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Hypersurfaces of extremal slope

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The hypersurface of extremal slope consists of extrema of the gradient magnitude along integral curves of the gradient vector field of a smooth function. We study the singularities, and other geometric features, of hypersurfaces of extremal slope associated with generic functions and with one-parameter families of functions representing generic solutions to the heat equation.

1. Introduction

Given a smooth function g on an open subset $\Omega \subset \mathbb{R}^n$, let Γ denote the locus of points in which the (squared) gradient magnitude along the integral curves of the gradient vector field associated with g attains an extremum. For generic functions g, Γ is a hypersurface in Ω with isolated singular points. We are interested in the following questions. What types of singular points and other geometric features can Γ have for generically chosen functions g? What types of transitions can arise on Γ for generic one-parameter families of functions and for one-parameter families representing generic solutions to the heat equation? Here, *generic* means for an *open and dense* subset of the space of smooth functions (or one-parameter families of smooth functions or solutions to the heat equation).

The present study extends the work in [12] on curves of extremal slope in the plane to higher dimensions. The original motivation for the work in [12] was to study the possible evolutions of Canny edges [4] in scale-space, given a generic two-dimensional signal g (for example, the intensity function of a computer image).

It is time for some definitions. Given a smooth function

$$g : \mathbb{R}^n \supset \Omega \to \mathbb{R}, \qquad x \mapsto g(x),$$

let

$$\nabla g(x) := \sum_{i=1}^{n} g_{x_i} \cdot \frac{\partial}{\partial x_i}$$

denote the gradient-vector field in Ω and let $t \mapsto \alpha(t)$ be an integral curve of the gradient vector through $\alpha(0) = x$, i.e. $\mathrm{d}\alpha(t)/\mathrm{d}t = \nabla g(\alpha(t))$. Geometrically, such an integral curve is the projection of a line of steepest descent on the hypersurface $\{(x, g(x)) : x \in \Omega\}$ onto Ω. Define the following 'punctured' set (in case that g has isolated critical points):

$$\Omega_{\mathrm{reg}} := \Omega \setminus \{x \in \Omega : \nabla g(x) = 0\}.$$

The hypersurface of extremal slope is defined as

$$\Gamma := \overline{\{x \in \Omega : \gamma(x) = 0,\ \nabla g(x) \neq 0\}},$$

where $\bar{A}$ denotes the closure of A and

$$\begin{aligned}\gamma &:= \frac{1}{2} \cdot \left.\frac{\mathrm{d}\langle \nabla g(\alpha(t)), \nabla g(\alpha(t))\rangle}{\mathrm{d}t}\right|_{t=0} \\ &= \mathrm{d}^2 g(x)(\nabla g(x), \nabla g(x)) \\ &= \sum_{1 \leqslant i \leqslant n} g_{x_i}^2(x) g_{x_i^2}(x) + 2 \sum_{1 \leqslant i < j \leqslant n} g_{x_i}(x) g_{x_j}(x) g_{x_i x_j}(x)\end{aligned}$$

(here we set $g_{x^\alpha} := \partial^{|\alpha|} g / \partial x^\alpha$, where $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$). The set Γ of extremal slope in gradient direction can be divided up into subsets of maximal, minimal and 'transitional' slope. More precisely, we set

$$\Gamma := \Gamma^- \cup \Gamma^+ \cup \Gamma^0,$$

where a point x in Γ belongs to one of the three sets on the right-hand side according to the sign ($-$, $+$ or 0) of $\mathrm{d}^2\langle \nabla g(\alpha(t)), \nabla g(\alpha(t))\rangle / \mathrm{d}t^2$ at $x = \alpha(0)$.

Note that the set $\tilde{\Gamma} := \{(x, g(x)) : x \in \Gamma\}$ is contained in the region of non-positive curvature of the function graph $x \mapsto (x, g(x))$, $x \in \Omega$ (this is clear if we write the defining equation of Γ in local coordinates that diagonalize $\mathrm{d}^2 g$). The hypersurface of extremal slope Γ can therefore not reach the non-degenerate maxima and minima of g (i.e. the A_1 points of signature n of g).

We summarize our main results in the following statement.

THEOREM 1.1.

(i) *Excluding a subset of functions $g \in C^\infty(\Omega, \mathbb{R})$ of infinite codimension, $\Gamma \subset \Omega$ is a hypersurface with isolated singular points.*

(ii) *The generic geometry of $\Gamma \cap \Omega_{\mathrm{reg}}$ is the same as that of a general hypersurface in $\mathbb{R}^n$.*

(iii) *At singular points of g of type A_k, $k \geqslant 1$, Γ has A_{3k-2} points (furthermore, Γ has non-simple singular points at D_k, $k \geqslant 4$, and E_6, E_7 and E_8 points of g).*

(iv) *Generically (i.e. in codimension 0), Γ has only isolated A_1 points at A_1 points of g, and Γ^0 is a smooth submanifold in Γ of codimension 1 that divides Γ into Γ^+ and Γ^- regions.*

(v) *In codimension 1, Γ and Γ^0 can have A_1 points at regular points of g, and Γ can also have A_4 points at A_2 points of g.*

Proof. Statement (ii) is a (somewhat informal) description of the content of proposition 1.7, which, together with the fact that non-isolated singularities of g have infinite codimension, implies (i). Statement (iii) corresponds to lemma 3.1, and (iv) and (v) summarize the more detailed results stated in propositions 1.2 and 1.3. □

Table 1. *Singular points of Γ*

codim	$\gamma(x)$	signature(γ)	$g(x)$	signature(g)
0	A_1	$n-2, n-4, \ldots, (n \bmod 2)$	A_1	signature(γ)
1	A_1	$n, n-2, \ldots, (n \bmod 2)$	regular point	
1	A_4	$n-1, n-3, \ldots, (n-1 \bmod 2)$	A_2	signature(γ)

The generic geometry of hypersurfaces of extremal slope Γ is described in more detail in propositions 1.2 and 1.3 below: the former summarizes the relationship between the subsets $\Gamma^{\pm}$ and Γ^0 of Γ and the integral curves of the gradient vector field $\nabla g(x)$; and the latter describes the possible singularities of Γ. Both propositions have two parts: part (i) describes the properties of Γ that occur for generic (but fixed) functions g; and part (ii) describes those occurring for generic one-parameter families of functions g or for generic solutions to the heat equation.

Let $G : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \times \mathbb{R}$, $(t, x) \mapsto (t, g(t, x))$ be a one-parameter family of smooth functions such that g is a solution to the heat equation $\partial g/\partial t = \Delta g$ with initial condition $g(0, x) = f(x) \in L_2(\Omega, \mathbb{R})$. Let $T := (t_1, t_2) \subset \mathbb{R}_+$ be an open interval and let $\mathcal{H} \subset C^\infty(T \times \Omega, \mathbb{R})$ be the space of solutions to the heat equation on $T \times \Omega$. For $n := \dim \Omega \geqslant 2$, we then have the following result.

Proposition 1.2.

