Real double-points of deformations of $\mathcal{A}$-simple map-germs from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$

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Abstract

The only stable singularities of a real map-germ $f : \mathbb{R}^n \to \mathbb{R}^{2n}$ are isolated transverse double-points. All $\mathcal{A}$-simple germs f have a deformation with the maximal number $d(f)$ of real double-points (this is a partial generalization to higher n of the result of A'Campo [**1**] and Gusein-Zade [**13**] that all plane curve-germs have a deformation with δ real double points, with the extra hypothesis of $\mathcal{A}$-simplicity). The proof of this result is based on a classification of all $\mathcal{A}$-simple orbits.

1. *Introduction*

A fair number of classifications of $\mathcal{A}$-simple smooth germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^p, 0$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and where smooth means infinitely differentiable or real analytic in the former and complex analytic in the latter case) can be found in the literature: namely in dimensions $(n, p) = (1, 2)$ (see [**5**]), $(1, 3)$ (see [**12**]), $(1, n)$, $n \geq 2$ (up to stable equivalence, see [**2**]), $(2, 3)$ (see [**20**]), $(n, 2)$, $n \geq 2$ (see [**22**, **24**]), $(3, 3)$ (see [**17**]) and $(3, 4)$ (see [**15**]). In the present paper we consider the case $(n, 2n)$, $n \geq 2$, and $\mathbb{K} = \mathbb{R}$ (Theorems 1·1 and 1·2), see Remark 1·3, part (iii) for the necessary modifications in the classification over $\mathbb{C}$. In our classification procedure we construct the $\mathcal{A}$-simple orbits in dimensions $(n + 1, 2n + 2)$ from those in dimensions $(n, 2n)$, $n \geq 3$. The basic case $n = 3$ has been considered by the first named author in [**16**], for completeness we also give the classification for $n = 2$ (using his Transversal program Kirk has apparently carried out an extensive unpublished classification of map-germs from surfaces to 4-space, but we do not know whether this classification contains all $\mathcal{A}$-simple orbits). The classification will be used to show that all such $\mathcal{A}$-simple germs have a real deformation with the maximal number $d(f)$ of real double-points, which – in combination with a result by Houston [**14**] – implies that these germs have a good real deformation (see part (ii) of Remark 1·6). Also, combining the above classification with results on $\mathcal{A}_\Omega$-equivalence [**10**], we obtain the simple map-germs from a n-manifold into a $2n$-manifold with a volume form on it (see part (iv) of Remark 1·3).

It will turn out that all $\mathcal{A}$-simple germs have corank at most one. In order to rule out certain adjacencies of orbits it is useful to have available $\mathcal{A}$-invariants that are upper semi-continuous under deformations. Apart from the $\mathcal{A}$-codimension of a corank-1 germ

$$f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0, \quad f(\mathbf{x}, y) = (\mathbf{x}, g_n(\mathbf{x}, y), \dots, g_{2n}(\mathbf{x}, y)),$$

we consider the upper semi-continuous $\mathcal{A}$-invariants

$$m_f(0) := \dim C_n/f^*\mathcal{M}_{2n} \cdot C_n$$

(the local multiplicity of f), $d(f)$ (the double-point number of f) and $\rho(f)$. The latter invariants are defined as follows: consider the map-germ

$$G_{(0,0)} : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+1}, \quad (\mathbf{x}, y, \bar{y}) \mapsto (G_n, \dots, G_{2n}),$$

where $G_i := (g_i(\mathbf{x}, \bar{y}) - g_i(\mathbf{x}, y))/(\bar{y} - y)$. Then $d(f) := 1/2 \cdot m_{G_{(0,0)}}(0)$ and $\rho(f) := \mathrm{corank} G_{(0,0)}$. Notice: if f and f' are $\mathcal{A}$-equivalent then the corresponding maps $G_{(0,0)}$ and $G'_{(0,0)}$ are $\mathcal{K}$-equivalent (see Lemma 2.3 on equidimensional germs f together with the remarks in Section 3 on germs $f : \mathbb{R}^n \to \mathbb{R}^p$, $n < p$, in [**23**]) – the local multiplicity and the corank are clearly $\mathcal{K}$-invariants of $G_{(0,0)}$, and hence $\mathcal{A}$-invariants of f.

Finally, notice that for $n = 1$ the invariant $d(f)$ is the well-known δ-invariant of the plane curve-germ f. And $d(f)$ corresponds to the invariant $r_{(0,0)}(f)$ in the notation of [**23**] (in [**23**] the numbers of isolated stable singularities of corank-1 map-germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ are denoted by $r_{k(s,m)}(f)$, where $k(s,m) = (k_1, \dots, k_s)$ denotes a s-tuple for which there is an isolated stable s-germ consisting of A_{k_i}-points with $m = \sum_{i=1}^s k_i$; but for $(n,p) = (n, 2n)$ there is only one such invariant, namely $r_{(0,0)}(f) = d(f)$). For real germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ the invariant $d(f)$ measures the number of double points in a stable perturbation of the complexification of f, and $d(f)$ is an upper bound for the number of real double-points appearing in a deformation of the real germ f. Deformations of f with $d(f)$ real double-points (provided such deformations exist) are called maximal or M-deformations, for short. More precisely M-deformations of real map-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ are deformations for which the maximal numbers of real isolated stable singularities are simultaneously present in the image (for $n < p$) or in the discriminant (for $n \geq p$). In [**25**] it has been shown that all $\mathcal{A}$-simple singular germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, $n \geq p$, of minimal corank $n - p + 1$ have an M-deformation. The main result of the present paper is that all $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ also have an M-deformation (Theorem 1·5).

In dimensions $(n, 2n)$ the existence of an M-deformation is, by a result of Houston [**14**], equivalent to the existence of a good real perturbation as defined by Mond and his coworkers (see e.g. [**18**, **8**]). A good real perturbation of a map-germ f is a real perturbation for which the homology of the image (for $n < p$) or discriminant (for $n \geq p$) coincides with that of its complexification (this is analogous to the definition of an M-variety X in real algebraic geometry for which $b_*(X_\mathbb{R}) = b_*(X_\mathbb{C})$, where $X_\mathbb{K}$ is the set of $\mathbb{K}$-points, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and b_* is the sum of the Betti numbers).

In dimensions (n, p), $p < 2n$, it is *a priori* possible that there exist map-germs with M-deformations but without good real deformations and *vice versa*. At present the following facts about singular map-germs of minimal corank $\max(1, n - p + 1)$ can be found in the literature (see also the introduction of [**25**] for further references): (i) all $\mathcal{A}_e$-codimension 1 germs have a good real deformation (but there are $\mathcal{A}$-simple germs of higher $\mathcal{A}_e$-codimension without a good real deformation, see e.g. [**8**, **18**]) and (ii) all $\mathcal{A}$-simple germs with $n \geq p$ have an M-deformation [**25**]. And there are examples of $\mathcal{A}$-simple map-germs of higher corank without an M-deformation [**25**].

We begin the description of the main results of the present paper by stating our classifications of $\mathcal{A}$-simple orbits of map-germs $\mathbb{R}^n \to \mathbb{R}^{2n}$ for $n = 2$ and $n \geq 3$.

THEOREM 1·1. *Any $\mathcal{A}$-simple map-germ $f : \mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ is $\mathcal{A}$-equivalent to one of the following germs:*

$(x, y, 0, 0)$	(immersion)
$(x, xy, y^2, y^{2k+1}),\ k \geq 1$	(type I_k)
$(x, y^2, y^3, x^k y),\ k \geq 2$	(type II_k)
$(x, y^2, y^3 + (\pm 1)^{k+1} x^k y, x^l y),\ l > k \geq 2$	(type $\mathrm{III}_{k,l}$)
$(x, y^2, x^2 y \pm y^{2k+1}, xy^3),\ k \geq 2$	(type IV_k)
$(x, y^2, x^2 y, y^5)$	(type V)
$(x, y^2, x^3 y + y^5, xy^3)$	(type VI)
$(x, xy, xy^2 + y^{3k+1}, y^3),\ k \geq 1$	(type VII_k)
$(x, xy, xy^2 + y^{3k+2}, y^3),\ k \geq 1$	(type VIII_k)
$(x, xy + y^{3k+2}, xy^2, y^3),\ k \geq 1$	(type IX_k)
(x, xy, y^3, y^4)	(type X)
(x, xy, y^3, y^5)	(type XI)
$(x, xy + y^3, xy^2 + y^{2k+1}, y^4),\ k \geq 2$	(type XII_k)
$(x, x^2 y + y^4 \pm y^5, xy^2, y^3)$	(type XIII)
$(x, x^2 y + y^4, xy^2, y^3)$	(type XIV).

Table 1 contains a list of $\mathcal{A}$-invariants of the non-immersive germs of this classification.

Table 1. *$\mathcal{A}$-invariants for the non-immersive germs in Theorems 1·1 and 1·2 : $(*)$ for $n \geq 4$ only 22_2 is simple.*

Type ($n = 2$)	Type ($n \geq 3$)	$\mathrm{cod}(\mathcal{A}, f)$	$m_f(0)$	$d(f)$	$\rho(f)$
I_k	1_k	$k + n$	2	k	1
II_k	2_k	$2k + n - 1$	2	k	2
$\mathrm{III}_{k,l}$	$3_{k,l}$	$l + k + n - 1$	2	l	2
IV_k	4_k	$k + n + 3$	2	$k + 2$	2
V	5	$n + 6$	2	4	2
VI	6	$n + 7$	2	5	2
–	7	$n + 7$	2	4	3
–	8	$n + 8$	2	4	3
–	9	$n + 7$	2	4	3
–	10	$n + 8$	2	4	3
–	11	$n + 8$	2	5	3
–	12	$n + 9$	2	5	3
–	13	$n + 10$	2	5	3
VII_k	14_k	$(n + 1)(k + 1)$	3	$3k$	2
VIII_k	15_k	$(n + 1)(k + 1) + 1$	3	$3k + 1$	2
IX_k	16_k	$(n + 1)(k + 1) + 2$	3	$3k + 2$	2
–	$17_{k,l}$	$k + nl + 4$	3	$3l$	2
–	$18_{k,l}$	$k + nl + 5$	3	$3l + 1$	2
–	$19_{k,l}$	$k + nl + 6$	3	$3l + 2$	2
X	20	$3n + 1$	3	3	2
XI	21	$3n + 2$	3	4	2
XII_k	–	$k + 6$	3	$k + 3$	2
–	22_k $(*)$	$2n + k + 2$	3	$k + 3$	2
–	23	$3n + 2$	3	5	2
XIII	–	9	3	5	3
XIV	–	10	3	5	3

THEOREM 1·2. *Any $\mathcal{A}$-simple map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, $n \geq 3$, is $\mathcal{A}$-equivalent to one of the following germs. (Here $\mathbf{x}$ and $\mathbf{x}y$ denote $x_1, \dots, x_{n-1}$ and $x_1 y, \dots, x_{n-1} y$, respectively, and we set $\epsilon_i := \pm 1$.)*

