M-deformations of $\mathcal{A}$-simple germs from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$

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Abstract

All $\mathcal{A}$-simple corank-1 germs from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, where $n \neq 4$, have an M-deformation, that is a deformation in which the maximal numbers of isolated stable singular points are simultaneously present in the image.

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1. *Introduction*

In the present paper we study real deformations of C^∞ map-germs from $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ for which the maximal numbers of isolated stable singular points are simultaneously present in the image which we call M-deformations for short (M as in maximal), furthermore we call the maximal numbers of isolated stable singularities 0-stable invariants. Here "maximal", of course, in comparison with the upper bound given by the corresponding numbers of isolated stable singular points of each given type appearing in a stable perturbation of the complexified germ. For map-germs of target dimension no greater than the source dimension we replace in the definition of a M-deformation image by discriminant. This terminology is analogous to the concept of a M-morsification of a function-germ, which, for example, exist for singularities of type A_k and D_k [**2**, **8**], and also for those of type E_6, E_7 and E_8. For map-germs the following is presently known about M-deformations. The classical result by A'Campo [**1**] and Gusein-Zade [**11**] states that plane curve-germs always have M-deformations, i.e. deformations with δ real double-points (notice that the δ-number is the only 0-stable invariant in this case). More generally, the same is true for map-germs from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$ for any n with the extra hypothesis of $\mathcal{A}$-simplicity (which is not needed for $n = 1$), see [**16**]. M-deformations also exist for all $\mathcal{A}$-simple singular germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, $n \geq p$, of rank $p-1$ [**29**].

For map-germs from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$ the existence of an M-deformation is, by a result of Houston [**13**], equivalent to the existence of a good real perturbation – recall that a good real perturbation of a map-germ f is a real perturbation for which the homology of the image (for $n < p$) or discriminant (for $n \geq p$) coincides with that of its complexification (see [**19**, **5**]). On the other hand, amongst the $\mathcal{A}$-simple singular germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, $n \geq p$, of rank $p-1$ (all of which have an M-deformation by [**29**]) there are many without a good real perturbation, exceptional germs as well as germs forming a series. (A convenient source for finding examples of such germs without good real perturbations is the classification of corank-1 germs $f : \mathbb{R}^2 \to \mathbb{R}^2$, where the real bifurcation sets of all $\mathcal{A}$-simple germs and of all germs of topological $\mathcal{A}_e$-codimension less than 4 – and hence essentially all stable real perturbations of these germs – are known [**25**].) For general (n,p) it is known that every $\mathcal{A}_e$-codimension 1 orbit of singular map-germs $\mathbb{C}^n \to \mathbb{C}^p$ of minimal corank has a real representative which in turn has a good real perturbation [**5**, **14**, **21**]. And it is also known that every real $\mathcal{A}_e$-codimension 1 singular map-germ of minimal corank has an M-deformation [**30**] – notice that the second statement holds for a larger class of map-germs: for example, the complex $\mathcal{A}$-orbit of $f = (x, y^3 + x^2y)$ has representatives $f^{\pm} = (x, y^3 \pm x^2y)$ (distinct over the reals), both having an M-deformation, but only f^+ (the lip) has a good real perturbation (not the beak-to-beak f^-).

The main result of the present paper is that all $\mathcal{A}$-simple corank-1 germs from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, $n \neq 4$, have an M-deformation. We also show that in dimensions $(4,5)$ the open $\mathcal{A}$-orbit in A_3 is $\mathcal{A}$-simple and consists of germs that do not have an M-deformation and also do not have a good real perturbation. This is the first example of an $\mathcal{A}$-simple singular germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ of minimal corank without an M-deformation (notice that singular of minimal corank simply means corank-1 for $p \geq n$ and corank $n - p + 1$ for $n > p$). We discuss the question how the $\mathcal{A}$-simple singular germs of minimal corank and the class of germs having an M-deformation might be related in the final Section 5 of this paper.

The main technical improvement, compared to our study of M-deformations in dimensions (n,p) with $n \geq p$ in [**29**], is a technique for detecting positive $\mathcal{A}$-modality. With this technique one can also obtain the main result in [**29**] (the existence of M-deformations for all $\mathcal{A}$-simple singularities of minimal corank) by a much shorter argument, on the other hand one loses information about the $\mathcal{A}$-classification in this way.

For map-germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^{n+1}, 0$ of corank greater than one the following is known. By a simple counting argument one can show that there are no $\mathcal{A}$-simple map-germs of corank ≥ 2 for $n < 5$. And for $n = 5$ there is an $\mathcal{A}$-simple complex corank-2 germ of $\mathcal{A}_e$-codimension 1 which does not have a real representative with a good real perturbation [**22**] (but this map-germ does have an M-deformation).

2. *Notation and the main results*

A corank-1 germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^{n+1}, 0$ is given by the pre-normal form

$$(x, y) \mapsto (x, g_n(x, y), g_{n+1}(x, y)),$$

where $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_s) : \mathbb{R}^n, S \to \mathbb{R}^{n+1}$, $\tilde{f}(S) =: q$, $\tilde{f}_i(x, y_i) = (x, g_n(x, y_i), g_{n+1}(x, y_i))$, $i = 1, \ldots, s := |S|$, be an s-germ appearing in a deformation of f (here S is a finite set of source points being mapped to the point q in the target). The corank-1 $\mathcal{K}$-classes of germs $\mathbb{R}^n, 0 \to \mathbb{R}^{n+1}, 0$ are those of A_k, with representatives $(x, y^{k+1}, 0)$, and the $\mathcal{K}$-classes of s-germs $A_{(k_1,\ldots,k_s)}$ have an A_{k_i}-singularity at the ith source point. The stable corank-1 multi-germs are those transverse to their $\mathcal{K}$-class $A_{(k_1,\ldots,k_s)}$, where $k_i \geq 0$. Setting $m := \sum_{i=1}^{s} k_i$, the isolated stable (or 0-stable) singularities in dimensions $(n, n+1)$ amongst these are those with $2(m + s - 1) = n + s - 1$. Let $k(s, m) := (k_1, \ldots, k_s)$ be such a "partition" of m with s summands (recall that $k_i \geq 0$).

For map-germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ the number of isolated stable $A_{k(s,m)}$-points in a generic deformation of f, denoted by $r_{k(s,m)}(f)$, can be calculated by dividing the local multiplicity of a certain map-germ $G_{k(s,m)} : \mathbb{C}^{n+s-1}, 0 \to \mathbb{C}^{n+s-1}$ by some overcount factor (see [**27**] and Section 3·2 below).

For real germs f the invariants $r_{k(s,m)}(f)$ are defined by complexifying, but clearly the above geometric interpretation no longer holds: the number $r^{\mathbb{R}}_{k(s,m)}(f_t)$ of real $A_{k(s,m)}$-points in a deformation f_t of f now depends on the choice of deformation. One only has the obvious inequality $r^{\mathbb{R}}_{k(s,m)}(f_t) \leq r_{k(s,m)}(f)$.

We call a real deformation f_t of f an M-deformation, if the maximal numbers $r_{k(s,m)}(f)$ of 0-stable singularities (for all "partitions" $k(s,m)$ of m satisfying $2(m+s-1) = n+s-1$) are simultaneously present in the image of f_t.