(i) *For an open and dense subset of $g \in C^\infty(\Omega, \mathbb{R})$, Γ is a hypersurface in Ω with isolated singular points at $\{x \in \Omega : \nabla g(x) = 0\}$. The locus of transitional slope Γ^0 is a non-singular submanifold of Γ of codimension 1, which divides Γ into regions of maximal Γ^- and minimal slope Γ^+. Furthermore, Γ^0 is the locus of tangency of Γ and the integral curves of the gradient vector field of g. There are no singular points in Γ^-, and the only singular points of Γ^+ are of type $\gamma = \sum_{i=1}^{n} a_i x_i^2$, where all a_i are non-zero but at least two of them have distinct signs (i.e. A_1 singularities of signature different from n, see proposition 1.3(i) for more information about these singular points).*

(ii) *For an open and dense subset of $g(t, x) \in \mathcal{H}$ or $g(t, x) \in C^\infty(T \times \Omega, \mathbb{R})$, the features listed in (i) above occur for open sub-intervals of T. For isolated points $t \in T$, the set of transitional slope can have singular points of type A_1. (For information about the singular points of Γ see proposition 1.3(ii).)*

The following result describes the generic singularities of the hypersurface of extremal slope $\Gamma = \overline{\{x : \gamma(x) = 0,\ \nabla g(x) \neq 0\}}$ and relates them to the singularities of g. Multiplying γ by -1 does not change its zero-set Γ, and we therefore define the signature of γ as follows: signature(γ) $:= |r_+ - r_-|$, where $r_\pm$ is the number of positive/negative elements of the diagonalized Hessian $\mathrm{d}^2\gamma$ at x.

Proposition 1.3. *The list of singular points of $\gamma := \mathrm{d}^2 g(\nabla g, \nabla g)$ in table 1 is complete in the following sense.*

(i) *For an open and dense subset of $g \in C^\infty(\Omega, \mathbb{R})$, only the codimension-0 singularity of γ in table 1 can occur.*

(ii) *For an open and dense subset of $g(t,x) \in \mathcal{H}$ or $g(t,x) \in C^\infty(T \times \Omega, \mathbb{R})$, the codimension-0 singularity occurs for open sub-intervals of T and the codimension-1 singularities for isolated points $t \in T$.*

REMARK 1.4. Call a property generic if it holds for an open and dense set of solutions $g(t,x)$, $t \in T$, to the heat equation. A result by Damon [7, theorem 1] asserts that the set of initial conditions $f := g(0,x) \in L_2(\Omega, \mathbb{R})$, which give rise to solutions g that satisfy any given generic property in all points of $T \times \Omega$, is open and dense in $L_2(\Omega, \mathbb{R})$. The properties listed in part (ii) of propositions 1.2 and 1.3 do therefore arise for open and dense subsets of initial conditions to the heat equation.

REMARK 1.5. Clearly, the deformation of an A_4 singularity of γ, which is induced by a one-parameter deformation of an A_2 singularity of g, cannot be versal. We shall see in § 3 that the number of A_1 points appearing in deformations of γ induced by generic solutions g to the heat equation, or by general versal one-parameter deformations g, depends on the signature of the A_4 point. By contrast, it is shown in § 5 that, in dimension 3, the A_1 singularities of Γ and of Γ^0 deform versally. It seems reasonable to conjecture that this is true in any dimension.

REMARK 1.6. In the special case where $\Omega = [a,b] \times [c,d] \subset \mathbb{R}^2$, the arcs of maximal slope Γ^- are known as Canny curves and 'correspond' to edge curves in computer images (see [4, 12]).

The next result says that, loosely speaking, the generic geometry of Γ in the punctured set

$$\Omega_{\mathrm{reg}} := \Omega \setminus \{x \in \Omega : \nabla g(x) = 0\}$$

is the same as that of a general hypersurface $g^{-1}(0) \subset \Omega_{\mathrm{reg}}$. To be more precise, define the sets

$$W(g) := \{g : W \text{ holds for } g\}$$

and

$$W(\gamma) := \{g : W \text{ holds for } \gamma := \mathrm{d}^2 g(\nabla g, \nabla g)\}.$$

Here, W is some differential geometric property, such as, for example, '$h^{-1}(0)$, where $h = g$ or γ, has isolated umbilics or isolated A_2 singularities', which imposes certain conditions on the k-jet of h. To be more precise, let $U = \Omega_{\mathrm{reg}}$ or $T \times \Omega_{\mathrm{reg}}$, then these conditions are the defining equations of $W \subset J^k(U, \mathbb{R})$ and the classical transversality lemma says that the set of h, whose k-jet extension is transverse to some closed submanifold or, more generally, to some closed Whitney-stratified set $W \subset J^k(U, \mathbb{R})$ on a compact subset of the source, is open and dense in $C^\infty(U, \mathbb{R})$. (Note that our sets W will always be semi-algebraic, and hence Whitney stratifiable.)

The classical transversality lemma obviously does not apply to solutions to the heat equation, because open and dense subsets of $C^\infty(U, \mathbb{R})$ might fail to intersect $\mathcal{H} \subset C^\infty(U, \mathbb{R})$ in open dense sets. For solutions to the heat equation (and also for other partial differential equations, see [6]), there is the following transversality result by Damon [5, corollary 5.3]. It says that if the space $\mathcal{H}^k$ of k-jets of solutions to the heat equation is transverse to some closed Whitney-stratified set $W \subset J^k(U, \mathbb{R})_{(t_0,0),0}$, then the set

$$\{g \in \mathcal{H} : j^k g \pitchfork U \times \mathcal{H}^k \times \mathbb{R} \text{ on a compact subset of the source}\}$$

is open and dense (for the regular C^l-topology, $k+1 \leqslant l$). It is easy to see that, for $k \geqslant 1$, all submanifolds $W \subset J^k(U, \mathbb{R})$ whose defining equations involve spatial partial derivatives only, and not derivatives with respect to t, are transverse to $\mathcal{H}^k$. For almost all $g \in \mathcal{H}$, the k-jet extension $j^k g$ will therefore miss submanifolds W of codimension greater than $n+1$ and, for $c := \operatorname{codim} W \leqslant n+1$, the set $(j^k)^{-1}(W)$ has codimension c in U.

In the case of families of functions g that are solutions to the heat equation, we therefore assume that the defining conditions of W involve only spatial partial derivatives and not derivatives with respect to $t \in T$. (Also notice that, for $g \in \mathcal{H}$, the induced family γ is, in general, an element of $C^\infty(U, \mathbb{R})$ and not of $\mathcal{H}$.) We now have the following result.

Proposition 1.7.

(i) *$W(g)$ is an open and dense subset of $C^\infty(\Omega_{\text{reg}}, \mathbb{R})$ if and only if $W(\gamma)$ is.*

(ii) *$W(g)$ is an open and dense subset of $\mathcal{H} \cap C^\infty(T \times \Omega_{\text{reg}}, \mathbb{R})$ or $C^\infty(T \times \Omega_{\text{reg}}, \mathbb{R})$ if and only if $W(\gamma)$ is an open and dense subset of $C^\infty(T \times \Omega_{\text{reg}}, \mathbb{R})$.*

1.1. Organization of paper

The remainder of the present paper is organized as follows. Sections 1–4 contain the proofs of propositions 1.2, 1.3 and 1.7. In § 2 it is shown that, at regular points of g, the map sending the $(k+2)$-jet of g to the k-jet of γ is a submersion. In § 3 it is shown that A_k singularities of g correspond to A_{3k-2} singularities of γ, and § 4 concludes the proofs of propositions 1.2, 1.3 and 1.7. In the final § 5 we study the generic deformations of codimension-1 singularities of Γ and Γ^0 in the case where $\dim \Omega = 3$. We describe normal forms for these deformations up to an equivalence relation that preserves the decomposition of Γ into regions Γ^- and Γ^+ of maximal and minimal slope, respectively.