$(\mathbf{x}, y, 0, \dots, 0)$ (immersion)
$(\mathbf{x}, \mathbf{x}y, y^2, y^{2k+1})$, $k \geq 1$ (type 1_k)
$(\mathbf{x}, x_2 y, \dots, x_{n-1} y, y^2, y^3, x_1^k y)$, $k \geq 2$ (type 2_k)
$(\mathbf{x}, x_2 y, \dots, x_{n-1} y, y^2, y^3 + (\pm 1)^{k+1} x_1^k y, x_1^l y)$, $l > k \geq 2$ (type $3_{k,l}$)
$(\mathbf{x}, x_2 y, \dots, x_{n-1} y, y^2, x_1^2 y \pm y^{2k+1}, x_1 y^3)$, $k \geq 2$ (type 4_k)
$(\mathbf{x}, x_2 y, \dots, x_{n-1} y, y^2, x_1^2 y, y^5)$ (type 5)
$(\mathbf{x}, x_2 y, \dots, x_{n-1} y, y^2, x_1^3 y + y^5, x_1 y^3)$ (type 6)
$(\mathbf{x}, x_3 y, \dots, x_{n-1} y, y^2, x_1^2 y, x_2^2 y, y^3 + x_1 x_2 y)$ (type 7)
$(\mathbf{x}, x_3 y, \dots, x_{n-1} y, y^2, x_1^2 y, x_2^2 y, y^3)$ (type 8)
$(\mathbf{x}, x_3 y, \dots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 - x_2^2) y, y^3 \pm x_2^2 y)$ (type 9)
$(\mathbf{x}, x_3 y, \dots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 - x_2^2) y, y^3)$ (type 10)
$(\mathbf{x}, x_3 y, \dots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3 \pm x_2^2 y)$ (type 11)
$(\mathbf{x}, x_3 y, \dots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3 + x_2^3 y)$ (type 12)
$(\mathbf{x}, x_3 y, \dots, x_{n-1} y, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3)$ (type 13)
$(\mathbf{x}, \mathbf{x}y, x_1 y^2 + y^{3k+1}, y^3)$, $k \geq 1$ (type 14_k)
$(\mathbf{x}, \mathbf{x}y, x_1 y^2 + y^{3k+2}, y^3)$, $k \geq 1$ (type 15_k)
$(\mathbf{x}, x_1 y + y^{3k+2}, x_2 y, \dots, x_{n-1} y, x_1 y^2, y^3)$, $k \geq 1$ (type 16_k)
$(\mathbf{x}, x_1 y, x_2 y + y^{3k+2}, x_3 y, \dots, x_{n-1} y, x_1 y^2 + y^{3l+1}, y^3)$,
$l > k \geq 1$ (type $17_{k,l}$)
$(\mathbf{x}, x_1 y, x_2 y + y^{3k+2}, x_3 y, \dots, x_{n-1} y, x_1 y^2 + y^{3l+2}, y^3)$,
$l > k \geq 1$ (type $18_{k,l}$)
$(\mathbf{x}, x_1 y + y^{3l+2}, x_2 y + y^{3k+2}, x_3 y, \dots, x_{n-1} y, x_1 y^2, y^3)$,
$l > k \geq 1$ (type $19_{k,l}$)
$(\mathbf{x}, \mathbf{x}y, y^3, y^4)$ (type 20)
$(\mathbf{x}, \mathbf{x}y, y^3, y^5)$ (type 21)
$(\mathbf{x}, x_1 y + y^3, x_2 y, \dots, x_{n-1} y, x_1 y^2 + y^{2k+1}, x_2 y^2 + y^4)$,
$k \geq 2$ $(n = 3)$, $k = 2$ $(n \geq 4)$ (type 22_k)
$(\mathbf{x}, x_1 y + y^3, x_2 y, \dots, x_{n-1} y, x_1 y^2 + y^5, y^4)$, (type 23).

Table 1 contains a list of $\mathcal{A}$-invariants of the non-immersive germs of this classification.

Remark 1·3. (i) The normal forms f for $n \geq 3$ in Table 1 that appear in the same row as some normal form $h : \mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ have the form $f = (x_2, \dots, x_{n-1}, x_2 y, \dots, x_{n-1} y, h)$. The last three invariants in the table are the same for all normal forms f in a given row, and for those with $m_f(0) = 2$ the $\mathcal{A}_e$-codimension is also the same (notice that $\mathrm{cod}(\mathcal{A}_e, f) = \mathrm{cod}(\mathcal{A}, f) - n$, for non-immersive f), but for the others the $\mathcal{A}_e$-codimension increases with n.

(ii) All germs in our classification, except for types XII_k, 22_k and 23, are either semi-quasihomogeneous (type XIII) or quasihomogeneous (the remaining germs). (For $n > 3$ the germs 22_k and 23 are "weakly quasihomogeneous" in the sense of [**10**]: for integer weights the weighted degrees are non-negative and the total weighted degree is positive.) If $w_1, \dots, w_n$ are the weights of the variables $x_1, \dots, x_{n-1}, y$ and if $d_1, \dots, d_{2n}$ are the weighted degrees of the component functions of a quasihomogeneous corank-1 germ $f = (\mathbf{x}, g_n, \dots, g_{2n})$ or of the filtration-0 part of a semi-quasihomogeneous germ $f = f_0 + f_+$ then we have the following formula (apply the generalized Bezout formula to $G_{(0,0)}$ with

w_n the weight of the additional variable $\bar{y}$):

$$d(f) = \frac{\prod_{i=n}^{2n}(d_i - w_n)}{2w_n^2 \prod_{j=1}^{n-1} w_j}.$$

(iii) In the classification of $\mathcal{A}$-simple orbits over $\mathbb{C}$ we omit types $9^{\pm}$ and 10 (over $\mathbb{C}$, we have $7 \sim 9^{\pm}$ and $8 \sim 10$) and we identify all real $\mathcal{A}$-orbits in Theorems 1·1 and 1·2 that are distinguished by $\pm$ signs.

(iv) In [**10**] the (non-geometric in the sense of Damon [**9**]) subgroup $\mathcal{A}_\Omega$ of $\mathcal{A}$, in which the target diffeomorphisms are volume-preserving, has been studied. And it has been shown that, over $\mathbb{C}$, the $\mathcal{A}_\Omega$- and the $\mathcal{A}$-orbit of a weakly quasi-homogeneous germ coincide. As a corollary of our classification we obtain the list of $\mathcal{A}_\Omega$-simple orbits over $\mathbb{C}$ consisting of the following map-germs: for dimensions $(2,4)$ the germs I_k to XI and for dimensions $(n, 2n)$, $n \geq 3$, the germs 1_k to 8 and 11 to 21, and, for $n \geq 4$, in addition the germs 22_2 and 23. (Over $\mathbb{R}$ the volume-preserving diffeomorphisms are orientation-preserving, hence different connected components of a given $\mathcal{A}$-orbit can correspond to distinct $\mathcal{A}_\Omega$-orbits.)

(v) In dimensions (n,p), where $p \geq 2n$, a map-germ is $\mathcal{A}$-finite if and only if it is $\mathcal{L}$-finite (see e.g. [**26**]), hence any $\mathcal{A}$-finite germ $f : \mathbb{K}^n, 0 \to \mathbb{K}^{2n}, 0$ in the above classification gives rise to $\mathcal{A}$-finite germs $(f, h) : \mathbb{K}^n, 0 \to \mathbb{K}^{2n+c}$ (for some smooth germ $h : \mathbb{R}^n, 0 \to \mathbb{R}^c, 0$). It would be interesting to obtain the $\mathcal{A}$-simple orbits in dimensions (n,p), for any $p \geq 2n$, from the above classification and induction on c (similar to Arnol'd's classification of curves up to stable equivalence [**2**]).

(vi) Recently Zhitomirskii has introduced the notion of a fully $\mathcal{A}$-simple map-germ: f is fully $\mathcal{A}$-simple if only a finite number of $\mathcal{A}$-classes of multi-germs can appear in a deformation of f (ordinary $\mathcal{A}$-simplicity of mono-germs only requires deformation finiteness with respect to mono-germs), see [**27**]. And he found that the fully $\mathcal{A}$-simple curve-singularities form a small subset of the set of $\mathcal{A}$-simple curve-singularities [**27**]. We do not know the fully $\mathcal{A}$-simple orbits in our classification above.

The following result is an analogue of a key property in the construction of M-deformations of Σ^{n-p+1}-germs $f : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$, in [**25**]. For map-germs in dimension $(n,p) = (n, 2n)$ this property ensures the existence of M-deformations only in certain cases, hence additional techniques are required (see Section 4).

PROPOSITION 1·4. *For any $\mathcal{A}$-simple germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, where $n \geq 2$, there exists another such germ g (of lower $\mathcal{A}$-codimension) such that $[f] \to [g]$ and $d(f) - d(g) \leq 1$.*

Notice that this proposition implies the following lower bound for the $\mathcal{A}_e$-codimension: $\mathrm{cod}(\mathcal{A}_e, f) \geq d(f)$.

THEOREM 1·5. *All $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ have a deformation with $d(f)$ real double-points.*

Remark 1·6. (i) For germs of parameterized plane curves f $(n = 1)$ the statement of Theorem 1·5 holds even without the hypothesis $\mathcal{A}$-simplicity of f (see [**1**, **13**]), but an analogue of Proposition 1·4 does not hold for $n = 1$ (for example, the $\mathcal{A}$-simple curve-germ $f = (y^4, y^6 + y^7)$ with $\delta(f) = 8$ cannot be deformed to any mono-germ g of lower codimension with $\delta(g) = 7$ or 8).

(ii) A result of Houston (Theorem A in [**14**]) states that for the pair of dimensions $(n, 2n)$ the disentanglement of an $\mathcal{A}$-finite map-germ f has the homotopy type of a wedge of circles (roughly speaking, the disentanglement is the intersections of the image of a stable perturbation of f with a suitable neighborhood in the target). Theorem 1·5 therefore implies that any $\mathcal{A}$-simple germ f has a good real deformation as defined in [**18**] (i.e. the homology of the the image of a deformation f_t of f with $d(f)$ real double-points coincides with that of the complexification of f_t).