The main result on the existence of M-deformations in the present paper is the following

THEOREM 2·1. *All $\mathcal{A}$-simple corank-1 germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{n+1}, 0$, where $n \neq 4$, have an M-deformation.*

In dimensions $(4, 5)$ we have

PROPOSITION 2·2. *The $\mathcal{A}$-simple corank-1 germs $f : \mathbb{R}^4, 0 \to \mathbb{R}^5, 0$ have local multiplicity $m_f(0) \leq 4$. And all such germs of local multiplicity at most three have an M-deformation, but the germs in the open $\mathcal{A}$-orbit in A_3 do not have an M-deformation and furthermore they do not have a good real perturbation.*

Remark 2·3. We are using the usual notion of $\mathcal{A}$-simplicity of mono-germs in the above proposition, i.e. f is $\mathcal{A}$-simple if only mono-germs from a finite number of $\mathcal{A}$-classes appear in any deformation of f. Following Zhitomirskii [**33**], we say that a mono-germ f is fully $\mathcal{A}$-simple if only mono- and multi-germs from a finite number of $\mathcal{A}$-classes appear in any deformation of f. However, it is easy to see that the germ $f = (x_1, x_2, x_3, y^4 + x_1 y, y^6 + y^7 + x_2 y + x_3 y^2)$, whose $\mathcal{A}$-orbit is open in A_3 and which does not have an M-deformation, is also fully simple (it has $\mathcal{A}_e$-codimension 2 and all the $\mathcal{A}_e$-codimension 1 multi-germs appearing in its versal deformation are simple).

We now fix some notation. Let C_n denote the local ring of smooth (or complex-analytic) function germs $f : \mathbb{K}^n, 0 \to \mathbb{K}, 0$ and $\mathcal{M}_n$ its maximal ideal. For the groups $\mathcal{A}$ and $\mathcal{K}$ (of left-right and of contact equivalence, respectively) acting on the space of smooth map-germs and for the tangent spaces to the $\mathcal{A}$- and $\mathcal{K}$-orbits we use the usual notation, such as $T\mathcal{A} \cdot f = tf(\mathcal{M}_n \cdot \theta_n) + wf(\mathcal{M}_p \cdot \theta_p)$ and $T\mathcal{K} \cdot f = tf(\mathcal{M}_n \cdot \theta_n) + f^*\mathcal{M}_p \cdot \theta_f$ (a basic reference for these concepts is the survey on determinacy [**32**] by Wall). For map-germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^{n+1}, 0$ of corank 1 we use source coordinates $(x, y) = (x_1, \ldots, x_{n-1}, y)$ such that $f(x, y) = (x, g_n(x, y), g_{n+1}(x, y))$, and target coordinates $(X_1, \ldots, X_{n+1})$. In describing elements of $T\mathcal{A} \cdot f$ we sometimes use the shorter notation e_i for the target and source vector fields $\partial/\partial X_i$ and $\partial/\partial x_i$ (where $x_n = y$).

3. *Techniques*

The principal techniques in the proofs of our results will be a counting argument for detecting positive $\mathcal{A}$-modality and splitting off 0-stable real singular points from the origin in the target by origin preserving deformations of f to map-germs g whose 0-stable invariants $r_{k(s,m)}$ differ from those of f by at most one. The invariants $r_{k(s,m)}(f)$ are (up to some overcount factor) equal to the multiplicity of certain maps $G_{k(s,m)}$ associated with f whose 0-sets define the closure of the $A_{k(s,m)}$ locus in the source of the s-germ of f. The maps $G_{k(s,m)}$ have been studied in [**27**] (and they have been used before in [**26**] in the study of bifurcation sets). Alternative ways of defining the $A_{k(s,m)}$ locus have been described in [**18**] for corank-1 germs from $\mathbb{C}^n$ to $\mathbb{C}^p$, $p \geq n$, and in [**10**] for corank $n-p+1$ germs from $\mathbb{C}^n$ to $\mathbb{C}^p$, $n \geq p$. In Section 3·2 we recall some properties of the maps $G_{k(s,m)}$ for map-germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^{n+1}, 0$, but first we describe the counting technique.

3·1. *Detecting positive $\mathcal{A}$-modality*

In the proofs we use certain sufficient conditions for positive $\mathcal{A}$-modality that are based on counting arguments in $\mathcal{A}$-tangent spaces filtered by weighted degrees.

Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^p, 0$ be weighted-homogeneous with weights $w = (w_1, \ldots, w_n)$ and weighted degrees $\delta = (\delta_1, \ldots, \delta_p)$, and let $(\theta_n)_s$, $(\theta_p)_s$ and $(\theta_f)_s$ denote the weighted degree s parts of the modules of source-, target-vector fields and vector fields along f, respectively. (Recall that the monomial vector fields $u \cdot e_i \in \theta_n$, $v \cdot e_i \in \theta_p$ and $m \cdot e_i \in \theta_f$, with exponent vectors α_u, α_v and α_m, have weighted degree s if $\langle \alpha_u, w\rangle - w_i$, $\langle \alpha_v, \delta\rangle - \delta_i$ and $\langle \alpha_m, w\rangle - \delta_i$ are equal to s.) For integers $s \geq 0$ consider the linear maps

$$\gamma_s(f) : (\theta_n)_s \oplus (\theta_p)_s \to (\theta_f)_s, \quad (a, b) \mapsto tf(a) - wf(b),$$

of $\mathbb{K}$-vector spaces. Notice that (E_w, E_δ), where $E_w := \sum_i w_i x_i \cdot e_i$ (where $x_n = y$) and $E_\delta := \sum_j \delta_j X_j \cdot e_j$ are the Euler vector fields in source and target, is in the kernel of $\gamma_0(f)$, and we call $e(f) := \gamma_0(f)(E_w, E_\delta)$ the Euler relation. From the C_p-module generated by $e(f)$ we get further relations in the higher weighted degree-s parts ($s > 0$) of $T\mathcal{A} \cdot f$ (notice: $e(f) = 0$ implies $\gamma_s(f)(f^* X^\alpha \cdot E_w, X^\alpha E_\delta) = 0$ for target monomials X^α with $\langle \alpha, \delta\rangle = s$).

Now let f be a corank-1 germ in dimensions $(n, p) = (n, n+1)$ such that $w_n = 1$ and let H_n^r and H_{n-1}^r denote the vector spaces (possibly 0-dimensional) of monomials in x, y and in x, respectively, of weighted degree r, then we can write

$$(\theta_f)_s = H_n^{s+\delta_1} \partial/\partial X_1 \oplus \ldots \oplus H_n^{s+\delta_{n+1}} \partial/\partial X_{n+1}, \quad H_n^r = \bigoplus_{i=0}^{r} y^i H_{n-1}^{r-i}.$$

And (since $w_i = \delta_i$, $i < n$, and $w_n = 1$)

$$(\theta_n)_s = H_n^{s+\delta_1} \partial/\partial x_1 \oplus \ldots \oplus H_n^{s+\delta_{n-1}} \partial/\partial x_{n-1} \oplus H_n^{s+1} \partial/\partial y.$$

Let K_{n+1}^r denote the vector space generated by target monomials X^α of weighted degree r and set $B^r := \{X^\alpha \in K_{n+1}^r : \alpha_n + \alpha_{n+1} > 0\}$, then we have the decomposition

$$(\theta_p)_s = K_{n+1}^{s+\delta_1} \partial/\partial X_1 \oplus \ldots \oplus K_{n+1}^{s+\delta_{n+1}} \partial/\partial X_{n+1}, \quad K_{n+1}^r \cong H_{n-1}^r \oplus B^r$$

(remark: substitute $x_1, \ldots, x_{n-1}$ in H_{n-1}^r by $X_1, \ldots, X_{n-1}$). And we can decompose

$$B^r \cong \bigoplus_{0 < \alpha_n \delta_n + \alpha_{n+1}\delta_{n+1} \leq r} X_n^{\alpha_n} X_{n+1}^{\alpha_{n+1}} H_{n-1}^{r - \alpha_n \delta_n - \alpha_{n+1}\delta_{n+1}}.$$

Finally, note that we have a vector space $e(f) K_{n+1}^s$ of relations of weighted degree s from the Euler relation.