2. Regular points of g

The following lemma is the key result in the proof of proposition 1.7.

Lemma 2.1. *The map $\phi_k : J^{k+2}(\Omega_{\text{reg}}, \mathbb{R}) \to J^k(\Omega_{\text{reg}}, \mathbb{R})$, with $\Omega_{\text{reg}} \subset \mathbb{R}^n$ an open subset of regular points of g, given by $j^{k+2}g \mapsto j^k\gamma$, is a submersion for all $0 \leqslant k < \infty$, $1 \leqslant n < \infty$.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index, and make a linear coordinate change such that $\nabla g = (g_{x_1}, 0, \dots, 0)$ with $g_{x_1} \neq 0$. Then

$$\gamma_{x^\alpha} = g_{x_1}^2 g_{x^{\alpha+2e_1}} + R,$$

where $e_1, \dots, e_n$ is the standard basis in $\mathbb{R}^n$ and where R is a polynomial in the partial derivatives g_{x^δ} with $|\delta| < |\alpha| + 2$ (here, δ is another multi-index). Hence, for all δ with $|\delta| < |\alpha| + 2$ and all δ with $|\delta| = |\alpha| + 2$ but $\delta \neq \alpha + 2e_1 := (\alpha_1 + 2, \alpha_2, \dots, \alpha_n)$, we get

$$\frac{\partial \gamma_{x^\alpha}}{\partial g_{x^\delta}} = 0.$$

Furthermore,

$$\frac{\partial \gamma_{x^\alpha}}{\partial g_{x^{\alpha+2e_1}}} = g_{x_1}^2.$$

Therefore, fixing bases

$$\{\gamma_{x^\alpha} : k \geqslant |\alpha| \geqslant 0\} \quad \text{and} \quad \{g_{x^\delta} : k + 2 \geqslant |\delta| \geqslant 0\}$$

for $J^k(\Omega_{\mathrm{reg}}, \mathbb{R})$ and $J^{k+2}(\Omega_{\mathrm{reg}}, \mathbb{R})$, respectively, such that the multi-indices α and δ are strictly decreasing with respect to some suitable order on $\mathbb{N}^n$ (see below), we see that $\mathrm{d}\phi_k = [A \mid B]$, where A is an upper diagonal $m \times m$ matrix, where $m = \dim J^k(\Omega_{\mathrm{reg}}, \mathbb{R})$, with the non-zero elements $g_{x_1}^2$ on its diagonal. Hence ϕ_k is a submersion.

For α we take any total degree lexicographical order (i.e. we first compare degrees and for equal degrees break ties using some lexicographical order for the components), and for $\delta = (\delta_1, \dots, \delta_n)$ we apply the induced order from α to $\delta - 2e_1$ in case $\delta_1 \geqslant 2$, and we give the δ with $\delta_1 < 2$ lower weight than any other δ (they correspond to the B block in $\mathrm{d}\phi_k = [A \mid B]$). □

Remark 2.2. The fact that $\phi_k : J^{k+2}(\Omega_{\mathrm{reg}}, \mathbb{R}) \to J^k(\Omega_{\mathrm{reg}}, \mathbb{R})$ is a submersion implies the following. Let W be some kth-order differential geometric property of a hypersurface $g^{-1}(0) \subset \Omega_{\mathrm{reg}}$, and consider the following submanifolds of $J^k(\Omega_{\mathrm{reg}}, \mathbb{R})$ and $J^{k+2}(\Omega_{\mathrm{reg}}, \mathbb{R})$:

$$W^k(g) := \{j^k g : W \text{ holds for } g\}$$

and

$$W^{k+2}(\gamma) := \{j^{k+2} g : W \text{ holds for } \gamma := \mathrm{d}^2 g(\nabla g, \nabla g)\}.$$

Then the codimensions of $W^k(g)$ and of $W^{k+2}(\gamma) = \phi_k^{-1}(W^k(g))$ are equal. This fact, together with the classical transversality lemma or with Damon's transversality results for the heat equation (see § 1), implies that the generic geometry of the hypersurfaces $\gamma^{-1}(0)$, $g^{-1}(0) \subset \Omega_{\mathrm{reg}}$ is the same.

The following analogue of lemma 2.1 will be useful later on.

Lemma 2.3. *Set $\beta := \langle \nabla\gamma, \nabla g \rangle$. Then the map $\phi_k : J^{k+3}(\Omega_{\mathrm{reg}}, \mathbb{R}) \to J^k(\Omega_{\mathrm{reg}}, \mathbb{R})$, with $\Omega_{\mathrm{reg}} \subset \mathbb{R}^n$ an open subset of regular points of g, given by $j^{k+3} g \mapsto j^k \beta$, is a submersion for all $0 \leqslant k < \infty$, $1 \leqslant n < \infty$.*

Proof. Analogous to the proof of lemma 2.1. □

3. Critical points of g

In this section we study the relationship between the singularities of g and those of γ. It is easy to see that if g' is another function that is $\mathcal{R}$-equivalent (or $\mathcal{K}$-equivalent) to g, then $\gamma' := \mathrm{d}^2 g'(\nabla g', \nabla g')$ will, in general, not be $\mathcal{R}$-equivalent (or $\mathcal{K}$-equivalent) to γ. Coordinate changes that do not change the type of singular point of g can change the singularity type of γ. Nevertheless, we have the following relationship between A_k singularities of g and γ.

LEMMA 3.1. *Let x be a critical point of g. Then the function g has an A_k, $k \geqslant 1$, singularity of signature m at x if and only if γ has an A_{3k-2} singularity of the same signature at x. Furthermore, if g has a simple singularity of type D_k, $k \geqslant 4$, or E_k, $k = 6$, 7 or 8, at x then γ has a non-simple singularity at x.*

Proof. We can assume that x is the origin. Apply a linear change of coordinates in the source (a rotation) that simultaneously diagonalizes $\mathrm{d}^2 g$ and $\mathrm{d}^2\gamma$ and leaves the singularity types of g and γ fixed (notice that $\gamma_{x_i x_j} = 0$, $i \neq j$, for $g_{x_i x_j} = 0$ and $\nabla g = 0$.) Assign the following weights to the variables:

$$\begin{aligned} \mathrm{wt}(x_i) &= k+1, \quad i < n, \\ \mathrm{wt}(x_n) &= 2. \end{aligned}$$

For the first implication ($\Rightarrow$), assume that g has an A_k singularity at the origin and hence is of the form

$$g = \sum_{i<n} \tfrac{1}{2} a_i x_i^2 + c x_n^{k+1} + R,$$

where $c, a_i \neq 0$ and where the terms of R have weighted degree greater than or equal to $2k+2$ (by the hypothesis that g is an A_k singularity).