(iii) For dimensions (n, p), where $n \geq p$, there are examples of $\mathcal{A}$-simple singularities of non-minimal corank that fail to have an M-deformation (and these are also fully $\mathcal{A}$-simple in the sense of [**27**], e.g. the corank-2 germ $f = (x^2 - y^2 + x^3, xy)$ in [**25**], Remark 2.2). For dimensions $(n, 2n)$ we do not know whether there are germs f of corank greater than one (that are necessarily of positive $\mathcal{A}$-modality) and without a deformation with $d(f)$ real double-points. For map-germs $f = (f_1(\mathbf{x}), \ldots, f_{2n}(\mathbf{x})) : \mathbb{C}^n, 0 \to \mathbb{C}^{2n}, 0$, where $\mathbf{x} = (x_1, \ldots, x_n)$, of any corank there is a double-point formula

$$\epsilon(f) = 2d(f) = \dim_{\mathbb{C}} \frac{\langle x_1 - \bar{x}_1, \ldots, x_n - \bar{x}_n \rangle}{\langle f_1(\mathbf{x}) - f_1(\bar{\mathbf{x}}), \ldots, f_{2n}(\mathbf{x}) - f_{2n}(\bar{\mathbf{x}}) \rangle}$$

of Artin and Nagata [**3**] (counting the double points in the source, i.e. twice the number of double points in the target) and the work of Gaffney [**11**] relating $2d(f)$ to other invariants of f. But for the example of a corank-2 germ $f = (x^2, y^2, x^3 - xy, y^3 + 2xy)$ from [**11**] (which is bi-modal and for which $2d(f) = 12$) one calculates that the quotient of ideals appearing in the formula for $\epsilon(f)$ is isomorphic to $C_4/\mathcal{I}$, for some ideal $\mathcal{I}$ that fails to define the source double point set as a complete intersection. For maps f of higher corank $2d(f)$ is therefore, in general, not given by the multiplicity of a $\mathcal{K}$-finite map analogous to $G_{(0,0)}$ in the corank-1 case, which makes the study of M-deformations more difficult.

The plan of the present paper is as follows. In Section 2 we introduce some notation and illustrate some classification techniques using one series in our classification as an example. All such details will be suppressed in the actual classification in Section 3: the results in 3·1 reduce the classification in dimensions $(n, 2n)$, $n \geq 2$, to that in dimensions $(2, 4)$ and $(3, 6)$ of corank-1 and multiplicity 2 and 3, and the latter is described in 3·2 and 3·3. And 3·4 describes some partial adjacencies of $\mathcal{A}$-simple orbits that are sufficient to establish Proposition 1·4. Finally, in Section 4 it is shown that all the $\mathcal{A}$-simple germs f have a deformation with $d(f)$ real double-points.

2. *Notation and techniques*

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a C^∞-germ, the group $\mathcal{A} = \mathrm{Diff}(\mathbb{R}^n, 0) \times \mathrm{Diff}(\mathbb{R}^p, 0)$ acts on the space of smooth germs f as follows: $(h, k) \cdot f = h \circ f \circ k^{-1}$, $(k, h) \in \mathcal{A}$. Let C_n and C_p denote rings of function-germs at the origin in source and target, and let m_n and m_p denote the corresponding maximal ideals. We write $J^k(n, p)$ for the space of kth-order Taylor polynomials at the origin, and $j^k f$ for the k-jet of the map f. Similarly $\mathcal{A}^k = j^k(\mathcal{A})$ denotes k-jets of elements of $\mathcal{A}$. The Lie group $\mathcal{A}^k$ acts smoothly on $J^k(n, p)$, and when we speak of equivalence of k-jets we shall always mean $\mathcal{A}^k$-equivalence. Instead of writing $T_{j^k f(0)} \mathcal{A}^k \cdot j^k f(0)$ we shall write $T\mathcal{A}^k \cdot f$. A map-germ f is said to be k-determined (for some given group of equivalences) if every map g with the same k-jet as f is equivalent to f, in that case any jet $j^l f$ with $l \geq k$ is said to be sufficient.

Let θ_f denote the C_n-module of vector fields over f (i.e. sections of $f^*T\mathbb{R}^p$), which is a free C_n-module of rank p which can be identified with $(C_n)^{\times p}$ (below we shall refer to the jth component of θ_f meaning the jth copy of C_n in $(C_n)^{\times p}$). Set $\theta_n = \theta(1_{\mathbb{R}^n})$ and $\theta_p = \theta(1_{\mathbb{R}^p})$; then the homomorphisms tf and wf are defined as follows:

$$tf : \theta_n \to \theta_f, \quad tf(a) = df \cdot a,$$

(where df is the differential of f), and

$$wf : \theta_p \to \theta_f, \quad wf(b) = b \circ f.$$

Apart from $\mathcal{A}$, we need the groups $\mathcal{A}_1$, $\mathcal{A}_e$ and $\mathcal{K}_e$: $\mathcal{A}_1$ is the subgroup of $\mathcal{A}$ of elements whose 1-jet is the identity, $\mathcal{A}_e$ is the extended pseudo-group of non-origin-preserving diffeomorphisms, and $\mathcal{K}_e$, resp. $\mathcal{K}$, is the (pseudo-) group obtained by allowing invertible $p \times p$ matrices with entries in C_n to act on the left, the right action is the same as for $\mathcal{A}_e$, resp. $\mathcal{A}$. The following tangent spaces are associated with these latter groups: $T\mathcal{A}_e \cdot f = tf(\theta_n) + wf(\theta_p)$ and $T\mathcal{K}_e \cdot f = tf(\theta_n) + f^*m_p \cdot \theta_f$, for $\mathcal{A}$ and $\mathcal{K}$ one multiplies by the first and for $\mathcal{A}_1$ by the second powers of the relevant maximal ideals, respectively.

To find the $\mathcal{A}^k$-orbits over a given $(k-1)$-jet we use a combination of coordinate changes, Mather's Lemma (Lemma 3.1 in [**21**]) and complete transversals (Theorem 2.9 in [**7**]), to determine the order of $\mathcal{A}$-determinacy we use Theorem 2.1 in [**6**] (for introductions to determinacy theory see the surveys in [**4**, **26**]). For many of the more complicated normal forms in our classification it is useful to consider filtrations by weighted degrees, the following example illustrates this technique for the series of germs XII_k. In Section 3 such details will be suppressed. Notice that this example is not even semi-quasihomogeneous (the initial part not being $\mathcal{A}$-finite), but it is still useful to work with weight filtrations using the weights of the initial part.

EXAMPLE. Claim: all the $\mathcal{A}$-finite germs over the 4-jet $f = (x, xy + y^3, xy^2, y^4)$ belong to the series $\mathrm{XII}_k = (x, xy + y^3, xy^2 + y^{2k+1}, y^4)$ for some $k \geq 2$. And XII_k is $(2k+1)$-determined.

In order to prove the claim we assign weights $2, 1$ to x, y so that the component functions $X_1, \dots, X_4$ have weighted degrees $(d_1, \dots, d_4) = (2, 3, 4, 4)$. Then the $e_j := \partial/\partial X_j$ have weighted degree $-d_j$ and $\partial/\partial x$ and $\partial/\partial y$ (which we also denote by e_1, e_2, which should cause no confusion because the source vector fields appear in $tf(\cdot)$ and the target vector fields in $wf(\cdot)$) have the weights -2 and -1. For the above weights let $(\theta_n)_s$, $(\theta_{2n})_s$ and $(\theta_f)_s$ denote the filtration s parts of the modules of source-, target-vector fields and vector fields along f, respectively, and consider the linear map

$$\gamma_s : (\theta_n)_s \oplus (\theta_{2n})_s \to (\theta_f)_s, \quad (a, b) \mapsto tf(a) + wf(b),$$

of $\mathbb{R}$-vector spaces. We claim that γ_s is surjective for all $s = 2r \geq 2$ and that for all $s = 2r + 1 \geq 1$ the cokernel of γ_s is spanned by $y^{2r+5} \cdot \partial/\partial X_3$. This implies that the complete transversals for f of even filtration are empty and those of odd filtration are spanned by the one element above.

For even filtration $s = 2r$ it is easy to see that the elements of $(\theta_f)_s$ in the first, third and fourth component are in $wf((\theta_{2n})_s)$. And modulo these we obtain the elements of the second component from generators $tf(m \cdot e_1)$, where m has weighted degree $s + 3$.

For odd filtration $s = 2r + 1$ the elements of the second component of $(\theta_f)_s$ are in the image of $wf((\theta_{2n})_s)$. For $s = 2r+1 \geq 5$ we get from any element of each of the remaining components of $(\theta_f)_s$, together with $wf((\theta_{2n})_s)$, all the remaining elements of that component. Selecting $y^{2r+5} \cdot \partial/\partial X_3$ as such an element of the third component, we obtain (working modulo elements of the second component) from $tf(y^{2r+3} \cdot e_1)$ and from $tf(y^{2r+2} \cdot e_2)$ such elements in the first and fourth component. Hence $y^{2r+5} \cdot \partial/\partial X_3$ spans the cokernel of γ_s for $s \geq 5$, as desired. For $s = 1, 3$ we have to use some more generators from tf in order to obtain the same conclusion.

Next, it is clear that f and $f_c := f + cy^{2r+5} \cdot \partial/\partial X_3$ are not $\mathcal{A}$-equivalent for $c \neq 0$, because the latter is $\mathcal{A}$-finite but not the former (maybe the easiest way to see this is to check that the double-point numbers are $d(f) = \infty$ and $d(f_c) < \infty$, respectively). The coefficient $c \neq 0$ can be scaled to 1: from the "Euler generator" $e(f_c) := tf_c(2x \cdot e_1 + y \cdot e_2) - wf_c(\sum_{j=1}^{4} d_j X_j \cdot e_j) = (2r+1)cy^{2r+5} \cdot \partial/\partial X_3$ and Mather's lemma we conclude that $\{f_c : c > 0\}$ and $\{f_c : c < 0\}$ lie in a single $\mathcal{A}^{2r+5}$-orbit, hence we can scale c to ± 1, but f_1 and f_{-1} are clearly $\mathcal{A}$-equivalent.

Finally, we have to show that $g_k := (x, xy + y^3, xy^2 + y^{2k+1}, y^4)$, $k \geq 2$, is $(2k+1)$-determined. The local multiplicity of g_k is three, hence it suffices to show that $\mathcal{M}_n^{2k+2} \cdot \theta_{g_k} \subset T\mathcal{A}_1 \cdot g_k + \mathcal{M}_n^{2k+5} \cdot \theta_{g_k}$ or that the three successive r-transversals, $r = 2k+2, 2k+3, 2k+4$, are empty for g_k (we are using transversal by degree here as g_k is no longer weighted homogeneous). But from the weighted-degree transversal calculations above it is clearly sufficient to check that $y^{2k+3} \cdot \partial/\partial X_3 \in T\mathcal{A}_1 \cdot g_k + \mathcal{M}_n^{2k+4} \cdot \theta_{g_k}$: notice that, at the r-jet level, we can use the initial part f of g_k to "push" terms T of filtration greater than $r - d_j$ in the jth component into $\mathcal{M}_n^{r+1} \cdot \theta_{g_k}$ (using a sequence of generators $tg_k(a) + wg_k(b)$ such that $tf(a) + wf(b) = T$ and $\deg(tg_k(a) + wg_k(b) - T) > \deg T$); and for $r = 2k+2, 2k+4$ we obtain all terms of filtration $r - d_j$ from the initial part f (as in the weighted-degree transversal calculations above). For $r = 2k+3$, we can eliminate the terms of filtration 2 between $tg_k(g_k^* X_3 \cdot e_1)$, $tg_k(y^3 \cdot e_2)$, $tg_k(y^4 \cdot e_1)$ and some obvious generators of type $wg_k(a)$ and obtain an additional generator

$$(2ky^{2k+3} - 3xy^{2k+1} - 2x^2y^{2k-1}) \cdot \partial/\partial X_3$$

of filtration $2k-1$, which together with the generators of filtration $2k-1$ from the initial part f of g_k yields (modulo terms of higher filtration, and hence modulo $\mathcal{M}_n^{2k+4} \cdot \theta_{g_k}$) the term $y^{2k+3} \cdot \partial/\partial X_3$, as desired.