The counting arguments involve comparing the source dimensions of the maps $\gamma_s(f)$, minus $\dim e(f) K_{n+1}^s$, and the target dimensions of these maps. And we can cancel most of the direct summands (isomorphic to $H^i := H_{n-1}^i$ for some i) in the above decompositions and count only the dimensions of the few remaining terms.

The most basic form of the counting argument is then as follows. Suppose the kernel of $\gamma_0(f)$ is 1-dimensional and therefore generated by (E_w, E_δ), and that

$$c_s := \dim(\theta_f)_s - \dim(\theta_n)_s \oplus (\theta_p)_s + \dim e(f)K^s_{n+1} \geq 2,$$

for some $s > 0$, then $\dim(\theta_f)_s/\mathrm{im}\gamma_s(f) \geq 2$ (so that the weighted degree s complete transversal of f is at least 2-dimensional). Then $g := f + c_1m_1 + c_2m_2 + \ldots$ (where the monomial vectors m_i generate the weighted degree s complete transversal) is at least unimodal, because the weighted degree-0 part of any generator $tg(a) - wg(b)$, $(a, b) \in \theta_n \oplus \theta_p$, relating the m_i must lie in the kernel of $\gamma_0(f)$, which is 1-dimensional.

In order to apply this counting argument we typically find a germ f as above – that is, f weighted homogeneous and with essentially unique weights (i.e. unique up to common multiplicative factor) such that $\gamma_0(f)$ has 1-dimensional kernel generated by (E_w, E_δ) – which is "best possible" within its $\mathcal{K}$-orbit in the sense that $\gamma_0(f)$ is surjective, so that the $F^0\mathcal{A}$-orbit of f is open in its $F^0\mathcal{K}$-orbit. (Here F^s refers to the filtration on $C_n^{\times p}$, $p = n + 1$, induced by the weights and weighted degrees of f: for $\mathcal{G} = \mathcal{A}$ or $\mathcal{K}$, $F^s\mathcal{G}$ denotes the subgroup of $\mathcal{G}$ of elements of weighted degree at most s.)

Finally, notice that $\gamma_0(f)$ fails to be surjective if $c_0 \geq 1$. In this case there is a modulus in weighted degree zero (this trivial version of the counting argument is required in the proof of Proposition 4·6 (v)).

3·2. *The maps $G_{k(s,m)}$* [**27**]

To a given corank-1 germ $f : \mathbb{K}^n, 0 \to \mathbb{K}^{n+1}, 0$, $f(x, y) = (x, g_n(x, y), g_{n+1}(x, y))$ we associate map-germs $G_{k(s,m)} : \mathbb{K}^{n+s-1}, 0 \to \mathbb{K}^{2(m+s-1)}$ whose component functions define the closure of $A_{k(s,m)} := A_{(k_1,\ldots,k_s)} \subset \mathbb{K}^{n+s-1}$, $m = \sum_i k_i$, $k_i \geq 0$. For corank-1 germs we can identify the $\mathbb{K}^{n+s-1}$ with coordinates $x, y, \epsilon_2, \ldots, \epsilon_s$ with the space $(\mathbb{K}^n)^s$ of s-fold points in the source $(x, y_1), \ldots, (x, y_s)$, whose f-images coincide, by setting

$$y_1 = y, y_2 = y + \epsilon_2, \ldots, y_s = y + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_s.$$

For $r = n, n + 1$ set

$$g^{(i)}_{r,1} := \partial^i g_r/\partial y^i, \quad i \geq 1$$

and define by iteration for $j = 1, \ldots, s - 1$

$$g^{(0)}_{r,j+1} := \sum_{\alpha \geq k_j+1} g^{(\alpha)}_{r,j}\, \epsilon^{\alpha-k_j-1}_{j+1}/\alpha!, \qquad g^{(i)}_{r,j+1} := \partial^i g^{(0)}_{r,j+1}/\partial\epsilon^i_{j+1}, \quad i \geq 1.$$

Then the component functions $G_1, \ldots, G_{2(m+s-1)}$ of $G_{k(s,m)}$ are given (in this order) by

$$g^{(1)}_{n,1}, \ldots, g^{(k_1)}_{n,1}; \quad g^{(0)}_{n,j}, \ldots, g^{(k_j)}_{n,j} \ (j = 2, \ldots, s)$$

$$g^{(1)}_{n+1,1}, \ldots, g^{(k_1)}_{n+1,1}; \quad g^{(0)}_{n+1,j}, \ldots, g^{(k_j)}_{n+1,j} \ (j = 2, \ldots, s),$$

where $\{g^{(1)}_{r,1}, \ldots, g^{(0)}_{r,1}\}$ denotes the empty set.

For $2(m + s - 1) = n + s - 1$ the 0-sets of the $G_{k(s,m)}$ are zero-dimensional, and the corresponding 0-stable invariant $r_{k(s,m)}(f)$ of f is equal to the local multiplicity of $G_{k(s,m)}$ divided by an overcount factor $c = \prod^t_{i=1}(b_i!)$ counting those permutations of the s source points that permute subsets of b_i points of the same type A_{k_i}, $s = \sum^t_{i=1} b_i$. The following facts will be useful.

First, if $c = 1$ and f is adjacent to g (over $\mathbb{R}$) with $r_{k(s,m)}(f) - r_{k(s,m)}(g) = 1$ then, in an origin-preserving deformation from f to g, one real $A_{k(s,m)}$-point splits off the origin (notice that for real maps f the imaginary $A_{k(s,m)}$-points in the source occur in conjugate pairs). If one can show that all $\mathcal{A}$-simple germs f are adjacent to some germ g whose 0-stable invariants differ from that of f by at most one then one can inductively split off real 0-stable points (with $c = 1$) one by one. This was the main strategy in constructing M-deformations of equidimensional map-germs in [**29**]. We shall see that this strategy also works for $\mathcal{A}$-simple germs from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ for even $n \geq 6$ and for odd $n = 2l + 1$ and l. For odd $n = 2l + 1$ with l even some more work is required (due to the presence of 0-stable invariants with $c > 1$).

Second, if f is weighted homogeneous (or is the weighted homogeneous initial part of a semi-quasihomogeneous germ) with weights $w_1, \ldots, w_n$ and weighted degrees $\delta_1, \ldots, \delta_{n+1}$ then $G_{k(s,m)}$ is weighted homogeneous with weights $w_1, \ldots, w_n, \ldots, w_n$ (i.e. giving the extra variables ϵ_i the weight $w_n = wt(y)$) and weighted degrees

$$\delta_n - iw_n,\ i = 1, \ldots, m + s - 1;\quad \delta_{n+1} - iw_n,\ i = 1, \ldots, m + s - 1,$$

hence (using the generalized Bezout formula):

$$r_{k(s,m)}(f) = \frac{\prod_{i=1}^{m+s-1}(\delta_n - iw_n)(\delta_{n+1} - iw_n)}{cw_n^s \prod_{j=1}^{n-1} w_j}.$$

Finally, notice that (by a result of Mather) f is stable if the $G_{k(s,m)}$ are submersive (then the multi-jet extension of f is transverse to the $\mathcal{K}$-classes $A_{k(s,m)}$ in the space of multi-jets of sufficiently high order). Here we consider all $k(s, m)$ with $m + s \leq m_f(0)$ [**27**], and we let $B_{k(s,m)}$ denote the set of parameters of an unfolding F of f for which the induced unfolding of $G_{k(s,m)}$ is non-submersive at some point $(x, y, \epsilon_2, \ldots \epsilon_s)$. Using the sets $B_{k(s,m)}$ we can determine the codimension-1 components of the bifurcation set of f in an efficient way (without having to know the $\mathcal{A}_e$-codimension-1 classification of multi-germs). This will be used to show that in dimensions $(4, 5)$ there are $\mathcal{A}$-simple corank-1 germs without an M-deformation.