For the corresponding γ, we use the weights

$$\begin{aligned} \mathrm{wt}(x_i) &= 3k-1, \quad i < n, \\ \mathrm{wt}(x_n) &= 2, \end{aligned}$$

and make the decomposition $\mathrm{d}^2 g = D + S$, where D is a diagonal matrix with diagonal elements $a_1, \ldots, a_{n-1}, k(k+1)c x_n^{k-1}$ and where $S = (s_{uv})$ is a matrix of higher weight terms. Relative to the weights for γ, the entries of S have the following weighted degrees:

$$\begin{aligned} \deg s_{uv} &> 0, \quad u, v < n, \\ \deg s_{nv}, \deg s_{un} &> 3k-3, \\ \deg s_{nn} &> 6k-6. \end{aligned}$$

The first term of

$$\gamma = \mathrm{d}^2 g(\nabla g, \nabla g) = \langle D \cdot \nabla g, \nabla g \rangle + \langle S \cdot \nabla g, \nabla g \rangle$$

is given by

$$\sum_{i<n} a_i (a_i x_i + R_{x_i})^2 + k(k+1) c x_n^{k-1} ((k+1) c x_n^k + R_{x_n})^2.$$

This term, in turn, consists of the weighted-homogeneous initial part

$$\sum_{i<n} a_i^3 x_i^2 + k(k+1)^3 c^3 x_n^{3k-1}$$

of degree $6k-2$ and a remainder of higher weighted degree. One checks that the second term $\langle S \cdot \nabla g, \nabla g \rangle$ has weighted degree greater than $\min(6k-2, 7k-3)$, which, for $k>1$, is greater than $6k-2$. Hence γ has an A_{3k-2} singularity at x, as required. Also note that

$$\operatorname{sgn}(\tfrac{1}{2}a_i) = \operatorname{sgn}(a_i^3) \quad \text{and} \quad \operatorname{sgn}(c) = \operatorname{sgn}(k(k+1)^3 c^3).$$

Hence, if g is of type A_k and signature m, then γ is of type A_{3k-2} and signature m.

For the converse, it is essential that $\nabla g = 0$ at the origin, so that altering g by any linear term leaves γ fixed. But, assuming that g is singular at the origin, it is an easy matter to reverse the above argument.

For the last statement, note that D_k and E_k singularities have zero 2-jet but non-zero 3-jet. But, for zero 2-jets of g, the corresponding γ has zero 3-jet. □

We conclude this section by considering deformations of A_4 points of γ that are induced by one-parameter deformations of A_2 points of g, either by general versal one-parameter deformations or generic solutions to the heat equation. In particular, we are interested in counting A_1 points of some given signature. This will be useful in § 5.

Consider a versal one-parameter deformation of an A_2 singularity with deformation parameter t, or a generic solution to the heat equation near an A_2 singularity. Depending on the sign of t, the A_2 point of g deforms into zero or two A_1 points in both cases (in Damon's classification of generic solutions to the heat equation, there are two normal forms for A_2: one corresponds to the creation of two A_1 points; the other to the annihilation (see [5, list II, § 2])). Lemma 3.1 tells us that the A_1 points of some given signature that appear in a deformation of g are also A_1 points of the same signature of the deformation of γ. Recall that only A_1 points of signature different from n belong to Γ. The number of singular points of type A_1 that appear in a deformation of γ that is induced by a deformation of g is therefore equal to the number of A_1 points of signature different from n that appear in a deformation of g. In particular, we have the following relation. Consider a versal one-parameter deformation of an A_2 point of g. If the corresponding A_4 point of γ has signature m, then it either deforms into zero or two A_1 points, and the A_1 points have signature $m+1$ and $m-1$ (for $m>0$) or $m+1$ and 1 (for $m=0$). (Note that a small versal deformation preserves the signs of the non-zero eigenvalues of the Hessian and makes the zero eigenvalue positive at one A_1 point and negative at the other.) We therefore have the following result.

Lemma 3.2. *Consider a deformation of an A_4 point of Γ of signature m induced by a versal deformation of an A_2 point of g. There are two types of stabilizations, Γ' and Γ'', that appear in such a deformation of Γ:* (i) *Γ' is a smooth hypersurface germ; and* (ii) *Γ'' has either two A_1 points of signature $m \pm 1$ (for $m \neq 0, n-1$), two A_1 points both of signature* 1 *(for $m=0$) or one A_1 point of signature $n-2$ (for $m=n-1$).*

4. Proofs of the main propositions

For the proof of statements (i) and (ii) of proposition 1.3, we either use proposition 1.7 (for the regular points of g) or the relationship between the singularities of g and γ (lemma 3.1) and apply the classical transversality lemma or Damon's results for the heat equation. Proposition 1.7 says that, at a regular point x of g, the hypersurface Γ has the generic singularities of a general hypersurface: it is smooth in general and, in one-parameter families, it can have A_1 singularities for isolated points $t \in T$ (at an A_1 point, the defining equation of a hypersurface and its gradient have to vanish, which imposes $n+1$ conditions on its 1-jet). At singular points x of g, we have the following. The functions g having an A_1 singularity at x, and hence also an A_1 of γ, correspond to a codimension-n submanifold in jet-space. A_2 points of g, which are A_4 points of γ, correspond to codimension-$(n+1)$ behaviour (and hence can occur for isolated points t), and more degenerate singularities of g of codimension greater than or equal to $n+2$ do not occur in generic one-parameter families nor in generic solutions to the heat equation.

For the proofs of propositions 1.2 and 1.7, we do not directly apply the transversality lemma, or Damon's generalization, to the space of jets $J^{k+s}(\Omega_{\mathrm{reg}}, \mathbb{R})$ of g. Instead, we use the transversality of the maps $\phi : J^{k+s}(\Omega_{\mathrm{reg}}, \mathbb{R}) \to J^k(\Omega, \mathbb{R}^m)$, sending the $(k+s)$-jet of g to the k-jet of the defining equation of Γ or of Γ^0, to certain submanifolds $W \subset J^k(\Omega, \mathbb{R}^m)$ (lemmas 2.1, 2.3, and 4.1 below), together with the basic fact that, in this case, $\phi^{-1}(W) \subset J^{k+s}(\Omega, \mathbb{R})$ is a submanifold of the same codimension as W or is empty.

In fact, for the defining equation γ of Γ, the map ϕ is a submersion (lemma 2.1) and hence transverse to any submanifold in its target. This implies proposition 1.7. For the defining equations of Γ^0, the situation is slightly more complicated, because ϕ is, in general, not a submersion. Here, the situation is as follows.

Proof of proposition 1.2 (conclusion). The boundary Γ^0 between the regions of maximal and minimal slope of Γ is given by the vanishing of γ and $\beta := \langle \nabla\gamma, \nabla g \rangle$; more precisely,

$$\Gamma^0 := \overline{\{x \in \Omega : \gamma(x) = \beta(x) = 0,\ \nabla g(x) \neq 0\}}.$$

One checks that, at a singular point x of g, the tangent cones of $\gamma^{-1}(0)$ and $\beta^{-1}(0)$ coincide and that $\gamma^{-1}(0) \cap \beta^{-1}(0) = \{x\}$. The singular points of g are therefore isolated solutions of $\gamma = \beta = 0$ that do not belong (by the definition above) to Γ^0. Hence Γ^0 is a subset of Ω_{reg}. Lemma 2.3 implies that $\beta^{-1}(0) \subset \Omega_{\mathrm{reg}}$ is a smooth hypersurface for open intervals in T and has isolated A_1 singularities for isolated points $t \in T$. The type of singularity, or the regularity, of Γ^0 depends on the $\mathcal{K}$-class of the map $F = (\gamma, \beta) : \Omega \to \mathbb{R}^2$, whose zero-set is Γ^0. One checks that the sets $\Gamma^0 \cap \{x \in \Omega_{\mathrm{reg}} : \nabla\gamma(x) = 0\}$ and $\Gamma^0 \cap \{x \in \Omega_{\mathrm{reg}} : \nabla\beta(x) = 0\}$ correspond to behaviour of codimension greater than or equal to $n+2$, and hence can be avoided in generic one-parameter families or in generic solutions to the heat equation (even for isolated t). The map $(\gamma, \beta) : \mathbb{R}^n \to \mathbb{R}^2$ is therefore regular or has corank 1 (if one excludes behaviour of codimension greater than or equal to $n+2$). Let $\Sigma^1[J^k(\Omega_{\mathrm{reg}}, \mathbb{R}^2)]$ denote the space of k-jets of maps of corank less than or equal to 1. The next lemma then completes the proof of proposition 1.2. □