3. *Proof of classification*

We will see that all $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, $n \geq 2$, have corank at most one and local multiplicity at most three. Furthermore, for each $\mathcal{A}$-simple germ $h : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, $h(\mathbf{x}, y)$, where $n \geq 3$, there is at most one $\mathcal{A}$-class of $\mathcal{A}$-simple germs $f : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{2n+2}, 0$, with $f = (x_n, x_n y, h(\mathbf{x}, y))$. (In fact, the only pairs h and f, where h is $\mathcal{A}$-simple but not f, come from the series 22_k for $k \geq 3$.) And any $\mathcal{A}$-simple f can be obtained from such an h, and clearly we have $d(f) = d(h)$, $\rho(f) = \rho(h)$ and $m_f(0) = m_h(0)$. (Also, for $m_f(0) = m_h(0) = 2$ there is an isomorphism of normal spaces $N\mathcal{A}_e \cdot f \cong N\mathcal{A}_e \cdot h$, but not for $m_f(0) = m_h(0) = 3$.) Hence the basic case in the classification of germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, $n \geq 3$, is that of $n = 3$. (The classification of map-germs from $\mathbb{R}^3$ to $\mathbb{R}^6$ was the subject of the MSc thesis of the first named author, see [**16**]. Below we shall correct some of the adjacencies described in [**16**], this leads to a slightly bigger list of $\mathcal{A}$-simple orbits). The relation between the classifications for $(2, 4)$ and $(3, 6)$ follows in many cases the same pattern as for $(n, 2n)$, $n \geq 3$, and $(n+1, 2n+2)$, namely for the

$n = 2$ germs h in Table 1 appearing in the same row as the corresponding $n = 3$ germ $f = (x_2, x_2y, h)$ – but here we have some additional exceptional germs (not appearing for $n = 3$ and *vice versa*) which can be classified without too much extra effort.

The remainder of the present section consists of the following parts: in 3·1 we obtain restrictions on the invariants of $\mathcal{A}$-simple orbits in dimensions $(n, 2n)$, $n \geq 2$, in 3·2 and 3·3 we describe the structure of the classifications of $\mathcal{A}$-simple orbits of multiplicity 2 and 3, respectively, and in 3·4 we describe some partial adjacencies of $\mathcal{A}$-orbits (which are sufficient to prove Proposition 1·4 and the simplicity of the orbits in Theorem 1·1 and 1·2).

3·1. *Restrictions on $m_f(0)$ and $\rho(f)$ for $\mathcal{A}$-simple germs f*

We first rule out $\mathcal{A}$-simple germs of corank greater than one.

LEMMA 3·1. *All $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ have corank at most one.*

Proof. It is enough to show that all Σ^2-germs have positive $\mathcal{A}$-modality (then the same is true for all $\overline{\Sigma}^2$-germs). Let $\Sigma^2 J^3(n, 2n)$ denote the space of 3-jets of corank-2 of the form

$$f = (x_1, \dots, x_{n-2}, g_{n-1}, \dots, g_{2n}), \quad g_i = g_i(x_1, \dots, x_n) \in \langle x_{n-1}, x_n \rangle \cap (\mathcal{M}_n^2/\mathcal{M}_n^4).$$

Consider the subspace U of $\Sigma^2 J^3(n, 2n) = U \oplus V$ of dimension $(n + 2)(2n + 3)$ spanned by the following monomial vectors:

$$x_{n-1}^a x_n^b \cdot e_i \quad \text{and} \quad x_j x_k \cdot e_i,$$

where $i = n - 1, \dots, 2n$, $a + b = 2, 3$, $j = 1, \dots, n - 2$ and $k = n - 1, n$. Modulo V, the $\mathcal{A}^3$-tangent space of any element $f \in U \oplus V$ contains the following generators for the subspace U: $wf(X_i \cdot e_j)$ $(n - 1 \leq i, j \leq 2n)$, $tf(x_i \cdot e_j)$ $(i = 1, \dots, n,\ j = n - 1, n)$, $tf(x_{n-1}^a x_n^b \cdot e_j)$ $(a + b = 2,\ j = n - 1, n)$, $tf(x_i \cdot e_j) - wf(X_i \cdot e_j)$ $(1 \leq i, j \leq n - 2)$ and in addition $h_{a,b,j} := tf(x_{n-1}^a x_n^b \cdot e_j) - x_{n-1}^a x_n^b \cdot e_j$ $(a + b = 2,\ j = 1, \dots, n - 2)$, provided $x_{n-1}^a x_n^b \cdot e_j \in T\mathcal{A}^3 \cdot f + V$. (Remark: there are no additional generators $h_{a,b,j}$ with $a + b = 1$, because $tf(x_i \cdot e_j)$ is the only generator for the monomial vector $x_i \cdot e_j \notin U \oplus V$ for $i = n - 1, n$, $j = 1, \dots, n - 2$.) These are at most $2n^2 + 5n + 8$ generators for a subspace of dimension $(n + 2)(2n + 3)$, hence the modality at the 3-jet level is at least $2n - 2 \geq 2$ (for $n \geq 2$). $\square$

Remark 3·2. For corank-2 germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ with $n \geq 10$ already the $\mathcal{A}^2$-orbits are at least uni-modal, as the following argument shows. A complete 2-transversal T for

$$\sigma := j^1 f(x_1, \dots, x_n) = (x_1, \dots, x_{n-2}, 0, \dots, 0)$$

is given by all degree 2 monomials divisible by x_{n-1} or x_n in the last $n + 2$ component functions, hence $\dim T = (n + 2)(2n - 1)$. Setting $f := \sigma + \sum_{m_i \in T} a_i m_i$, we have the following generators for the subspace $T \cap T\mathcal{A}^2 \cdot f$ of $T\mathcal{A}^2 \cdot f$:

$$wf(X_i \cdot e_j), n - 1 \leq i, j \leq 2n; \quad tf(x_i \cdot e_j), 1 \leq i, j \leq n - 2;$$

$$tf(x_i \cdot e_j), i = 1, \dots, n, \quad j = n - 1, n.$$

These are $(n + 2)^2 + (n - 2)^2 + 2n = 2n^2 + 2n + 8$ generators. But f is weighted homogeneous, so we have at least one relation (Euler's relation) between these generators.

Hence $\dim T > 2n^2 + 2n + 7 \iff n \geq 10$, so we get at least one modulus at the 2-jet level for $n \geq 10$.

LEMMA 3·3. *There are no open $\mathcal{A}$-orbits of corank-1 map-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, $n \geq 2$, in the $\mathcal{K}$-orbit A_3 (i.e. with local multiplicity $m_f(0) = 4$).*

Proof. For $n \geq 3$ one checks that over the open $\mathcal{A}^4$-orbit in $A_3 \cap J^4(n, 2n)$, $n \geq 3$, one has the unimodal $\mathcal{A}^5$-orbits

$$(\mathbf{x}, x_1y + y^4, x_2y, \ldots, x_{n-1}y, x_1y^2 + \alpha y^5, x_2y^2 + y^4).$$

For $n = 2$ the $\mathcal{A}^6$-orbit in A_3 of minimal codimension has the representative $f_\alpha = (x, xy, xy^2 + y^5 + y^6, y^4 + \alpha y^6)$. The $\mathcal{A}^6$-codimension of f_α is 10 (and the modular stratum has codimension 9). $\square$

Next consider germs of local multiplicity two.

LEMMA 3·4. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, $n \geq 3$, be $\mathcal{A}$-simple, of local multiplicity $m_f(0) = 2$ and of corank-1. Then (i) $\rho(f) \leq 3$ and (ii)*

$$f \sim_{\mathcal{A}} (x_3, \ldots, x_{n-1}, x_3y, \ldots, x_{n-1}y, h),$$

for some $\mathcal{A}$-simple germ $h : \mathbb{R}^3, 0 \to \mathbb{R}^6, 0$ of corank one and 2-jet $j^2h(x_1, x_2, y) = (x_1, x_2, y^2, g_4, g_5, g_6)$, $g_i \in \mathcal{M}_3^2$.

Proof. We claim that for all $n \geq 4$ there are no simple $\mathcal{A}^3$-orbits over $\sigma := j^2f = (\mathbf{x}, y^2, 0, \ldots, 0)$. The complete 3-transversal T for σ is spanned by the cubic monomials containing odd powers of y in the last n component functions, hence $\dim T = n + n^2(n-1)/2$. Considering for the weighted homogeneous germ $f := \sigma + \sum_{m_i \in T} a_i m_i$ the $1 + (n-1)^2 + n^2$ generators of $T \cap T\mathcal{A}^3 \cdot f \subset T\mathcal{A}^3 \cdot f$, namely $tf(y \cdot e_n)$, $tf(x_i \cdot e_j)$ $(1 \leq i, j \leq n-1)$ and $wf(X_i \cdot e_j)$ $(n+1 \leq i, j \leq 2n)$, and taking into account the Euler relation between them, we see that $\dim T > n^2 + (n-1)^2$ for all $n \geq 4$, as desired.

Statement (i) now follows (for f with 2-jet σ as above the associated $G_{(0,0)}$ has the 1-jet $(y + \bar{y}, 0, \ldots, 0)$ and therefore has corank at least four). For (ii) we observe that the $\mathcal{A}_e$-normal spaces of f and h are isomorphic (see Section 3·2 for more details). $\square$

Remark 3·5. For $f : \mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ with $m_f(0) = 2$ we, of course, have $\rho(f) \leq 2$. Also notice that the $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ with $m_f(0) = 2$ and $\rho(f) = i \leq 3$ can be obtained from an $\mathcal{A}$-simple germ $h : \mathbb{R}^i, 0 \to \mathbb{R}^{2i}, 0$ with $m_h(0) = 2$ by adding to h component functions $x_i, x_iy, \ldots, x_{n-1}, x_{n-1}y$ in y and the additional variables $x_i, \ldots, x_{n-1}$.

Finally, consider germs of local multiplicity three.