4. *Proofs*

Proposition 4·1. *Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^{n+1}, 0$ be a corank-1 $\mathcal{A}$-finitely determined germ. Suppose that the $\mathcal{A}$-orbit of f is open in its $\mathcal{K}$-orbit. Then $m_f(0) \leq [n/2] + 2$.*

Proof. Our first proof of this statement made use of the necessary and sufficient condition for the openness of an $\mathcal{A}$-orbit in its $\mathcal{K}$-orbit in Theorem 5.1 of [**31**] (see also Proposition 4.6 and the proof of Lemma 4.5 in [**29**]). Alternatively, we can use the counting technique from Section 3·1, as follows.

For even source dimension $n = 2l \geq 6$ Proposition 4·2 below improves the statement that A_{l+2} does not contain open $\mathcal{A}$-orbits to A_{l+1} (using the counting technique). For $n = 4$ we do have an open $\mathcal{A}$-orbit in A_3, but there is no such open $\mathcal{A}$-orbit in A_4, because all $\mathcal{A}^6$-orbits in A_4 lie in the closure of the bi-modal $\mathcal{A}^6$-orbit of

$$(x_1, x_2, x_3, x_1y + x_3y^3 + y^5 + \alpha y^6, x_2y + x_3y^2 + x_1y^4 + \beta y^6).$$

The result therefore follows if we can show that for odd $n = 2l + 1$ there is no open $\mathcal{A}$-orbit in A_{l+2}.

Initially using complete transversal by degree (see [**4**]) and Mather's lemma (up to degree $l+3$) we find the following "best possible" $(l+3)$-jet in A_{l+2}

$$g = (x, x_1 y + \ldots + x_l y^l + y^{l+3}, x_{l+1} y + \ldots + x_{2l} y^l),$$

which is weighted homogeneous for the non-unique choice of weights given by $w = (l + 2, l+1, \ldots, 3, l+2, l+1, \ldots, 3, 1)$. The weighted degree 1 complete transversal for g is generated by $y^{l+4} \cdot e_{2l+2}$, and the "best possible" filtration-1 jet in A_{l+2} is that of

$$f = (x, x_1 y + \ldots + x_l y^l + y^{l+3}, x_{l+1} y + \ldots + x_{2l} y^l + y^{l+4}).$$

Changing weights to $w = (l+2, l+1, \ldots, 3, l+3, l+2, \ldots, 4, 1)$ we see that f is weighted homogeneous (but now the weights are essentially unique). The map $\gamma_0(f)$ is surjective, its kernel is 1-dimensional and we then apply the counting argument described in 3·1. For $s = 1, 2$ there are no relations from the C_p-module generated by $e(f)$. Cancelling direct summands $H^i := H^i_{n-1}$ in the source and target of $\gamma_1(f)$ we see that in the target $(\theta_f)_1$ two H^3 and one H^4 summand remain, while three H^0 summands remain in the source, and $2 \dim H^3 + \dim H^4 - 3 \dim H^0 = 2 + 2 - 3$. Hence $\dim(\theta_f)_1/\mathrm{im}\gamma_1(f) \geq 1$. For $s = 2$, two H^4 and one H^5 summand remain in the target, and three H^0 summands in the source, hence $2 \dim H^4 + \dim H^5 - 3 \dim H^0 = 4 + 2 - 3$ and $\dim(\theta_f)_2/\mathrm{im}\gamma_2(f) \geq 3$. If $\dim(\theta_f)_1/\mathrm{im}\gamma_1(f) = 1$ then the kernel dimension of $\gamma_1(f)$ is zero, in which case $\dim(\theta_f)_2/\mathrm{im}\gamma_2(f) \geq 3$ implies tri-modality at filtration 2. For $\dim(\theta_f)_1/\mathrm{im}\gamma_1(f) \geq 2$ there is already a modulus at filtration 1. □

It follows from the classifications in [**20**] and [**15**] that all $\mathcal{A}$-simple corank-1 germs in dimensions $(2, 3)$ and $(3, 4)$ have an M-deformation (see the concluding remarks in [**29**]). In Section 4·1 we consider corank-1 germs in dimensions $(n, n+1)$ for even $n \geq 6$, in Section 4·2 those for odd $n \geq 5$ and the remaining $n = 4$ case is dealt with in Section 4·3.

4·1. *Even source dimensions $n \geq 6$*

For even source dimensions $n \geq 6$ we can improve the above bound on the local multiplicity as follows.

PROPOSITION 4·2. *Consider germs $f : \mathbb{K}^{2l}, 0 \to \mathbb{K}^{2l+1}, 0$ of even source dimension $n := 2l \geq 6$. There are no open $\mathcal{A}$-orbits in A_{l+1}.*

Proof. Using complete transversals, one shows that the "best possible" $\mathcal{A}^{l+4}$-orbit in A_{l+1} has the representative

$$f := (x, y^{l+2} + x_1 y + \ldots + x_{l-1} y^{l-1}, y^{l+4} + x_l y + \ldots + x_{2l-1} y^l),$$

which is weighted-homogeneous for $w = (l+1, l, \ldots, 3, l+3, l+2, \ldots, 4, 1)$. The map $\gamma_0(f)$ is surjective and its kernel is 1-dimensional, and for $s = 1, 2$ there are no relations from the C_p-module generated by $e(f)$. Cancelling direct summands $H^i := H^i_{n-1}$ in the source and target of $\gamma_1(f)$ we see that in the target $(\theta_f)_1$ one H^3 and one H^4 summand remain, while two H^0 summands remain in the source, and $\dim H^3 + \dim H^4 - 2 \dim H^0 = 1 + 2 - 2$. Hence $\dim(\theta_f)_1/\mathrm{im}\gamma_1(f) \geq 1$. For $s = 2$ one H^4 and one H^5 summand remain in the target, and three H^0 summands in the source, hence $\dim H^4 + \dim H^5 - 3 \dim H^0 = 2 + 2 - 3$ and $\dim(\theta_f)_2/\mathrm{im}\gamma_2(f) \geq 1$. If $\dim(\theta_f)_1/\mathrm{im}\gamma_1(f) = 1$ then the kernel dimension of $\gamma_1(f)$ is zero, in which case $\dim(\theta_f)_2/\mathrm{im}\gamma_2(f) \geq 1$ implies that there is a modulus at filtration 2. For $\dim(\theta_f)_1/\mathrm{im}\gamma_1(f) \geq 2$ there is already a modulus at filtration 1. □

Notice that the above argument fails for $l = 2$. For $l = 2$ we have that $2(l+2) = l+6$, which gives an extra H^0 term in $B^8 \cong X_{2l}H^4 \oplus X_{2l}^2 H^0$. And, in fact, for $l = 2$ the $\mathcal{K}$-orbit A_{l+1} contains $\mathcal{A}$-simple orbits, as we shall see later.