Lemma 4.1. *Let $W(\bar{A}_1)$ and $W(\bar{A}_2)$ denote the closures of the $\mathcal{K}$-orbits of maps $\mathbb{R}^n \to \mathbb{R}^2$ of type A_1 and A_2 in $\Sigma^1[J^2(\Omega_{\mathrm{reg}}, \mathbb{R}^2)]$. The sets $W(\bar{A}_1)$ and $W(\bar{A}_2)$ are semi-algebraic subsets of $\Sigma^1[J^2(\Omega_{\mathrm{reg}}, \mathbb{R}^2)]$ of codimension $n+1$ and $n+2$, respectively, and the image of the map $\phi : J^5(\Omega_{\mathrm{reg}}, \mathbb{R}) \to \Sigma^1[J^2(\Omega_{\mathrm{reg}}, \mathbb{R}^2)]$, given by $j^5 g \mapsto j^2(\gamma, \beta)$, is transverse to all the strata of a Whitney stratification of these semi-algebraic sets.*

Proof. After a coordinate change, we can assume that

$$(\gamma(x), \beta(x)) = (x_1, \beta(x_2, \dots, x_n)).$$

The conditions for an A_1 singularity (or worse) are then

$$\gamma = \beta = \beta_{x_2} = \dots = \beta_{x_n} = 0,$$

and for an A_2 singularity (or worse) we have the additional condition $|\mathrm{d}^2\beta| = 0$. The sets of regular points, of A_1 points and of A_2 points are therefore semi-algebraic subsets of $\Sigma^1[J^2(\Omega_{\mathrm{reg}}, \mathbb{R}^2)]$ of codimension 2, $n+1$ and $n+2$, respectively.

To verify the second assertion, it is sufficient to check that the restriction, $\tilde{\phi}$, of ϕ to the linear subspace

$$L := \mathbb{R}\{\gamma, \beta_{x^\alpha} : 0 \leqslant \alpha \leqslant 2\} \subset \Sigma^1[J^2(\Omega_{\mathrm{reg}}, \mathbb{R}^2)]$$

(which is spanned by the constant term of γ and by the constant, linear and quadratic terms of β) is a submersion. The highest-order derivatives appearing in the kth derivatives of γ and β are of order $k+2$ and $k+3$, respectively. As in the proof of lemma 2.1, we can see that the map $\tilde{\phi} : J^5(\Omega_{\mathrm{reg}}, \mathbb{R}) \to L$ is a submersion by choosing suitable term orders in the source and target. For $x \in \Omega_{\mathrm{reg}}$,

$$\nabla g(x) = (0, \dots, 0, g_{x_\ell}(x), \dots, g_{x_n}(x)),$$

where $g_{x_\ell}(x) \neq 0$. (The above condition for an A_1 singularity already involved a source coordinate change, so we cannot, by a further coordinate change (as in lemma 2.1), assume that $\nabla g = (g_{x_1}, 0, \dots, 0)$.) If

$$\{\beta_{x^\alpha}, \gamma : 2 \geqslant \alpha \geqslant 0\} \quad \text{and} \quad \{g_{x^\delta} : 5 \geqslant \delta \geqslant 0\}$$

are bases for L and $J^5(\Omega_{\mathrm{reg}}, \mathbb{R})$, then the multi-index α is strictly decreasing with respect to some total degree lexicographical order. For the δ with $\delta_1 \geqslant 3$, we apply the order for α to $\delta - 3e_\ell$, and all δ with $\delta_1 < 3$ have lower weight than any other δ. Now we see that $\mathrm{d}\tilde{\phi}$ is surjective. □

5. Deformations of codimension-1 singularities of Γ and Γ^0 for $\dim \Omega = 3$

In the present section, Γ is a two-dimensional hypersurface in Ω, and its subregions of maximal and minimal slope, Γ^- and Γ^+, are drawn in grey and white, respectively. Also, $A_k(m)$ will denote an A_k singularity of signature m. Figure 1 summarizes the stable features of Γ for $\dim \Omega = 3$ (recall part (i) of propositions 1.2 and 1.3): the locus of transitional slope Γ^0 is a smooth space-curve dividing the surface Γ into smooth regions Γ^- of maximal slope and regions Γ^+ of minimal

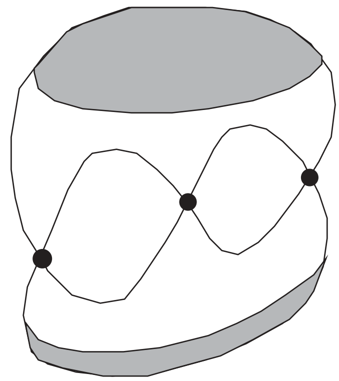

Figure 1. The surface Γ of extremal slope (for $n = 3$). Regions of maximal slope Γ^- are shown in grey. The three double-cones of type $A_1(1)$ belong to the region of minimal slope Γ^+.

slope that have $A_1(1)$ singularities at the saddle-points of g. In the complement of the $A_1(1)$ points, the surface Γ has the same generic features as a general surface in 3-space (proposition 1.7(i)). For example, the parabolic curves $\mathcal{P} \subset \Gamma$ are non-singular, and the image of the parabolic curves under the Gauss map is a curve in the sphere with isolated cusps and crossings [1]. The ridge curves $\mathcal{R} \subset \Gamma$ intersect in umbilical points of Γ, and there are three different types of umbilics according to Porteous [11].

In generic one-parameter families of functions g, or in generic solutions g to the heat equation, the codimension-1 singularities of Γ^0 and Γ described in part (ii) of propositions 1.2 and 1.3 can occur for isolated points $t \in T$, namely (1) A_1 points of Γ^0, (2) A_1 points of Γ at regular points of g, and (3) A_4 points of Γ at A_2 points of g. Furthermore, by proposition 1.7(ii), $\Gamma \in \Omega_{\text{reg}}$ can have the same generic transitions as a generic one-parameter family of general hypersurfaces in 3-space. For example, the parabolic and ridge curves on Γ can have certain singular points for isolated points $t \in T$ whose deformations in generic one-parameter families of general hypersurfaces have been classified by Bruce *et al.* in [2] and [3], respectively.

To understand the generic deformations of the codimension-1 singularities of Γ^0 and Γ, we have to study versal one-parameter deformations of the maps $F = (\gamma, \beta)$ and γ for some appropriate equivalence relation. A list of normal forms for these one-parameter deformations can be obtained by adapting an argument given by Damon in [8], that is, by using a somewhat finer equivalence relation than in [8].