LEMMA 3·6. *Suppose $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$, $n \geq 3$, $\mathcal{A}$-simple, of corank-1 and $m_f(0) = 3$. Then $\rho(f) = 2$, i.e. $j^2f \sim_{\mathcal{A}^2} (\mathbf{x}, \mathbf{x}y, 0, 0)$.*

Proof. All $\rho(f) \geq 3$ and $m_f(0) = 3$ germs f lie in the closure of the $\mathcal{A}^2$-orbit of

$$\sigma = (\mathbf{x}, x_1y, \ldots, x_{n-2}y, 0, 0, 0).$$

The complete 3-transversal T for σ is spanned by $y^3, x_{n-1}y^2, x_{n-1}^2y$ in the last $n+1$ component functions, call this part $T_0 \subset T$, and by x_iy^2, $1 \leq i \leq n-2$, in the last three component functions, call this subspace T_1. Then write $f_0 := \sigma + \sum_{m_i \in T_0} a_i m_i$ and

$f = f_0 + f_1 := f_0 + \sum_{m_i \in T_1} b_i m_i$, where f_0 has 0 weighted degree and f_1 has weighted degree one. The Euler generator $e(f)$ together with $tf(x_i \cdot e_n)$ $(i = 1, \dots, n-2)$ are the only generators for the subspace $T_1 \cap T\mathcal{A}^3 \cdot f \subset T\mathcal{A}^3 \cdot f$ of positive filtration, hence $\dim T_1 = 3n - 6 > n - 1 \iff n \geq 3$, as desired. $\square$

Remark 3·7. For $f : \mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ with $m_f(0) = 3$ we could (and actually do) have $\mathcal{A}$-simple germs with $\rho(f) = 3$.

3·2. *Germs of local multiplicity two*

Let $h : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0, h(\mathbf{x}, y)$ be a corank-1 germ with $m_h(0) = 2$, then $x_1, \dots, x_{n-1}, y^2$ are amongst the component functions of h. For $f : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{2n+2}, 0$, where $f = (x_n, x_n y, h(\mathbf{x}, y))$, we have the inclusion

$$\langle x_n \rangle \cdot \theta_f \subset T\mathcal{L} \cdot f \subset T\mathcal{A} \cdot f.$$

Using this inclusion and generators of the type

$$tf(p(\mathbf{x}, y) \cdot e_n), \quad wf(q(X_3, \dots, X_{2n+2}) \cdot e_i), \ i = 1, 2$$

we see that the first two components of $T\mathcal{A}_e \cdot f$ are each equal to C_{n+1}. Then, modulo the first two components of $T\mathcal{A}_e \cdot f$ and $\langle x_n \rangle \cdot \theta_f \subset T\mathcal{A} \cdot f$, the last $2n$ components of $T\mathcal{A}_e \cdot f$ are equal to $T\mathcal{A}_e \cdot h$.

Lemma 3·4 – and the remark following it – imply that the $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ of local multiplicity two and $\rho(f) = i$ can be obtained for $i = 1$ from the series of curve germs

$$(y^2, y^{2k+1}), \quad k \geq 1,$$

for $i = 2$ (and $n \geq 2$) from the $\mathcal{A}$-simple germs $\mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ (setting $x := x_1$)

$$(x, y^2, x^2 y \pm y^{2k+1}, xy^3), \quad (x, y^2, x^2 y, y^5), \quad (x, y^2, y^3, x^k y), \quad k \geq 2$$

$$(x, y^2, y^3 + (\pm 1)^{k+1} x^k y, x^l y), \quad l > k \geq 2, \quad (x, y^2, x^3 y + y^5, xy^3)$$

and for $i = 3$ (and $n \geq 3$) from the $\mathcal{A}$-simple germs $\mathbb{R}^3, 0 \to \mathbb{R}^6, 0$

$$(x_1, x_2, y^2, x_1^2 y, x_2^2 y, y^3 + x_1 x_2 y), \quad (x_1, x_2, y^2, x_1^2 y, x_2^2 y, y^3)$$

$$(x_1, x_2, y^2, x_1 x_2 y, (x_1^2 - x_2^2) y, y^3 \pm x_2^2 y), \quad (x_1, x_2, y^2, x_1 x_2 y, (x_1^2 - x_2^2) y, y^3)$$

$$(x_1, x_2, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3 \pm x_2^2 y), \quad (x_1, x_2, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3 + x_2^3 y)$$

and

$$(x_1, x_2, y^2, x_1 x_2 y, (x_1^2 + x_2^3) y, y^3).$$

The case $\rho(f) = 1$, where $n = 1$, is well-known from the classification of plane curves. Hence consider the

Case $\rho(f) = 2$**,** $n = 2$. Here we have the following $\mathcal{A}^3$-orbits over the 2-jet $(x, y^2, 0, 0)$:

1. $(x, y^2, y^3, x^2 y)$, $\quad$ cod $= 5$
2. $(x, y^2, y^3 \pm x^2 y, 0)$, $\quad$ cod $= 6$
3. $(x, y^2, y^3, 0)$, $\quad$ cod $= 7$

4. $(x, y^2, x^2y, 0)$, $\quad$ cod $= 7$
5. $(x, y^2, 0, 0)$, $\quad$ cod $= 9$.

The orbit in 1. is 3-determined, and the $\mathcal{A}$-orbits over 2. are completely classified by the series $(x, y^2, y^3 \pm x^2y, y^l)$, $l \geq 3$. The $\mathcal{A}$-orbits over the 3-jet in 3. are completely classified by the two series $(x, y^2, y^3, x^k y)$, $k \geq 3$, and $(x, y^2, y^3 + (\pm 1)^{k+1}x^k y, x^l y)$, $l > k \geq 3$. The first series can be combined with the orbit in 1. by taking $k \geq 2$ (type II_k) and the second (doubly-indexed) series can be combined with the series in 2. by taking $l > k \geq 2$ (type $\mathrm{III}_{k,l}$).

Over the 3-jet in 4. we get:

4.1 $(x, y^2, x^2y, xy^{2l+1})$, $l \geq 1$, $\quad$ cod $= 3l + 4$
4.2 (x, y^2, x^2y, y^{2l+1}) , $l \geq 2$, $\quad$ cod $= 3l + 5$
4.3 $(x, y^2, x^2y \pm y^{2l+1}, 0)$, $l \geq 2$, $\quad$ cod $= 3l + 6$.

In the "best case", $l = 1$ in 4.1, we find the $\mathcal{A}$-simple series $(x, y^2, x^2y \pm y^{2k+1}, xy^3)$, $k \geq 2$ (type IV_k), and in the next best case, $l = 2$ in 4.2, we get the $\mathcal{A}$-simple germ (x, y^2, x^2y, y^5) (type V). For $l = 2$ in 4.3 (containing the orbits corresponding to $l \geq 2$ in 4.1, $l \geq 3$ in 4.2 and $l \geq 3$ in 4.3 in its closure) we obtain non-simple orbits lying in the closure of the uni-modal germ $f_\alpha = (x, y^2, x^2y \pm y^5, xy^5 + \alpha y^7)$.

Finally, over the 3-jet in 5. we obtain five $\mathcal{A}^4$-orbits: the first is that of (x, y^2, x^3y, xy^3), which leads to the series $(x, y^2, x^3y + y^{2k+1}, xy^3)$, $k \geq 2$. For $k \geq 3$, this series is non-simple, because it lies in the closure of the orbit of f_α above. The first member of this series, $(x, y^2, x^3y + y^5, xy^3)$ (type VI) has double-point number 5 and lies in the closure of the orbit of (x, y^2, x^2y, y^5). The second $\mathcal{A}^4$-orbit is that of $(x, y^2, x^3y + xy^3, 0)$. This orbit does not contain any $\mathcal{A}$-simple orbits, and the remaining three $\mathcal{A}^4$-orbits lie in its closure. All the $\mathcal{A}$-orbits over the $\mathcal{A}^4$-orbit $(x, y^2, x^3y + xy^3, 0)$ lie in the closure of $g_\alpha = (x, y^2, x^3y + xy^3, y^5 + \alpha x^4 y)$.

Finally, we record for future reference the following uni-modal "bordering germs" (whose orbits contain together all non-simple orbits with $m_f(0) = \rho(f) = 2$ in their closures):

B.1 $f_\alpha = (x, y^2, x^2y \pm y^5, xy^5 + \alpha y^7)$, $\quad d(f_\alpha) = 6$.
B.2 $g_\alpha = (x, y^2, x^3y + xy^3, y^5 + \alpha x^4 y)$, $\quad d(g_\alpha) = 6$.

Next, consider the

Case $\rho(f) = 3$**,** $n = 3$. The 2-jet of such a germ is equivalent to $(x_1, x_2, y^2, 0, 0, 0)$. A complete 3-transversal is spanned by the cubic monomials with y^3 or y as a factor in the last three components. We are only interested in $\mathcal{A}^3$-orbits leading to simple or bordering germs. Assuming that some y^3 coefficient is non-zero (say, the one in the last component), we can reduce to

$$(x_1, x_2, y^2, q(x_1, x_2)y, q'(x_1, x_2)y, y^3 + q''(x_1, x_2)y),$$

where q, q', q'' are functions in x_1, x_2 of degree at least two. By linear left-changes and right-changes in x_1, x_2 we can reduce the fourth and fifth components to x_1x_2y, $(x_1^a \pm x_2^b)y$, $2 \leq a \leq b$, where the $\pm$ cases coincide unless a, b are both even. For $a = b = 2$ we can take in the $+$ case the equivalent form x_1^2y, x_2^2y and reduce $q''(x_1, x_2)y$ to rx_1x_2y with $r = 1$ or 0 (types 7 and 8, respectively). These have codimension 10 and 11, respectively, and double-point number 4.

In the other cases we reduce to

$$f = (x_1, x_2, y^2, x_1x_2y, (x_1^a \pm x_2^b)y, y^3 + rx_2^c y), \quad d(f) = a + b,$$

where $2 \leq a, c \leq b$. For $a \geq 2, b \geq 4$ we obtain non-simple orbits in the closure of the uni-modal bordering germ g_α (type B.2) and for $a \geq 3, b \geq 3$ we obtain non-simple orbits in the closure of the uni-modal bordering germ f_α (type B.1). The remaining orbits are simple: for $a = b = 2$ we obtain in the $-$ case (the $+$ case gave types 7 and 8 above) the types 9 and 10 (with $r = 1$ or 0 and codimensions 10 and 11, respectively, and $c = 2$), and for $a = 2, b = 3$ we get types 11 (with $c = 2$, $r = \pm 1$ and codimension 11), 12 (with $c = 3$, $r = 1$ and codimension 12) and 13 (with $r = 0$ and codimension 13).

Finally, consider the case where all three y^3 coefficients in the 3-transversal of the 2-jet $(x_1, x_2, y^2, 0, 0, 0)$ vanish. This leads to non-simple germs in the closure of the orbit of the following tri-modal bordering germ:

B.3 $$h_{\alpha\beta\gamma} = (x_1, x_2, y^2, x_1^2y + (x_1 + x_2)y^3 + \alpha y^5, x_2^2y + x_1y^3 + \beta y^5, x_1x_2y + \gamma y^5), \qquad d(h_{\alpha\beta\gamma}) = 8,$$

which has codimension 16 (the codimension of the stratum being 13).