For germs f of local multiplicity $l+1$ there is only one 0-stable invariant, namely $r_{(l)}(f)$. We then have the following

Proposition 4·3. *A corank-1 germ $f : \mathbb{K}^{2l}, 0 \to \mathbb{K}^{2l+1}, 0$ with $m_f(0) = l+1$ is $\mathcal{A}$-simple if and only if the corresponding map $G_{(l)} : \mathbb{K}^{2l}, 0 \to \mathbb{K}^{2l}, 0$ is $\mathcal{K}$-simple. Now consider $\mathbb{K} = \mathbb{R}$: for each real $\mathcal{K}$-simple $G_{(l)}$ there is a real deformation $G^t_{(l)}$ of $G^0_{(l)} = G_{(l)}$ with $m_{G_{(l)}}(0)$ real points in the fibre $(G^t_{(l)})^{-1}(p)$ for some p near 0, and these points correspond to real $A_{(l)}$-points in a deformation f^t of f induced by $G^t_{(l)}$.*

Proof. The argument here is analogous to that for multiplicity $n+1$ germs from n-space to n-space (see [**29**]). Consider the pre-normal form

$$f = (x, y^{l+1} + P_1(x)y + \ldots + P_{l-1}(x)y^{l-1}, P_l(x)y + \ldots + P_{2l-1}(x)y^l + R(x,y)y^{l+1}),$$

then $G_{(l)}$ is (up to a suspension) $\mathcal{K}$-equivalent to the map-germ $P = (P_1, \ldots, P_{2l-1}) : \mathbb{K}^{2l-1}, 0 \to \mathbb{K}^{2l-1}, 0$. The $\mathcal{K}$-simplicity of the $G_{k(s,m)}$ in general follows from the $\mathcal{A}$-simplicity of f (see [**27**]), and the implication $G_{(l)}$ not $\mathcal{K}$-simple $\implies$ f not $\mathcal{A}$-simple follows from the fact that we can lift a deformation P^t of the above map P to a deformation f^t of f (as in Lemma 4.7 of [**29**]). The remaining statements then follow from Lemmas 3.2 and 4.8 in [**29**]. □

Remark 4·4. In the proof of Lemma 4.8 in [**29**] it was stated that there is no complete published reference for the $\mathcal{K}$-classification and the adjacencies of simple real equidimensional germs. However, Chapter 8 of [**23**] contains a very extensive classification of $\mathcal{K}$-orbits and their adjacencies, which in particular include the adjacencies that are required to show that each $\mathcal{K}$-simple germ f can be deformed to another germ g such that $m_f(0) - m_g(0) \leq 1$ (the notation in [**23**] for the $\mathcal{K}$-classes differs from that in [**29**], the latter uses a combination of Giusti's and Mather's notation).

4·2. *Odd source dimensions $n \geq 5$*

Proposition 4·5. *The $\mathcal{K}$-orbit A_{l+1} of corank-1 map-germs $\mathbb{K}^{2l+1}, 0 \to \mathbb{K}^{2l+2}, 0$, $l \geq 2$, contains two types of $\mathcal{A}$-orbits: those that are $\mathcal{A}^{l+2}$-equivalent to*

$$\tilde{f} := (x, y^{l+2} + x_1 y + \ldots + x_l y^l, x_{l+1} y + \ldots + x_{2l} y^l)$$

or to

$$f' := (x, y^{l+2} + x_1 y + \ldots + x_l y^l, x_{l+1} y + \ldots + x_{2l-1} y^{l-1} + x_{2l} y^{l+1} + y^{l+2}),$$

and those that lie in the closure of the $\mathcal{A}$-orbit of

$$f := (x, y^{l+2} + x_1 y + \ldots + x_l y^l, x_{l+1} y + \ldots + x_{2l-1} y^{l-1} + x_{2l} y^{l+1} + y^{l+3}),$$

and the latter have $\mathcal{A}$-modality at least one.

Proof. Using complete transversals and Mather's lemma, one shows that the germs with $(l+2)$-jet equivalent to $\tilde{f}$ or f' are the only ones that do not lie in the closure of $\mathcal{A} \cdot f$, and f is weighted-homogeneous for $w = (l+1, l, \ldots, 2, l+2, l+1, \ldots, 4, 2, 1)$. The map $\gamma_0(f)$ is surjective and its kernel is 1-dimensional. For $s = 1$ we have no relation coming from $e(f)$, and cancelling direct summands $H^i := H^i_{n-1}$ in the source and target of $\gamma_1(f)$ we see that for $l \geq 3$ there remains one H^4 summand in the target $(\theta_f)_1$, while three H^0 summands remain in the source, hence $\dim H^4 - 3\dim H^0 = 5 - 3$. And for $l = 2$ we find $\dim H^4 - 2\dim H^0 = 4 - 2$. In both cases we conclude that f is at least uni-modal. $\square$

Hence we have to consider the map-germs f and f' further.

PROPOSITION 4·6. *Set*

$$f_{l+i} := (x, y^{l+2} + x_1 y + \ldots + x_l y^l, x_{l+1} y + \ldots + x_{2l} y^l + y^{l+i}), l \geq 2.$$

(i) The $\mathcal{A}_e$-codimension-1 germ f_{l+3} is simple, $(l+3)$-determined and has the 0-stable invariants

$$r_{(l-i,i)}(f_{l+3}) = 2, \quad 0 \leq i < l/2, \quad \text{and for even } l, \quad r_{(l/2,l/2)}(f_{l+3}) = 1.$$

(ii) The map-germ f' (in the notation of Prop. 4·5) lies in the closure of the $\mathcal{A}$-orbit of f_{l+3} and has the same 0-stable invariants as f_{l+3}.

(iii) Any germ h with $(l+4)$-jet equivalent to f_{l+4} has the 0-stable invariants

$$r_{(l-i,i)}(h) = r_{(l-i,i)}(f_{l+4}) = 3, \quad 0 \leq i < l/2.$$

For odd l all the 0-stable invariants of h are therefore determined by its $(l+4)$-jet f_{l+4}.

(iv) For even $l = 2m$ any germ with $(l+4)$-jet equivalent to f_{l+4} is either at least uni-modal or lies in the orbit of $g_k := f_{l+4} + x_l^{k-m-1} y^{l+3} \cdot e_{2l+2}$, for some $k \geq m+2$. The g_k have the additional invariant

$$r_{(l/2,l/2)}(g_k) = k - m$$

(the other 0-stable invariants are all equal to three, see (iii) above).

(v) The remaining germs (i.e. those not of type f_{l+i}, $i = 3, 4$, and f') lie in the closure of the $\mathcal{A}^{l+5}$-orbit of $f_{l+5} + a x_l y^{l+3} \cdot e_{2l+2}$ and are at least uni-modal.

Proof. In part (i) the determinacy and the codimension follow from standard calculations, and since f_{l+3} is weighted homogeneous and $\mathcal{A}$-finite we obtain the invariants from the formula in Section 3·2. (And the $\mathcal{A}_e$-codimension-1 germ f_{l+3} is simple, because it is adjacent only to stable corank-1 germs, which are simple.)

In part (ii) we obtain the adjacency $[f'] \to [f_{l+3}]$ from the obvious deformation. The 0-stable invariants of f' can either be calculated directly, alternatively one can check that f' is $\mathcal{A}$-finite and apply the formula in Section 3·2 to the weighted homogeneous germ f'.

For part (iii) we use the definition of the maps $G_{(l-i,i)} : \mathbb{K}^{n+1}, 0 \to \mathbb{K}^{n+1}$, $n = 2l+1$, associated with $h = f_{l+4} + H$ (H of higher filtration) in Section 3·2, which for $i < l/2$ turn out to be semi-quasihomogeneous with $\mathcal{K}$-finite initial part the $G_{(l-i,i)}$ determined by f_{l+4}. (Notice that we do not know *a priori* that this initial part is $\mathcal{K}$-finite, otherwise we could obtain the invariants from the formula in Section 3·2.) Let $\epsilon := \epsilon_2$ and consider the $G_{(l-i,i)} = (G_1, \ldots, G_{2l+2})$ associated with f_{l+4}: the interesting component functions are the $(l+1)$st and the last (the other component functions are simply $G_j = c_j x_j + q$, $j = 1, \ldots, l$ and $G_j = c_j x_{j-1} + q$, $j = l+2, \ldots, 2l+1$, where $c_j \neq 0$ and $q \in \mathcal{M}^2$). The y^{l+2} term in f_{l+4} yields $G_{l+1} = i!((l+2)y + (i+1)\epsilon)$ and the y^{l+4} term yields a cubic form $G_{2l+2}(y, \epsilon)$, which after a linear coordinate change eliminating the y term from G_{l+1}

becomes

$$-\frac{(l+2)(2i-l)(i-1-l)i!}{3(i+1)^2}y^3.$$

For $i < l/2$ the y^3 coefficient is non-zero (in fact, negative), hence $G_{(l-i,i)} \sim_{\mathcal{K}} (x, \epsilon, y^3)$ and all the invariants are three, as desired.