First, we briefly sketch Damon's argument (see [8, part IV] for details). Note that the jet extensions of the maps F and γ, for t fixed, will be transverse to any given submanifold V for almost all $t \in T$, but for certain isolated t they can be non-transverse to certain submanifolds of codimension less than or equal to 4 (codimension-5 behaviour can be avoided in generic one-parameter families of functions in three variables). Damon has argued that the relevant one-parameter deformations of such non-transverse maps are $\mathcal{K}_V$ versal, where V is a some sub-manifold in jet-space. (Recall that, for an analytic variety V in the target of a smooth map-germ f, there is a refinement of the standard $\mathcal{K}$-equivalence for f whose target coordinate changes preserve V. For the standard contact group $\mathcal{K}$, consult, for example, [10].) Using the facts that the submanifolds V are smooth products $\{0\} \times \mathbb{R}^i$, $i \leqslant 4$, that the $\mathcal{K}_V$ codimension is invariant under suspension and that, for $V = \{0\}$, $\mathcal{K}_V$-equivalence reduces to $\mathcal{K}_e$-equivalence, Damon obtains the following normal forms for the versal one-parameter deformations,

$$\left(x_1, \ldots, x_i, x_{i+1}^2 + \sum_{j=i+2}^{3} \epsilon_j x_j^2 + t\right),$$

where $0 \leqslant i \leqslant 2$ and $\epsilon_j = \pm 1$, and (x_1, x_2, x_3, t). Setting the deformation parameter $t = 0$, these normal forms are $\mathcal{A}$-stable corank-1 germs of $\mathcal{K}_e$-codimension 1. Finally, considering the jets of these $\mathcal{K}_e$-codimension-1 germs, these jets form Zariski open subsets of the space of algebraic maps that are non-transverse to V. It is therefore enough to construct an example for each normal form.

In our context, we have to refine this classification as follows. At regular points of Γ, the equivalence relation has to preserve the curve Γ^0 and the decomposition of Γ into regions of maximal and minimal slope ($\mathcal{K}^+$-equivalence below). At singular points of Γ, we have the additional restriction that the equivalence has to preserve the level sets $\gamma =$ const. of the map $F = (\gamma, \beta)$ ($\mathcal{D}^+$-equivalence below).

Below, x, y, z are the source coordinates of F and γ, and C_{xyz} denotes the ring of smooth function-germs at the origin. For the A_1 points of Γ^0 and of Γ (at regular points of g), we first give refined normal forms for the relevant versal one-parameter deformations of F and γ. We then construct one-parameter families g whose associated F and γ are equivalent to these normal forms (under the above refined equivalences).

For the A_4 points of Γ, corresponding to A_2 points of g, the induced deformations of γ and $F = (\gamma, \beta)$ cannot be versal. But we can use Damon's normal forms [5] for the deformation of g, together with the correspondence of A_1 points of g and γ, to describe the generic transitions of Γ.

5.1. A_1 points of Γ^0

Proposition 1.2 says that the curve of transitional slope Γ^0 can have A_1 singularities of signature 0 or 2, for isolated $t \in T$. The curve Γ^0 is the zero-set of the map $F : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto F := (\gamma(x, y, z), \beta(x, y, z))$, and the generic transitions at an A_1 point of F are given by a $\mathcal{K}$-versal deformation of F. Recall, however, that Γ^0 is the boundary between the white Γ^+ and the grey Γ^- regions. To keep track of the possible transitions in the 'colouring' of Γ, we have to use a finer equivalence than $\mathcal{K}$-equivalence that preserves the boundary Γ^0 and that does not exchange white and grey regions.

If (u, v) are coordinates in the target of F, then the target coordinate changes for ordinary $\mathcal{K}$-equivalence are of the form

$$(h_1, h_2) := (a_{11}(x, y, z)u + a_{12}(x, y, z)v, a_{21}(x, y, z)u + a_{22}(x, y, z)v),$$

where $a_{11}a_{22} - a_{12}a_{21}$ is non-zero in a neighbourhood of the origin. A target coordinate change (h_1, h_2) preserves the half-space $S^+ := \{(u, v) : v \geqslant 0\}$ if it is of the form $h_2(u, v) = v\tilde{h}_2(u, v)$ with $\tilde{h}_2(0, 0) > 0$. For elements of $\mathcal{K}$ this means that $a_{22}(0, 0, 0) > 0$. The subset $\mathcal{K}^+$ of elements of $\mathcal{K}$ satisfying this condition does not form a subgroup, but it turns out that it is sufficient to work with a restricted subset $\mathcal{K}_r^+$ of $\mathcal{K}^+$ whose target coordinate changes are elements of the form $A \cdot (u, v)^{\mathrm{T}}$,

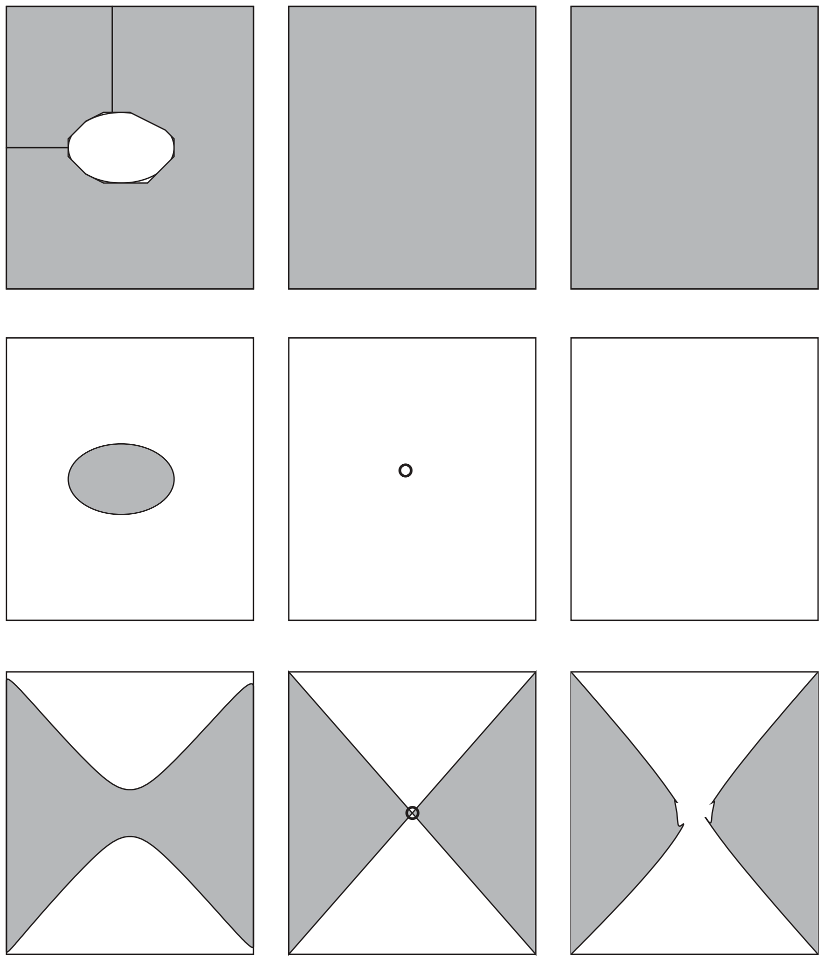

Figure 2. The singular points of $\Gamma^0 \subset \Gamma$ (shown in the centre of each row) and their deformations (on the left and right): (i) an $A_1(2)$ point in a Γ^- region (top row); (ii) an $A_1(2)$ point in a Γ^+ region (middle row); and (iii) an $A_1(0)$ point (bottom row).

where A belongs to the unipotent subgroup of $GL(2, C_{xyz})$ of lower-diagonal matrices, with ones on the diagonal. Note that $\mathcal{K}_r^+$ is a subgroup of $\mathcal{K}$.