This completes the classification of local multiplicity 2 germs. Also notice that any non-simple germ of multiplicity 2 lies in the closures of the orbits of one of three bordering germs (B.1, B.2, B.3, which have double-point numbers 6, 6 and 8), hence any germ of multiplicity 2 with double-point number at most 5 is simple.

3·3. *Germs of local multiplicity three*

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ be an $\mathcal{A}$-simple germ with $m_f(0) = 3$. First consider the

Case $n \geq 3$. From Lemma 3·6 we have that $j^2f \sim (\mathbf{x}, \mathbf{x}y, 0, 0)$ and a complete 3-transversal is spanned by

$$y^3 \cdot e_i \ (i = n, \dots, 2n), \qquad x_iy^2 \cdot e_j \ (i = 1, \dots, n-1;\ j = 2n-1, 2n).$$

1. Provided that one of the y^3 coefficients in the last two component functions of f is non-zero (say, the one in the last component) one can reduce to $j^3f = (\mathbf{x}, \mathbf{x}y, y^2\sum_{i=1}^{n-1} a_ix_i, y^3)$. If some $a_i \neq 0$ (say, a_1) we obtain the $\mathcal{A}^3$-orbit in 1.1 (of codimension $2n+2$), otherwise we obtain the orbit in 1.2 (of codimension $3n+1$):

1.1 $(\mathbf{x}, \mathbf{x}y, x_1y^2, y^3)$, cod $= 2n + 2$
1.2 $(\mathbf{x}, \mathbf{x}y, 0, y^3)$, cod $= 3n + 1$

Both $\mathcal{A}^3$-orbits lead to $\mathcal{A}$-simple orbits, which will be listed later.

2. If the y^3 coefficients in the last two components of f are zero then (because $m_f(0) = 3$) some other y^3 coefficient (say, the one in the nth component) must be non-zero. By direct coordinate changes we reduce

$$(\mathbf{x}, x_1y + y^3, x_2y + a_2y^3, \dots, x_{n-1}y + a_{n-1}y^3, y^2\sum_{i\geq 1} b_ix_i, y^2\sum_{i\geq 1} c_ix_i)$$

to

$$(\mathbf{x}, x_1y + y^3, x_2y, \dots, x_{n-1}y, y^2(Bx_1 + \sum_{i\geq 2} b_ix_i), y^2(Cx_1 + \sum_{i\geq 2} c_ix_i)),$$

where $B := b_1 - a_2b_2 - \ldots - a_{n-1}b_{n-1}$ and $C := c_1 - a_2c_2 - \ldots - a_{n-1}c_{n-1}$. For $B \neq 0$ (or $C \neq 0$) and $c_2 \neq 0$ (or some $c_i \neq 0$, $i > 2$) we reduce to 2.1, for $B \neq 0$ (or $C \neq 0$) and all $c_i = 0$ we reduce to 2.2 and for $B = C = 0$ the "best possible" $\mathcal{A}^3$-orbit (containing the others in its closure) is the one in 2.3.

2.1 $(\mathbf{x}, x_1y + y^3, x_2y, \ldots, x_{n-1}y, x_1y^2, x_2y^2)$, cod $= 2n + 4$
2.2 $(\mathbf{x}, x_1y + y^3, x_2y, \ldots, x_{n-1}y, x_1y^2, 0)$, cod $= 3n + 2$
2.3 $(\mathbf{x}, x_1y + y^3, x_2y, \ldots, x_{n-1}y, x_2y^2, x_3y^2)$, cod $= 2n + 6$

One shows that 2.3 leads to non-simple orbits lying in the closure of the following uni-modal bordering germ

B.4 $k_\alpha = (\mathbf{x}, x_1y + y^3, x_2y, \ldots, x_{n-1}y, x_2y^2 + x_1y^3 + \alpha y^5, x_3y^2 + y^4)$, $d(k_\alpha) = 6$

having codimension $2n + 7$ (codimension of modular stratum being $2n + 6$).

Hence we have to consider 1.1, 1.2, 2.1 and 2.2 further.

For the 3-jet in 1.1 and $k \geq 1$ the $(3k+1)$-transversal is spanned by $y^{3k+1} \cdot e_{2n-1}$, the $(3k+2)$-transversal by $y^{3k+2} \cdot e_i$, $i = n, \ldots, 2n-1$ and the $(3k+3)$-transversal is empty. Hence we can reduce to the following cases

1.1.1 $(\mathbf{x}, \mathbf{x}y, x_1y^2 + y^{3k+1}, y^3)$, cod $= (n+1)(k+1)$
1.1.2 $(\mathbf{x}, \mathbf{x}y, x_1y^2 + y^{3k+2}, y^3)$, cod $= (n+1)(k+1) + 1$
1.1.3 $(\mathbf{x}, x_1y + y^{3k+2}, x_2y, \ldots, x_{n-1}y, x_1y^2, y^3)$, cod $= (n+1)(k+1) + 2$
1.1.4 $(\mathbf{x}, x_1y, x_2y + y^{3k+2}, x_3y, \ldots, x_{n-1}y, x_1y^2, y^3)$, cod $= (n+1)(k+1) + 3$

The first three germs are sufficient and correspond to types 14_k (with d-number $3k$), 15_k (d-number $3k+1$) and 16_k (with d-number $3k+2$), respectively. The last germ in 1.1.4 has infinite d-number, for $l > k$ we find the following transversals for this $(3k+2)$-jet: $y^{3l+1} \cdot e_{2n-1}$ spans the $(3l+1)$-transversal, $y^{3l+2} \cdot e_i$, $i = n, 2n-1$ span the $(3l+2)$-transversal and the $(3l+3)$-transversal is empty. Hence we obtain

1.1.4.1 $(\mathbf{x}, x_1y, x_2y + y^{3k+2}, x_3y, \ldots, x_{n-1}y, x_1y^2 + y^{3l+1}, y^3)$, cod $= k + nl + 4$
1.1.4.2 $(\mathbf{x}, x_1y, x_2y + y^{3k+2}, x_3y, \ldots, x_{n-1}y, x_1y^2 + y^{3l+2}, y^3)$, cod $= k + nl + 5$
1.1.4.3 $(\mathbf{x}, x_1y + y^{3l+2}, x_2y + y^{3k+2}, x_3y, \ldots, x_{n-1}y, x_1y^2, y^3)$, cod $= k + nl + 6$

These germs are sufficient and correspond to types $17_{k,l}$ (with d-number $3l$), $18_{k,l}$ (d-number $3l+1$) and $19_{k,l}$ (with d-number $3l+2$), respectively.

For the 3-jet in 1.2 a 4-transversal is spanned by $y^4 \cdot e_{2n-1}$ and a 5-transversal by $y^5 \cdot e_i$, $i = n, \ldots, 2n-1$. Hence we have the following cases

1.2.1 $(\mathbf{x}, \mathbf{x}y, y^4, y^3)$, cod $= 3n + 1$
1.2.2 $(\mathbf{x}, \mathbf{x}y, y^5, y^3)$, cod $= 3n + 2$
1.2.3 $(\mathbf{x}, x_1y + y^5, x_2y, \ldots, x_{n-1}y, 0, y^3)$, cod $= 3n + 3$
1.2.4 $(\mathbf{x}, \mathbf{x}y, 0, y^3)$, cod $= 4n + 1$

The first two germs are sufficient and are of type 20 (with d-number 3) and 21 (d-number 4). All $\mathcal{A}$-orbits over the jets in 1.2.3 and 1.2.4 lie in the closure of the following uni-modal bordering germ

B.5 $l_\alpha = (\mathbf{x}, x_1y + y^5, x_2y, \ldots, x_{n-1}y, x_1y^5 \pm y^7 + \alpha y^8, y^3)$, $d(l_\alpha) = 6$

For the 3-jet in 2.1 a 4-transversal is spanned by y^4 in the last two component functions and by x_1y^3 in the last (which could also be replaced by $y^4 \cdot e_{n+1}$). Provided the coefficient of $y^4 \cdot e_{2n}$ is non-zero, one can reduce to $j^4 f = (\mathbf{x}, x_1y + y^3, x_2y, \dots, x_{n-1}y, x_1y^2, x_2y^2 + y^4)$. The $\mathcal{A}$-sufficient orbits over $j^4 f$ belong to the series (type 22_k)

$$(\mathbf{x}, x_1y + y^3, x_2y, \dots, x_{n-1}y, x_1y^2 + y^{2k+1}, x_2y^2 + y^4), \quad k \geq 2$$

with $d(22_k) = k + 3$ and codimension $2n + k + 2$. For $n \geq 4$, the deformation of the x_3y component by $t \cdot y^2$ gives for $t \neq 0$ a germ at the origin which lies in the closure of type 6, for $k = 2$ we get type 6 and for $k \geq 3$ a non-simple germ adjacent to type 6 and which lies in the closure of the bordering germ B.2 (here and in the following we mean, of course, the higher-dimensional analogue $(x_2, \dots, x_{n-1}, x_2y, \dots, x_{n-1}y, g_\alpha(x_1, y))$ of the germ g_α in B.2). In case the $y^4 \cdot e_{2n}$ coefficient is zero, we obtain non-simple orbits in the closure of the following uni-modal $\mathcal{A}^4$-orbit:

$$(\mathbf{x}, x_1y + y^3, x_2y, \dots, x_{n-1}y, x_1y^2 + \alpha y^4, x_2y^2 + x_1y^3),$$

which has codimension $2n + 6$ (the modular stratum has codimension $2n + 5$) and, for generic values of α is 5-determined. At the 5-jet level we get the following bi-modal bordering germ

B.6 $f_{\alpha\beta} := (\mathbf{x}, x_1y + y^3, x_2y, \dots, x_{n-1}y, x_1y^2 + \alpha y^4, x_2y^2 + x_1y^3 + \beta y^5)$,
$d(f_{\alpha\beta}) = 6$.

Finally, a complete 4-transversal for the 3-jet in 2.2 is spanned by y^4 in the last two components and by x_1y^3 in the last component. If the y^4 coefficient in the last component is zero, we obtain non-simple orbits in the closure of the bordering germ B.2, otherwise the $\mathcal{A}$-orbits are classified by the series

$$f = (\mathbf{x}, x_1y + y^3, x_2y, \dots, x_{n-1}y, x_1y^2 + y^{2k+1}, y^4), \quad k \geq 2.$$

For $k = 2$ we obtain the simple germ of type 23 (of codimension $3n + 2$ and double-point number 5), for $k \geq 3$ the germs lie in the closure of the orbit of the bordering germ B.2 and hence are non-simple.

