), \quad (x_1, x_2, u), \quad (x_1, x_2, 0). \qquad (*)$$

The remaining $\mathcal{A}(H)^1$-orbits lie in the closure of $\mathcal{A}^1 \cdot (x_1, u, 0)$ and have codimension at least 4 and can therefore be discarded.

The first germ in $(*)$ is 1-$\mathcal{A}(H)$-determined, and a complete 2-transversal for the second germ σ is spanned by x_3^2, x_1x_3 and x_2x_3 in the last component. From the "general" 2-jet $(x_1, x_2, u + ax_3^2 + bx_1x_3 + cx_2x_3)$ we obtain over the 1-jet σ the $\mathcal{A}(H)^2$-orbits represented by $(x_1, x_2, u \pm x_3^2)$, $f = (x_1, x_2, u + x_1x_3)$ and (x_1, x_2, u). The first of these has codimension 1 and is 2-determined, the third has codimension 4, and the second has codimension 2 and has to be considered further. Over f we find three $\mathcal{A}(H)^3$-orbits given by

$$(x_1, x_2, u + x_1x_3 + x_3^3), \quad (x_1, x_2, u + x_1x_3 + x_2x_3^2), \quad (x_1, x_2, u + x_1x_3).$$

The first has codimension 2 and is 3-determined, the third has codimension 4, and over the second (which has codimension 3) there is one $\mathcal{A}(H)^4$-orbit that has codimension 3 (the others have higher codimension), which is 4-determined and given by $(x_1, x_2, u + x_1x_3 + x_2x_3^2 \pm x_3^4)$. This completes the classification of the $\mathcal{A}(H)$-stable germs over the second 1-jet in $(*)$.

Finally, consider the third 1-jet $\sigma = (x_1, x_2, 0)$ in $(*)$. A complete 2-transversal consists of all degree 2 monomials in the third component involving x_3 or u. We consider two cases: the x_3^2 coefficient is non-zero or zero.

In the first case we can reduce, up to $\mathcal{A}(H)^2$-equivalence, to $(x_1, x_2, x_3^2 + au^2 + bx_3u + cx_1u + dx_2y)$. Assuming that c and d are not both zero (otherwise the $\mathcal{A}(H)^2$-codimension is greater than 3) we can reduce to the case where $c = 1$ and $d = 0$, and letting $x_3 \mapsto x_2 - bu/2$ we get:

$$(x_1, x_2, x_3 + x_1u + (a - b^2/4)u^2).$$

Hence we get the following three $\mathcal{A}(H)^2$-orbits: $(x_1, x_2, x_3 \pm u^2)$ (these are 2-determined and have $\mathcal{A}(H)$-codimension 2) and $\sigma = (x_1, x_2, x_3^2 + x_1u)$ (of $\mathcal{A}(H)^2$-codimension 3). A complete 3-transversal for σ is given by u^3 and x_2u^2 in the third component, and when the coefficients of these two terms are both non-zero we can reduce to $f = (x_1, x_2, x_3^2 + x_1u + u^3 + x_2u^2)$. Some more substantial calculations then show that f is 3-determined and $\mathrm{cod}(\mathcal{A}(H), f) = 3$. The other $\mathcal{A}(H)^3$-orbits over σ (for which the product of the two coefficients vanishes) have codimension at least four.

In the second case (zero x_3^2 coefficient) the $\mathcal{A}(H)^2$-codimension is at least three, and in the best possible case the coefficients of x_3u and x_1x_3 are non-zero (any degeneration would lead to $\mathcal{A}(H)^2$-orbits of codimension at least four) so that we can reduce to the 2-jet $\sigma = (x_1, x_2, x_3u + x_1x_3)$. A complete 3-transversal for σ is given by x_3^3 and $x_2x_3^2$ in the third component, and using

Mather's lemma one shows one can reduce to $f = (x_1, x_2, x_3u+x_1x_3+x_3^3+x_2x_3^2)$ provided that the coefficients of both terms are non-zero. Finally one shows that f is 3-determined and $\mathrm{cod}(\mathcal{A}(H), f) = 3$. □

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