One checks that there are three equivalence classes of maps F at 0 with $\mathrm{d}F = 0$ and $|\mathrm{d}^2F| \neq 0$ (i.e. A_1 points) both for $\mathcal{K}^+$- and for $\mathcal{K}_r^+$-equivalence. The normal forms for a versal deformation of these A_1 points (where t is the deformation parameter) are given by (i) $(y, -x^2 - z^2 + t)$, (ii) $(y, x^2 + z^2 + t)$ and (iii) $(y, x^2 - z^2 + t)$.

On the other hand, starting with

$$g = x + \tfrac{1}{2}z^2 + x^2y + ax^3y + \tfrac{1}{5}bx^5 + \tfrac{1}{3}tx^3,$$

we can reduce $F = (\gamma, \beta)$ to one of the normal forms above by $\mathcal{K}_r^+$ coordinate changes: for $a > \frac{2}{3}$, $b < -\frac{7}{6}$, we reduce to (i); for $a < \frac{2}{3}$, $b < -\frac{7}{6}$ to (ii); and for $a > \frac{2}{3}$, $b > -\frac{7}{6}$, to (iii). The three types of A_1 singularities and their deformations are shown in figure 2.

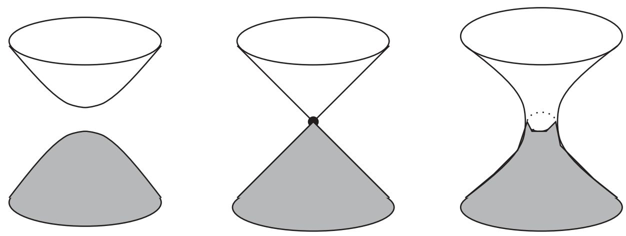

Figure 3. Singularities of Γ at regular points of g: (i) first type of $A_1(1)$ point (centre) and its deformations (left and right).

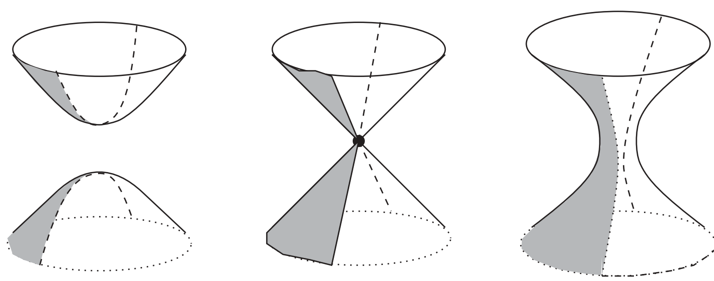

Figure 4. Singularities of Γ at regular points of g: (ii) second type of $A_1(1)$ point (centre) and its deformations (left and right).

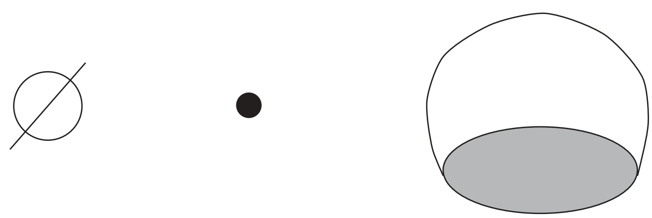

Figure 5. Singularities of Γ at regular points of g: (iii) an $A_1(3)$ point (centre) and its deformations (left and right).

5.2. A_1 points of Γ at regular points of g

At regular points of g, Γ can now have isolated A_1 singularities of signature 1 and 3 (proposition 1.3). The $\mathcal{K}$-versal deformations of the $A_1(m)$ points, $m = 3$ and 1, are given by $x^2 \pm y^2 \pm z^2 + t$, respectively. If

$$g = x + \tfrac{1}{2}tx^2 + \tfrac{1}{4}ax^4 + \tfrac{1}{2}bx^2y^2 + \tfrac{1}{2}cx^2z^2, \quad a, b, c \neq 0,$$

then $j^2\gamma = 3ax^2 + by^2 + cz^2 + t$. One checks that γ is $\mathcal{K}$-equivalent to $x^2+y^2+z^2+t$ for $\mathrm{sgn}(a) = \mathrm{sgn}(b) = \mathrm{sgn}(c)$ and to $x^2 - y^2 - z^2 + t$ otherwise.

The above normal forms for the A_1 singularities of γ do not take into account the decomposition of Γ into Γ^+ and Γ^- regions. On the other hand, $\mathcal{K}^+$-equivalence for the maps $F = (\gamma, \beta)$ does not preserve the singular points of $\gamma^{-1}(0)$. The following refinement of $\mathcal{D}$-equivalence ($\mathcal{D}$-equivalence has been introduced by Mancini and Ruas in [9]) preserves both the set S^+ in the target of F as well as the level sets of γ. Consider the composition $\pi_1 \circ F : \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$, where $\pi_1(u, v) = u$. Then $\mathcal{D}^+$ is the subgroup of $\mathcal{A}$ of germs of diffeomorphisms $k \in \mathrm{Diff}(\mathbb{R}^3, 0)$, $h \in \mathrm{Diff}(\mathbb{R}^2, 0)$

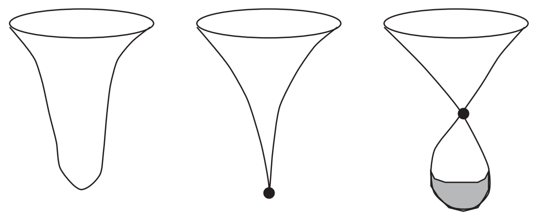

Figure 6. Singularities of Γ at A_2 points of g: an $A_4(2)$ point of Γ (centre) and its deformations (left and right).

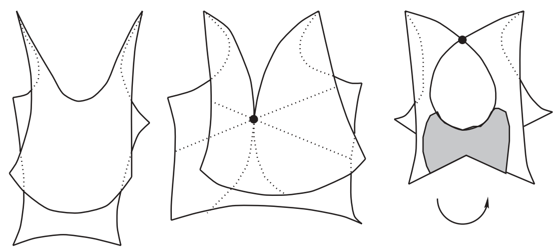

Figure 7. Singularities of Γ at A_2 points of g: an $A_4(0)$ point of Γ (centre) and its deformations (left and right). The surface on the right represents only the 'upper half' of a deformation of the $A_4(0)$ surface: the surface can be completed by turning the 'upper half' upside down, turning it by 90° (as indicated) and by gluing the resulting 'lower half' to the 'upper half'. Figure 8 below shows a sequence of horizontal sections, from top to bottom, of this surface.