This completes the classification of simple orbits of multiplicity 3 for $n \geq 3$. Finally consider the

Case $n = 2$. For $\rho(f) = 2$ the classification of the simple $\mathcal{A}$-orbits over the 2-jet $(x, xy, 0, 0)$ leads at the 3-jet level to the cases (x, xy, xy^2, y^3), $(x, xy, 0, y^3)$, $(x, xy + y^3, xy^2, 0)$ and $(x, xy + y^3, 0, 0)$ that are analogous (or in many cases even identical) to the cases 1.1, 1.2, 2.2 and 2.3, respectively, for $n \geq 3$ above. We omit the details.

For $\rho(f) = 3$ we obtain additional $\mathcal{A}$-simple orbits. Over the 2-jet $(x, 0, 0, 0)$ we consider 3-jets of the form

$$(x, ax^2y + bxy^2, \bar{a}x^2y + \bar{b}xy^2, y^3)$$

(for germs of local multiplicity 3 one of the y^3 coefficients in the last three components must be non-zero, say the one in the third – we then can reduce to the 3-jet above).

Now if (a, b) and $(\bar{a}, \bar{b})$ are linearly independent one can reduce to (x, x^2y, xy^2, y^3) and a 4-transversal is spanned by y^4 in the second and third component. For non-zero $y^4 \cdot e_2$ coefficient one obtains the $\mathcal{A}$-orbits

$$f = (x, x^2y + y^4 \pm y^5, xy^2, y^3), \quad \text{cod} = 9, \quad d(f) = 5, \quad \text{type XIII},$$

and

$$g = (x, x^2y + y^4, xy^2, y^3), \quad \text{cod} = 10, \quad d(g) = 5, \quad \text{type XIV}.$$

For zero $y^4 \cdot e_2$ coefficient we obtain non-simple $\mathcal{A}$-orbits in the closure of the uni-modal $\mathcal{A}^5$-orbit of

B.7 $h_\alpha = (x, x^2y + xy^4 + \alpha y^5, xy^2 + y^4, y^3)$, $d(h_\alpha) = 6$

which has codimension 11 (the modular stratum has codimension 10).

Finally, when $(\bar{a}, \bar{b})$ is a multiple of (a, b), but $a \neq 0 \neq b$, then the 3-jet is equivalent to $j^3f = (x, x^2y + xy^2, 0, y^3)$. The $\mathcal{A}^4$-orbits over j^3f (and those of 3-jets in the closure of the $\mathcal{A}^3$-orbit of j^3f) are non-simple and lie in the closure of the uni-modal $\mathcal{A}^4$-orbit of

B.8 $g_\alpha = (x, x^2y + xy^2, y^4 + \alpha x^3y, y^3)$, $d(g_\alpha) = 6$ $(\alpha \neq -1)$

which has codimension 12 (the modular stratum has codimension 11).

3·4. *Some partial adjacencies*

We can now complete the proofs of Theorems 1·1 and 1·2 (by showing that none of the germs given in these lists is adjacent to some non-simple $\mathcal{A}$-orbit) and of Proposition 1·4, which states that for any $\mathcal{A}$-simple germ there exists another such germ g (of lower $\mathcal{A}$-codimension) such that $[f] \to [g]$ and $d(f) - d(g) \leq 1$. We will use the notation $[f] \to_i [g]$ if $d(f) - d(g) = i$ $(i = 0, 1)$. The adjacencies of type $X \to Y$ in 1. to 6. below are between $\mathcal{A}$-orbits X, Y consisting of germs f with the same invariants $m_f(0)$ and $\rho(f)$, the adjacencies denoted by $X \to (Z)$ indicate that at least one of the invariants is lower for Z. In the case where $d(f) - d(g) = 1$ there are three possibilities: the point splitting off 0 in the target is (i) a real transverse double-point, (ii) a complex transverse double-point (coming from a complex-conjugate pair of source points) or (iii) a "virtual" double-point of type I_1 (for $n = 2$) or 1_1 (for $n \geq 3$). For future reference we will note below some of the adjacencies $[f] \to_1 [g]$ that come from origin-preserving deformations from f to g with one real transverse double-point splitting off 0. This information will be used in Section 4 in the construction of a deformation of f with $d(f)$ real double-points (if such a deformation of f is constructed in a different way then nothing will be said here about the point splitting off 0 – it then can be of any of the three types).

In order to show that all the germs f in Theorems 1·1 and 1·2 are $\mathcal{A}$-simple it suffices to check that they are not adjacent to any of the bordering germs B.1 to B.8, which gives the following restrictions on invariants and dimensions:

B.1, B.2: $m_f(0) \geq 2$, $\rho(f) \geq 2$, $d(f) \geq 6$, $n \geq 2$
B.3: $m_f(0) \geq 2$, $\rho(f) \geq 3$, $d(f) \geq 8$, $n \geq 3$
B.4, B.5, B.6: $m_f(0) \geq 3$, $\rho(f) \geq 2$, $d(f) \geq 6$, $n \geq 3$
B.7, B.8: $m_f(0) \geq 3$, $\rho(f) \geq 3$, $d(f) \geq 6$, $n = 2$.

(If necessary, we again add component functions x_i, x_iy in additional variables x_i to the normal forms of these bordering germs so that the resulting germs have the required source and target dimensions.) Notice that all the exceptional germs in our classification (i.e. those not belonging to some series) have d-number less than six and hence cannot be adjacent to any of these bordering germs.

Now consider the following six sets of germs f from Theorems 1·1 and 1·2 having the same invariants.

1. $m_f(0) = 2$, $\rho(f) = 1$, $n \geq 2$: for $n = 2$

$$\mathrm{I}_k \to_1 \mathrm{I}_{k-1}, \quad \mathrm{I}_1 \to_1 \text{(immersion)}$$

and for $n \geq 3$ the same with 1_k in place of I_k.

None of the above germs can be adjacent to some bordering germ (because $\rho(f) = 1$), and the adjacencies between them follow from the obvious deformations.

2. $m_f(0) = 2$, $\rho(f) = 2$, $n \geq 2$: for $n = 2$

$$\mathrm{II}_k \to_0 (\mathrm{I}_k), \quad \mathrm{III}_{2,3} \to_1 \mathrm{II}_2$$

$$\mathrm{III}_{k,l} \to_1 \mathrm{III}_{k,l-1}, \quad \mathrm{III}_{k,l} \to_0 \mathrm{III}_{k-1,l}$$

$$\mathrm{IV}_2^{\pm} \to_1 \mathrm{III}_{2,3}^{\pm}, \quad \mathrm{IV}_k^{\pm} \to_1 \mathrm{IV}_{k-1}^{\mp}$$

$$\mathrm{V} \to_0 \mathrm{IV}_2, \quad \mathrm{VI} \to_1 \mathrm{V}$$

and for $n \geq 3$ the same with $2_k, \ldots, 6$ in place of $\mathrm{II}_k, \ldots, \mathrm{VI}$.

Here the relevant bordering germs are B.1 and B.2: that none of the above germs lies in the closure of B.1 or B.2 follows from the adjacencies of $\mathcal{A}^r$-orbits ($r = 3, 4, 5$) over the 2-jet $(x, y^2, 0, 0)$ in Section 3·2 (case $\rho(f) = 2$). Notice that a d-constant origin-preserving deformation from II_k to I_k is given by $(x, y^2, y^3 + txy, x^k y)$. And in the origin-preserving deformations from VI to V and from $\mathrm{IV}_k^{\pm}$ to $\mathrm{IV}_{k-1}^{\mp}$ a real transverse double-point splits off 0.

3. $m_f(0) = 2$, $\rho(f) = 3$, $n \geq 3$:

$$8 \to_0 7 \to_0 (5), \quad 10 \to_0 9 \to_0 (4_2)$$

$$13 \to_0 12 \to_0 11 \to_0 (6)$$

None of the above germs can be adjacent to some bordering germ (because $d(f) \leq 5$), and the adjacencies shown follow from the obvious deformations.

4. $m_f(0) = 3$, $\rho(f) = 2$, $n = 2$:

$$\mathrm{VII}_{k+1} \to_1 \mathrm{IX}_k \to_1 \mathrm{VIII}_k \to_1 \mathrm{VII}_k, \quad \mathrm{VII}_1 \to_1 (\mathrm{II}_2)$$

$$\mathrm{XI} \to_1 \mathrm{X} \to_0 \mathrm{VII}_1$$

$$\mathrm{XII}_k \to_1 \mathrm{XII}_{k-1}, \quad \mathrm{XII}_2 \to_1 \mathrm{VIII}_1$$

The relevant bordering germs are B.1 and B.2. Deforming the above multiplicity-3 germs to multiplicity-2 germs we see that the orbits VII_k to XI can only be adjacent to germs in the series I_k and II_k. From the adjacencies of $\mathcal{A}^4$-orbits it is also clear that the germs in the series XII_k are not adjacent to B.1 or B.2 (but from the deformation $(x, xy + y^3 + ty^2, xy^2 + y^{2k+1}, y^4)$ we see that $\mathrm{XII}_k \to \mathrm{IV}_k$). Finally, it is clear that one can make an origin-preserving deformation from XII_k to XII_{k-1} and from XII_2 to

VIII_1 such that a real transverse double-point splits off 0.

5. $m_f(0) = 3$, $\rho(f) = 3$, $n = 2$:

$$\mathrm{XIV} \to_0 \mathrm{XIII} \to_0 (\mathrm{XII}_2)$$

These have d-number equal to 5 and hence are $\mathcal{A}$-simple. And the above adjacencies are easily checked.

6. $m_f(0) = 3$, $\rho(f) = 2$, $n \geq 3$:

$$14_{k+1} \to_1 16_k \to_1 15_k \to_1 14_k, \quad 14_1 \to_0 (3_{2,3})$$

$$19_{k,k+1} \to_1 18_{k,k+1} \to_1 17_{k,k+1} \to_1 16_k$$

$$17_{k,l+1} \to_1 19_{k,l} \to_1 18_{k,l} \to_1 17_{k,l}$$

$$21 \to_1 20 \to_0 14_1$$

$$23 \to_0 22_2 \to_1 15_1$$

and for $n = 3$

$$22_k \to_1 22_{k-1}$$

First, notice that none of the above germs is adjacent to B.4, B.5 or B.6: using the $\mathcal{A}^3$-orbit structure – see 1.1, 1.2, 2.1 to 2.3 in Section 3·3, case $n \geq 3$ – we see that 1.1 leads to types 14_k to $19_{k,l}$, 1.2 to types 20, 21 and B.5, 2.1 to types 22_k and B.6, 2.2 to type 23 and 2.3 to type B.4. The adjacencies of $\mathcal{A}^3$-orbits imply that the series in 1.1 are not adjacent to B.4, B.5 or B.6 and that the series 22_k is not adjacent to B.4 or B.5. From the adjacencies of $\mathcal{A}^4$-orbits inside 2.1 it is also clear that 22_k is not adjacent to B.6. (And recall that the exceptional types 20, 21 and 23 have d-number less than six and therefore are simple.)