For part (iv), notice that for even $l = 2m$ and $i = l/2$ the y^3 coefficient vanishes. In fact, the linear right coordinate change $\epsilon \mapsto \epsilon - 2y$ that eliminates for $i = m$ the y term from the component function G_{l+1} of $G_{(m,m)}$ makes $G_{(m,m)}$ even in y, for any $h = f_{l+4} + H \cdot e_{2l+2}$, where H has positive filtration with respect to the weights $w = (l+1, l, \ldots, 2, l+3, l+2, \ldots, 4, 1)$ of f_{l+4}. We claim that for any H of even weighted degree the corresponding map $G_{(m,m)}$ is $\mathcal{K}$-equivalent to $(x, \epsilon, 0)$: first, notice that by right coordinate changes all the component functions of $G_{(m,m)}$ – except $G_{l+1} = \epsilon$ and G_{2l+2} – reduce to $x_j + c_j y^{wt(x_j)}$, where $c_j = 0$ for odd $wt(x_j)$ (recall that $G_{(m,m)}$ is even in y). And any monomial vector $M \cdot e_{2l+2}$ of even weight either contains an even power of y (corresponding to an odd power of y in $G_{(m,m)}$) or some power of some x_j of odd weight. Such a monomial vector therefore gives a term in G_{2l+2} that reduces to zero (up to $\mathcal{K}$-equivalence).

Next, we claim that the odd weight monomial vector $x_l^{k-m-1} y^{l+3} \cdot e_{2l+2}$, $k \geq m+2$, makes the corresponding $G_{(m,m)}$ $\mathcal{K}$-equivalent to $(x, \epsilon, y^{2(k-m)})$. We have (after the right change $\epsilon \mapsto \epsilon - 2y$) $G_l = (m-1)!(x_l + (m+1)y^2)$ and $G_{2l+2} = (m+1)! x_l^{k-m-1} y^2$, so we get a $y^{2(k-m)}$ term (with non-zero coefficient) in G_{2l+2} if we eliminate the y^2 term in G_l. So the invariant $r_{(m,m)}(g_k)$ of the deformation g_k of f_{l+4} by the above monomial is $k - m$, as desired.

To conclude the proof of part (iv), we observe that $\gamma_0(f_{l+4})$ is surjective and its 1-dimensional kernel is generated by (E_w, E_δ). If all the even weighted transversals for f_{l+4} are empty and all transversals of odd weight $2(k-m-1)-1$, $k \geq m+2$, are generated by the single elements $x_l^{k-m-1} y^{l+3} \cdot e_{2l+2}$ and if g_k is $\mathcal{A}$-sufficient then the members of the series g_k completely classify the $\mathcal{A}$-finite orbits over the $(l+4)$-jet f_{l+4} (or if the g_k are $\mathcal{A}$-sufficient for all $k \leq K$ then we have a "partial series" up to K). The remaining possibilities lead to orbits that are not $\mathcal{A}$-simple: (a) if some even $2r$-transversal is non-empty then we either have a modulus at filtration $2r$ (if the transversal is generated by two or more elements) or at the filtration $2r+1$ level (from $G_{(m,m)}$ we know that the $2r+1$ transversal will be non-empty); (b) if the odd $2r+1$ transversals are generated by more than one element we have at least one modulus at this filtration; finally (c) if some g_k is not $\mathcal{A}$-sufficient then some higher filtration transversal will be non-empty, giving a modulus at this higher level.

Finally, for part (v) consider the $\mathcal{A}^{l+5}$-orbits of the family of germs (parameterized by a and b)

$$h := (x, y^{l+2} + x_1 y + \ldots + x_l y^l, x_{l+1} y + \ldots + x_{2l} y^l + a x_l y^{l+3} + b y^{l+5}), l \geq 2,$$

whose closures contain the "remaining orbits" in this proposition (i.e. those not of type f_{l+i}, $i = 3, 4$, and f'). Notice that h is weighted-homogeneous for $w = (l+1, l, \ldots, 2, l+4, l+3, \ldots, 5, 1)$. Cancelling direct summands H^i we get $c_0 = \dim H^4 - \dim H^0 = 1$ (for $l \geq 3$) and $c_0 = \dim H^4 = 1$ (for $l = 2$), which implies that the map $\gamma_0(f)$ fails to be surjective. Hence we have at least one modulus at weighted degree zero. (In fact, there is exactly one modulus at weighted degree zero, see the normal form in the statement of part (v)). $\square$

PROPOSITION 4·7. *Now consider real map-germs $f : \mathbb{R}^{2l+1}, 0 \to \mathbb{R}^{2l+2}, 0$, $l \geq 2$, in A_{l+1}, and let f_{l+i}, f' and g_k be defined as in the previous proposition.*

(i) $f_{l+3} + t \cdot y^{l+1} \cdot e_{2l+2}$, $t < 0$, is an M-deformation of f_{l+3}. And f' can be deformed to f_{l+3} by an origin preserving deformation such that all 0-stable invariants remain constant, hence f' has an M-deformation as well.

(ii) Any h with $(l+4)$-jet equivalent to f_{l+4} can be deformed to f_{l+3} such that one real point of each type $A_{(l-i,i)}$, $0 \leq i < l/2$, splits off the origin. For odd l all such h have therefore an M-deformation.

(iii) For even $l = 2m$, we can deform each g_k, $k > m+2$, to g_{k-1} such that one real $A_{(m,m)}$-point splits off the origin (and such that the remaining invariants remain constant). Finally, we deform g_{m+2} to f_{l+3} in such a way that one real point of each 0-stable type $A_{(l-i,i)}$, $0 \leq i \leq m = l/2$, splits off the origin. Hence all the g_k have an M-deformation as well.

Proof. For part (i) we look at the induced deformations $G^t_{(l-i,i)}$ of the maps $G_{(l-i,i)}$ associated with f_{l+3}, where $i < l/2$ (for odd l) and $i \leq l/2$ (for even l). After the right change $\epsilon \mapsto \epsilon - \frac{l+2}{i+1} y$ that eliminates the y term from G^t_{l+1} we have

$$G^t_{2l+2} = i!\Big(t + \frac{(l+2)(l-i+1)}{2(i+1)} y^2\Big).$$

The sign of the y^2 term is positive for all $i = 0, \ldots, l/2$, hence f^t, $t < 0$, has two real singular points of each type $A_{(l-i,i)}$, $i < l/2$, and (for even l) one additional real $A_{(l/2,l/2)}$ point in the target (coming from a pair of real $A_{l/2}$ points in the source).

For part (ii) we take the obvious origin-preserving deformation from h to f_{l+3}, the remaining statements follow from Lemma 3.2 in [**29**].

For part (iii) the required deformations are also obvious, but we have to choose the appropriate sign for the deformation parameter t to ensure that the $A_{(m,m)}$ points splitting off the origin are real. □

4·3. *Source dimension $n = 4$*

We know already that for even source dimensions $n = 2l$ all the $\mathcal{A}$-simple germs of multiplicity $l+1$ have an M-deformation. But for $n = 4$ there are $\mathcal{A}$-simple orbits of multiplicity 4.