and $l \in \mathrm{Diff}(\mathbb{R}, 0)$ having the following properties:

$$h = (h_1, h_2 = v\tilde{h}_2) \quad \text{such that } \tilde{h}_2(0,0) > 0$$

and

$$\mathrm{d}\pi_1(h) + l(\pi_1) = 0.$$

The normal forms for $F = (\gamma, \beta)$, corresponding to A_1 points of γ, for $\mathcal{D}^+$-equivalence (and also for $\mathcal{D}$-equivalence) are given by $(x^2 + \epsilon_1 y^2 + \epsilon_2 z^2, x)$, $\epsilon_i = \pm 1$ (the 'tangent folds' [9, item $2'$ in table 1]). The normal forms are $\mathcal{D}^+$-stable and, by Damon's result, have $\mathcal{K}_e$-codimension 1. There are three normal forms for deformations of F, where γ has an A_1 point at $t = 0$: (i) $(x^2 - y^2 - z^2 + t, x)$; (ii) $(x^2 + y^2 - z^2 + t, x)$; and (iii) $(x^2 + y^2 + z^2 + t, x)$. Finally, starting with the function g above, we obtain $j^2F = (3ax^2 + by^2 + cz^2 + t, 6ax)$: the case $\mathrm{sgn}(b) = \mathrm{sgn}(c) \neq \mathrm{sgn}(a)$ corresponds to (i); $\mathrm{sgn}(b) \neq \mathrm{sgn}(c)$ to (ii); and $\mathrm{sgn}(a) = \mathrm{sgn}(b) = \mathrm{sgn}(c)$ to (iii). The three cases are shown in figures 3, 4 and 5, respectively.

5.3. A_4 points of Γ at A_2 points of g

For isolated points $t \in T$, Γ has an A_4 singularity at a degenerate A_2 saddle of g (proposition 1.3). The signature-2 case is shown in figure 6 and the (geometrically more complicated) signature-0 case in figures 7 and 8. Figure 8 shows a sequence of sections through the surface shown on the right of figure 7. The deformations of the A_4 points, shown on the left and right in figures 6 and 7, follow from lemma 3.2: the

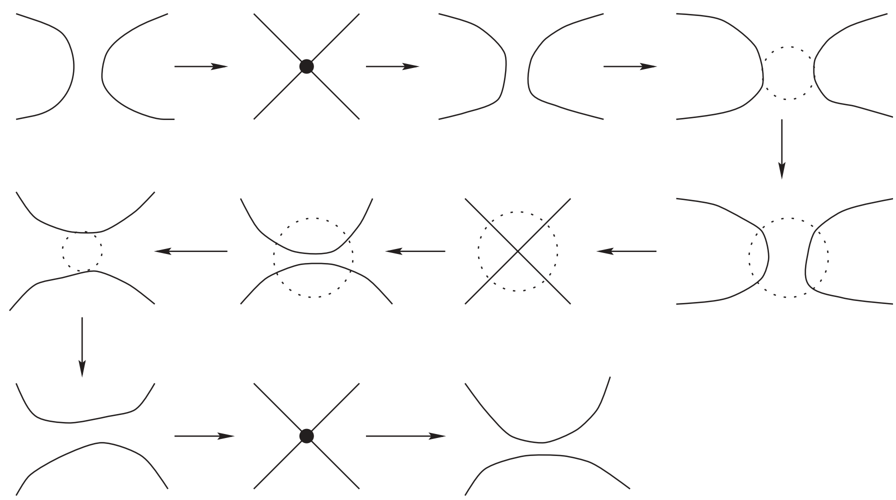

Figure 8. Horizontal sections through the right surface in figure 7, from top to bottom. The dotted circles in sections 4–8 belong to $\beta^{-1}(0)$, and their intersections with the other curves belong to Γ^0. The curve segments in the interiors of the circles belong to Γ^-, the other curve segments to Γ^+. There are five special sections: the two $A_1(1)$ double-cones are the line-crossings marked with solid dots (sections 2 and 10); the two sections where the circles are tangent to curve segments (sections 4 and 8); and the line-crossing corresponding to a horizontal saddle of Γ (section 6). The 'upper half' of Γ, the right surface in figure 7, is the union of all sections up to section 6 through the saddle, the 'lower half' starts with section 6. Notice the 90° turn.

single A_1 point (in the signature-2 case) and the pair of A_1 points (in the signature-0 case) in the deformations on the right in figures 6 and 7 all have signature 1.

The 'colouring' of the figures into white Γ^+ and grey Γ^- regions has been determined from Damon's normal forms for the deformation of an A_2 point of g, but other 'colourings' might arise generically. The deformations of γ and $F = (\gamma, \beta)$ induced by g are not $\mathcal{K}^+$ nor $\mathcal{D}^+$ versal, Damon's normal forms for the deformations of g therefore do not necessarily give all the generic possibilities for colouring the deformations of an A_4 point of Γ. However, one can check that, for any A_2 point 0 of g, the function β is positive on a punctured neighbourhood of 0 and has an A_5 point of signature 2 at 0. The A_4 point of Γ lies therefore in a white Γ^+ region. Furthermore, one checks that the deformations of Γ, corresponding to the non-singular surfaces shown on the left in figures 6 and 7, are also white Γ^+ regions, because the corresponding β is positive in a neighbourhood of 0. But, at this point, we do not know whether, generically, the deformations on the right in figures 6 and 7 can have different colourings from the ones shown there.

Finally, note that Damon gives two normal forms for solutions to the heat equation near an A_2 point of a given signature: one corresponds to the annihilation of critical points of g and the other to the creation. The latter evolution also corresponds to the creation of singular points of Γ and of a new Γ^- region (see the sequences of surfaces by moving from left to right in figures 6 and 7 (recall the discussion in § 3)).

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References

1 T. Banchoff, T. Gaffney and C. McCrory. *Cusps of Gauss mappings*. Pitman Research Notes in Mathematics, vol. 55 (Boston, MA: Pitman, 1982).

2 J. W. Bruce, P. J. Giblin and F. Tari. Families of surfaces: height functions, Gauss maps and duals. In *Real and complex singularities* (ed. W. L. Marar). Pitman Research Notes in Mathematics, vol. 333, pp. 148–178 (Boston, MA: Pitman, 1995).

3 J. W. Bruce, P. J. Giblin and F. Tari. Families of surfaces: focal sets, ridges and umbilics. *Math. Proc. Camb. Phil. Soc.* **125** (1999), 243–268.

4 J. Canny. A computational approach to edge detection. *IEEE PAMI* **8** (1986), 679–698.

5 J. Damon. Local Morse theory for solutions to the heat equation and Gaussian blurring. *J. Diff. Eqns* **115** (1995), 368–401.

6 J. Damon. Generic properties of solutions to partial differential equations. *Arch. Ration. Mech. Analysis* **140** (1997), 353–403.

7 J. Damon. Singularities with scale threshold and discrete functions exhibiting generic properties. Workshop on real and complex singularities (Sao Carlos, 1996). *Mat. Contemp.* **12** (1997), 45–65.

8 J. Damon. Generic structure of two dimensional images under Gaussian blurring. *SIAM J. Appl. Math.* **59** (1998), 97–138.

9 S. Mancini and M. A. S. Ruas. Bifurcations of generic one parameter families of functions on foliated manifolds. *Math. Scand.* **72** (1993), 5–19.

10 J. Martinet. *Singularities of smooth functions and maps*. LMS Lecture Note Series 58 (Cambridge University Press, 1982).

11 I. R. Porteous. The normal singularities of surfaces in $\mathbb{R}^3$. In *Proc. Symposia in Pure Mathematics*, vol. 40, part 2, pp. 379–393 (Providence, RI: American Mathematical Society, 1983).

12 J. H. Rieger. Generic evolutions of edges on families of diffused greyvalue surfaces. *J. Math. Imaging Vision* **5** (1995), 207–217.

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