PROPOSITION 4·8. *The open $\mathcal{A}$-orbit in A_3 has the representative*

$$f = (x_1, x_2, x_3, y^4 + x_1 y, y^6 + y^7 + x_2 y + x_3 y^2),$$

is $\mathcal{A}$-simple and has $\mathcal{A}_e$-codimension two. The 0-stable invariants of f are $r_{(1,0,0)}(f) = 3$ and $r_{(2)}(f) = 2$. The bifurcation set of f in the parameter (u,v)-plane (the unfolding is given in the proof below) is homeomorphic to the union of the v-axis, the non-positive part of the u-axis and the cuspidal curve $8u^3 + 27v^2 = 0$ and divides the parameter plane into 5 connected regions (see Figure 1). The numbers of real $A_{(1,0,0)}$- and $A_{(2)}$-points in these regions are 2/2 (in regions II and III), 0/2 (in regions I and IV) and 0/0 (in region V). Hence there is no M-deformation for f.

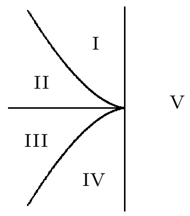

Figure 1: The (topological) bifurcation set $B(f)$

Proof. One checks that $f_0 = (x_1, x_2, x_3, y^4 + x_1 y, y^6 + x_2 y + x_3 y^2)$ is 7-determined and that $f_t = f_0 + ty^7 \cdot e_5$ is a C^0-$\mathcal{A}$-trivial deformation from f_0 to $f = f_1$ (for the 7-determinacy we use the techniques described in [**3**], and note that a deformation of an $\mathcal{A}$-finite weighted homogeneous map-germ f_0 by a term of positive weight is topologically trivial, see [**6**]). By Damon's duality criterion ([**7**], Theorem 5) the negative versal unfolding

$$F = (u, v, x, y^4 + x_1 y + uy^2, y^6 + x_2 y + x_3 y^2 + vy^3)$$

is a C^0-$\mathcal{A}_e$-versal unfolding of f_0, hence the bifurcation sets $B(f)$ and $B(f_0)$ are homeomorphic. (Here $B(f)$ and $B(f_0)$ are the bifurcation sets for the versal unfolding of f and the negative versal unfolding of f_0, respectively, both given by deforming by uy^2 and vy^3 in the nth and $n+1$st component.) The bifurcation set $B(f_0)$ is the union of sets $B_{k(s,m)}$ consisting of (u, v) for which $G_{k(s,m)}$ is non-submersive for some $(x, y, \epsilon_2, \ldots, \epsilon_s)$ (corresponding to the s-tuple of source points $(x, y), (x, y+\epsilon_2), \ldots, (x, y+\epsilon_2+\ldots+\epsilon_s)$). The map $G_{k(s,m)}$ defines the closure of the $A_{k(s,m)}$ set in the source, and we only have to consider "partitions" $k(s, m)$ with $m + s \leq m_{f_0}(0) = 4$ and $k(s, m) \neq (0)$ (see [**27**]). Furthermore, we can discard $\mathcal{K}$-classes $A_{k(s,m)}$ that do not contain s-germs of $\mathcal{A}_e$-codimension less than two. The remaining $A_{k(s,m)}$ can be further reduced using the inclusions

$$\bar{A}_{(1)} \subset \bar{A}_{(0,0)}, \quad \bar{A}_{(2)} \subset \bar{A}_{(1,0)} \subset \bar{A}_{(0,0,0)}$$

and

$$\bar{A}_{(1,1)} \cup \bar{A}_{(2,0)} \subset \bar{A}_{(1,0,0)} \subset \bar{A}_{(0,0,0,0)}$$

obtained by intersecting the sets on the RHS of an inclusion with suitable strata of the diagonal. For example, we have that $G_{(1,0,0)} = G_{(0,0,0,0)}|_{\epsilon_2=0}$ and it is clear that if $G_{(1,0,0)}$ is non-submersive for (u, v) then so is $G_{(0,0,0,0)}$ – hence the above inclusions of sets $A_{k(s,m)}$ yield corresponding inclusions of components $B_{k(s,m)}$ of the bifurcation set. Therefore it is sufficient to determine the three sets $B_{(0,0)}$, $B_{(0,0,0)}$ and $B_{(0,0,0,0)}$. And clearly $G_{(0,0)}$ is a submersion for all (u, v). Hence it is enough to determine the sets $B_{(0,0,0)}$ and $B_{(0,0,0,0)}$: $B_{(0,0,0)} = \{u = 0\}$ and in fact $u = 0$ is the $B_{(2)}$ subset of $B_{(0,0,0)}$. And $B_{(0,0,0,0)} \subset \{v(8u^3 + 27v^2) = 0\}$, more precisely $B_{(0,0,0,0)} = B_{(2,0)} \cup B_{(1,1)}$, where $B_{(2,0)} = \{8u^3 + 27v^2 = 0\}$ and $B_{(1,1)} = \{v = 0, u \leq 0\}$. Now one calculates the numbers of real $A_{(1,0,0)}$- and $A_{(2)}$-points of $F_{(u,v)}$ at a point (u, v) in each of the five regions in the complement of $\mathbb{R}^2 \setminus B(f_0)$. (For example, taking $(u, v) = (1, -10)$ in region II we find that the ideal generated by the component functions of $G_{(1,0,0)}$ has a standard basis containing $h := \epsilon_3^6 - 40\epsilon_3^4 + 32$ and one linear equation for each of the remaining variables, and h has 4 real and a pair of complex conjugate roots. Notice that $c = 2$, hence we get two real $A_{(1,0,0)}$ points. And a standard basis for the ideal generated by $G_{(2)}$ contains $3y^2-5$ and three linear equations for x_1, x_2, x_3, hence we get two real $A_{(2)}$ points.)

Finally, note that f and f_0 can (by the upper semi-continuity of $r_{(2)}(f) = r_{(2)}(f_0) = 2$) only be adjacent to some g with $m_g(0) = 3$ for which the corresponding $G_{(2)}$ is of type A_0 or A_1, and all such g are $\mathcal{A}$-simple. Hence f and f_0 are $\mathcal{A}$-simple too. $\square$

Finally, we complete the proof of Proposition 2·2 by showing that the above germ f (which does not have an M-deformation) also does not have a good real perturbation. First, we require some definitions.

Let $f : X \to Y$ be a continuous mapping of topological spaces. For $1 \leq k < \infty$, denote the k-fold multiple point space of f by

$$D^k(f) = \text{closure}\{(x_1, \ldots, x_k) \in X^k \,/\, f(x_1) = \ldots = f(x_k),\ x_i \neq x_j \text{ if } i \neq j\}.$$

The symmetric group S_k acts on $D^k(f)$ by permuting the factors. Thus S_k acts on $H_\ell(D^k(f); \mathbb{Q})$ by the permutation representation coming from the permutation action on $D^k(f)$. We denote the action of $\sigma \in S_k$ by σ^*. Define the alternating complex

$$Alt_k\, H_\ell(D^k(f); \mathbb{Q}) = \{c \in H_\ell(D^k(f); \mathbb{Q}) \,/\, \forall\, \sigma \in S_k,\ \sigma^*.\, c = \operatorname{sign} \sigma.\, c\}$$

Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ be a map-germ of corank-1 and $f_t : U_t \to \mathbb{C}^{n+1}$ a stable perturbation of f. If f is $\mathcal{A}$-finitely determined then for $2 \leq k \leq n+1$, $D^k(f)$ is an ICIS of dimension $n+1-k$. If t lies in the complement of the bifurcation set of f, $D^k(f_t)$ is smooth and is a Milnor fibre for $D^k(f)$. Then for each k, $D^k(f_t)$ has the homotopy type of a wedge of spheres of middle dimension ([**12**]).

Let Y_t be the image of f_t. It follows from Theorem 2.6 of [**9**] that

$$H_n(Y_t; \mathbb{Q}) \cong \bigoplus_{k=2}^{n+1} Alt_k\, H_{n+1-k}(D^k(f_t); \mathbb{Q})$$

The rank of $H_n(Y_t; \mathbb{Q})$ is called the image Milnor number of f, $\mu_I(f)$.

Suppose that $g : \mathbb{R}^n, 0 \to \mathbb{R}^{n+1}, 0$ is a real analytic map-germ of finite $\mathcal{A}_e$-codimension, with a stable perturbation g_t. Suppose also that the complexification $g_{\mathbb{C},t}$ of g_t is a stable perturbation of the complexification $g_{\mathbb{C}}$ of g. We say that g_t is a good real perturbation of g if $\operatorname{rank} H_n(Y_t; \mathbb{Q}) = \operatorname{rank} H_n(Y_{\mathbb{C},t}; \mathbb{Q})$ where Y_t is the image of g_t and $Y_{\mathbb{C},t}$ is the image of $g_{\mathbb{C},t}$.

Let $f : \mathbb{K}^4, 0 \to \mathbb{K}^5, 0$ as in Proposition 4·8,

$$f = (x_1, x_2, x_3, y^4 + x_1 y, y^6 + y^7 + x_2 y + x_3 y^2).$$

A stable perturbation of f is given by

$$f_t = (x_1, x_2, x_3, y^4 + x_1 y + u(t) y^2, y^6 + y^7 + x_2 y + x_3 y^2 + v(t) y^3)$$

where, for $t \neq 0$, the path $(u(t), v(t))$ lies in the complement of the bifurcation set of f. The defining equations of the sets $D^k(f_t)$ can either be obtained as in [**20**] (where these sets are denoted by $\tilde{D}^k(f_t)$) or by composing the maps $G_{(0,\ldots,0)}$ (k zeros), associated with f_t, with linear coordinate changes of the form $(x, y_1, \ldots, y_k) \mapsto (x, y_1, y_2 - y_1, \ldots, y_k - y_{k-1})$. We have that (recall that $x = (x_1, x_2, x_3)$):

- $D^2(f_t)$ is a contractible smooth surface,
- $D^3(f_t) \cong \{(x, y_1, y_2, y_3) \in \mathbb{K}^6 \,/\, (y_1+y_2)^2 + (y_2+y_3)^2 + (y_1+y_3)^2 + 2u(t) = 0,\ x = 0\}$,

- $D^4(f_t) \cong \{(x, y_1, y_2, y_3, y_4) \in \mathbb{K}^7 \,/\, x = 0,\ (y_1{+}y_2)^2{+}(y_2{+}y_3)^2{+}(y_1{+}y_3)^2{+}2u(t) = 0,\ (y_1 + y_2)(y_2 + y_3)(y_1 + y_3) - v(t) = 0,\ y_1 + y_2 + y_3 + y_4 = 0\}$
- $D^5(f_t)$ is empty.

When $\mathbb{K} = \mathbb{C}$, applying Theorem 4.14 of [**9**], we have that $\mu_I(f) = 3$. We can also obtain this number applying Marar's formulae for the Euler characteristic of Y_t ([**17**]), since $\chi(Y_t) = 1 + \mu_I(f)$.

Suppose $\mathbb{K} = \mathbb{R}$ and $(u(t), v(t))$ in either region II or III in Figure 1. It is easy to see that $Alt_3\, H_2(D^3(f_t); \mathbb{Q}) \cong \mathbb{Q}$. On the other hand, $D^4(f_t)$ has four real branches, each one diffeomorphic to S^1, and with the induced orientation from the ambient space they generate $H_1(D^4(f_t); \mathbb{Q})$. The sum of these classes generate $Alt_4\, H_1(D^4(f_t); \mathbb{Q})$. Therefore $Alt_4\, H_1(D^4(f_t); \mathbb{Q}) \cong \mathbb{Q}$. So $\operatorname{rank} H_4(Y_t; \mathbb{Q}) = 2$ and therefore f_t is not a good real perturbation of f. Since this perturbation is the best possible, f does not have a good real perturbation.

5. *$\mathcal{A}$-simple singular germs of minimal corank and M-deformations*

Examples show that the conditions *singular of minimal corank* and *$\mathcal{A}$-simple* are not necessary for the existence of an M-deformation, but – prior to the counterexample in dimensions $(4, 5)$ in Proposition 4·8 – one could hope that these conditions are sufficient. Notice that the map-germ f in Proposition 4·8 is open in its $\mathcal{K}$-orbit A_3 and that the invariant $r_{(1,0,0)}(f) = 3$ drops by three in any non-trivial deformation of f (because this invariant must be zero for any germ of lower local multiplicity). The phenomenon that a 0-stable invariant drops by more than one in any non-trivial deformation of a given germ can also be observed for germs of non-minimal corank that fail to have an M-deformation. For example, any non-trivial deformation of the $\mathcal{A}$-simple corank-2 germ $\mathrm{II}^1_{2,2}$ from the plane to the plane in [**28**] decreases the double-fold number by two, and this germ does not have an M-deformation.

Constructing deformations decreasing each 0-stable invariant by at most one is one of the main techniques in constructing M-deformations. We conjecture that any $\mathcal{A}$-simple singular map-germ of minimal corank, which is not open in its $\mathcal{K}$-orbit, can be deformed to a germ of lower $\mathcal{A}$-codimension and of the same local multiplicity such that all 0-stable invariants drop by at most one. If this property holds – at least for a given pair of dimensions (n, p) – then the existence of M-deformations can be shown by performing multiplicity preserving deformations decreasing each 0-stable invariant by at most one (splitting off stable singular points one by one and by showing that these points are real) and by analyzing the remaining open $\mathcal{A}$-orbits in each $\mathcal{K}$-orbit (notice that the $\mathcal{A}$-simple corank-1 germ f without an M-deformation in dimensions $(4, 5)$ is open in its $\mathcal{K}$-orbit).

Outside the class of $\mathcal{A}$-simple singular map-germs of minimal corank this property does not hold in general: there are germs (that are not open in their $\mathcal{K}$-orbit) for which such local multiplicity preserving, non-trivial deformations in which all 0-stable invariants decrease by at most one do not exist.

First, consider singular $\mathcal{A}$-simple germs of non-minimal corank. For the corank-2 germs in the series $\mathrm{II}^k_{2,2}$ in dimensions $(2, 2)$ from [**28**] the double-fold numbers drop by at least 2 in any deformation and for $k \geq 2$ the members of this series are not open in their $\mathcal{K}$-orbit (and none of the germs $\mathrm{II}^k_{2,2}$, $k \geq 1$, has an M-deformation).

Finally, we observe that for singular germs of positive $\mathcal{A}$-modality (and minimal corank) again some 0-stable invariant can drop by more than 1 in any local multiplicity preserving deformation. For singular germs of minimal corank and constant local multiplicity a "gap" for some 0-stable invariant i is (empirically) connected with a change of $\mathcal{A}$-modality. Here "gap" means the following: amongst the $\mathcal{A}$-orbits of a given local multiplicity there are some with $i = c$ and some with $i > c$, but none with $i = c + 1$. For example, the possible cusp numbers of corank-1 germs from the plane to the plane of local multiplicity 4 are $2, 3, 4, 6, \ldots$ and as the cusp number jumps from 4 to 6 the $\mathcal{A}$-modality increases from 0 to 1 (and any non-trivial deformation of a unimodal germ of type 19 in the notation of [**24**], which has cusp number 6 and whose orbit lies at the boundary of the $\mathcal{A}$-simple germs, decreases the cusp number by at least 2).

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