Then we observe that none of the above germs is adjacent to B.1 or B.2. These bordering germs have local multiplicity and ρ-invariant equal to two, hence we have to deform the $x_i y$-component functions of the above series by y^2 terms. The series 14_k to $19_{k,l}$ cannot be deformed to B.1 or B.2 in this way, because of the y^3 term in the last component. Finally, consider the members of the series 22_k: deforming, for any $n \geq 3$, the components n or $n+1$ by a y^2 term yields type 4_k (but never B.1, independent of $k \geq 2$), and deforming, for $n \geq 4$, the components $n+2, \ldots, 2n-2$ by y^2 yields type 6 for $k = 2$ and something in the closure of B.2 for any $k \geq 3$. Hence all of the above germs are simple.

There is an origin-preserving deformation from 22_k to 22_{k-1} (analogous to the one from XII_k to XII_{k-1} for $n = 2$) and from 22_2 to 15_1 (analogous to the one from XII_2 to VIII_1) such that in each case a real transverse double-point splits off 0. For the adjacency $14_1 \to 3_{2,3}$ we deform one of the $x_i y$ ($i \geq 2$) components of 14_1 by a y^2 term, the deformations for the remaining adjacencies are evident from the classification.

4. *Deformations with the maximal number of real double-points*

Set $h_i(y, \bar{y}) := (\bar{y}^{i+1} - y^{i+1})/(\bar{y} - y)$. In order to show that all $\mathcal{A}$-simple germs f have deformations with $d(f)$ real double-points we distinguish the following four cases:

(i) For the germs I_k, 1_k, VII_k, 14_k, VIII_k, 15_k, $17_{k,l}$, $18_{k,l}$, X, 20, XI and 21 the associated maps $G_{(0,0)}$ are suspensions of double-point maps $(\sum_i a_i h_i, \sum_j b_j h_j)$ of plane curve-germs $(\sum_i a_i y^{i+1}, \sum_j b_j y^{j+1})$. The result of A'Campo and Gusein-Zade [**1**, **13**] then implies that these have deformations with $d(f)$ real double-points.

(ii) For $\mathrm{III}_{k,l}$, $3_{k,l}$, IV_2 and 4_2 we can simply write down a suitable deformation.

(iii) For the remaining germs, except for IX_k, 16_k and $19_{k,l}$, we apply – as described in Section 3·4 – an origin-preserving deformation to another mono-germ (of lower codimension) having either the same d-number or d-number one less, and in the latter case a real double-point splits off the origin. (This was the strategy for constructing M-deformations in [**25**], but the lemma from this paper stating that the 0-stable singularities splitting off 0 are real does not apply to double-points.)

(iv) Finally, we have the remaining harder cases IX_k, 16_k and $19_{k,l}$, where an origin-preserving deformation to a mono-germ of d-number one less splits off 0 a complex double-point. Here we show that these are adjacent to a certain series N_r of bi-germs having the same d-number and having deformations with d real double-points.

First, consider the second case. For $\mathrm{III}_{k,l}$ (for $n \geq 3$, the case of $3_{k,l}$ is analogous) take the deformation

$$f^t := (x, y^2, y^3 + (\pm 1)^{k+1} x^k y - tcxy, y \prod_{i=1}^{l} (x - tc_i)).$$

Then $G^t_{(0,0)}$ is $\mathcal{K}$-equivalent to $(\bar{y}, y^2 + (\pm 1)^{k+1} x^k - tcx, \prod_{i=1}^{l} (x - tc_i))$. Hence we have for all $t \in (0, \epsilon)$, $\epsilon > 0$, exactly l distinct real pairs of solutions $(x, y, \bar{y}) = (x_i, \pm y_i, 0)$ by taking $c = 0$ and $0 > c_1 > \ldots > c_l$ for odd k and for even k in the $-$ case. For even $k = 2r$ in the $+$ case we take $0 < c_1 < \ldots < c_l$ and $c > \epsilon^{2r-2} c_l^{2r-1}$.

For IV_2 (for $n \geq 3$, the case of 4_2 is again analogous) take the deformation

$$f^t := (x, y^2, x^2 y \pm y^5 + txy, xy^3).$$

For $t \neq 0$ the corresponding map $G^t_{(0,0)} \sim_{\mathcal{K}} (\bar{y}, x^2 \pm y^4 + tx, xy^2)$ has a real solution $(0,0,0)$ of multiplicity 6 corresponding to a singularity of type I_3 and another real solution $(x, y, \bar{y}) = (-t, 0, 0)$ of multiplicity 2 corresponding to a singularity of type I_1. The I_3-point and the I_1-point can be further deformed into 3, respectively 1, real transverse double-points, giving the required 4 real double-points in a deformation of IV_2.

Finally, consider the fourth case. Consider the $\mathcal{A}_e$-classification of bi-germs with immersive component germs. By a left-change we obtain the following prenormal form:

$$F := \{f_1, f_2\} = \{(K_1(\mathbf{x}, y), \ldots, K_n(\mathbf{x}, y), \mathbf{x}, y), (0, \ldots, 0, \mathbf{x}', y')\}.$$

Notice that the $\mathcal{A}_e$-classification of such bi-germs F is given by the $\mathcal{K}_e$-classification of the associated maps $K = (K_1, \ldots, K_n)$ (which measure the order of contact of the immersion germs f_1 and f_2), and that the double-point number of F is given by the dimension of the local algebra Q_K of K. The mono-germs f in the fourth case have invariant $\rho(f) = 2$, hence, by upper semi-continuity, we can expect double-point algebras of corank ≤ 2 in a deformation of f. In fact, we will construct special deformations f_t such that the double-point algebra has at most corank-1 for $t \neq 0$. In the corank-1 case the following series of bi-germs gives the complete classification.

LEMMA 4·1. *Consider the following series N_k of bi-germs of non-transverse double-points from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$ of $\mathcal{A}_e$-codimension $k-1$:*

$$\{f_1, f_2\} := \{(x_1, \dots, x_{n-1}, y^k, 0, \dots, 0, y), (0, \dots, 0, x'_1, \dots, x'_{n-1}, y')\}, \quad k \geq 2.$$

Each N_k has a deformation with $d(N_k) = k$ real double-points.

Proof. The $\mathcal{A}_e$-normal space is spanned by $y, y^2, \dots, y^{k-1}$ in the nth component function of the first component germ f_1. And the deformation $\prod_{i=1}^{k}(y - a_i)$ of the nth component function of f_1 yields for pairwise distinct $a_i \in \mathbb{R}$ exactly k transverse double-points. □

PROPOSITION 4·2. *We have the following adjacencies:*

$$\mathrm{IX}_k, 16_k \to N_{3k+2}, \quad 19_{k,l} \to N_{3l+2},$$

for all $k \geq 1$ or all $l > k \geq 1$, respectively.

Proof. If we exchange the roles of k and l (so that $k > l$) then we can consider (up to a suspension) the same deformation of the associated $G_{(0,0)}$ for all three adjacencies. For 16_k and $19_{l,k}$ we set $x := x_1$. Take the following deformations of the $xy + y^{3k+2}$, xy^2 and y^3 components of the mono-germs on the left of the adjacencies (which are weighted homogeneous for $\mathrm{wt}(x) = 3k+1$ and $\mathrm{wt}(y) = \mathrm{wt}(t) = 1$): $y^3 - t^2 y$ and for odd k (setting $b_0 := 0$)

$$xy^2 + c_1 t^{3k+1} y^2 + c_2 t^{3k-1} y^4 + \dots + c_{k+1} t^2 y^{3k+1},$$

$$xy + y^{3k+2} + b_1 t^{3(k-1)} y^5 + b_2 t^{3(k-3)} y^{11} + \dots + b_{(k-1)/2} t^6 y^{3k-4},$$

and for even k

$$xy^2 + c_1 t^{3k+2} y + c_2 t^{3k-2} y^5 + \dots + c_{k+1} t^2 y^{3k+1},$$

$$xy + y^{3k+2} + b_1 t^{3k} y^2 + b_2 t^{3(k-2)} y^8 + \dots + b_{k/2} t^6 y^{3k-4},$$

where we don't deform in the xy^2 component by powers of y that are divisible by 3 (these would contribute $\mathcal{A}$-trivial deformation terms).

This deformation induces (up to a suspension and $\mathcal{K}$-equivalence) the following deformation (G_1, G_2) of $G_{(0,0)}$, where the h_i are the $\mathbb{Z}_2$-symmetric functions of degree i in $y, \bar{y}$ defined above: $G_1 = h_2 - t^2$ and for odd k

$$G_2 = -h_1 h_{3k+1} + c_1 t^{3k+1} h_1 + c_2 t^{3k-1} h_3 - b_1 t^{3k-3} h_1 h_4 + \dots + c_{k+1} t^2 h_{3k}$$

and for even k

$$G_2 = -h_1 h_{3k+1} + c_1 t^{3k+2} - b_1 t^{3k} h_1^2 + c_2 t^{3k-2} h_4 + \dots + c_{k+1} t^2 h_{3k}.$$

Composing (G_1, G_2) on the right with $(y, \bar{y}) = (u + v, u - v)$, we obtain even functions in v ($\mathbb{Z}_2$ now acts by a reflection in $v = 0$). Hence we can substitute w for v^2, and solve G_1 for $w = t^2 - 3u^2$. Substituting this solution w into G_2 (and scaling $u \mapsto u/2$) we get monic polynomials in u of the form:

$$(-1)^{k+1} u^{3k+2} + A_1 t^2 u^{3k} + A_2 t^4 u^{3k-2} + \dots,$$

where the A_l are (non-homogeneous) linear functions of the c_i, b_j.

First, we claim that for a unique choice of the c_i, b_j the A_l vanish simultaneously, so that $G_2 = \pm u^{3k+2}$ and $G_1 = v^2 - t^2 + 3u^2$. For $t \neq 0$ we then have a non-transverse double point $(u, v) = (0, \pm t)$ of multiplicity $3k+2$, as desired. And second, we claim that the bi-germ of f corresponding to this double point is of type N_{3k+2}.

For the first claim, we note that the functions h_i satisfy the recurrence

$$h_i = (y + \bar{y})h_{i-1} - y\bar{y}h_{i-2}, \quad h_0 = 1, h_1 = y + \bar{y}.$$

Let H_i denote the composition of h_i with

$$K(y, \bar{y}) := ((u + (4t^2 - 3u^2)^{1/2})/2, (u - (4t^2 - 3u^2)^{1/2})/2)$$

on the right, then H_i satisfies the recurrence

$$H_i = uH_{i-1} + (t^2 - u^2)H_{i-2}, \quad H_0 = 1, H_1 = u.$$

The H_i are homogeneous of degree i in u and t, and the coefficient of u^i is 0 (if $i+1$ is a multiple of 3) or ± 1 (otherwise): for $t = 0$ we get

$$H_i = u(H_{i-1} - uH_{i-2}) = u(u(H_{i-2} - uH_{i-3}) - uH_{i-2}) = -u^3 H_{i-3},$$

with $H_0 = 1$, $H_1 = u$ and $H_2 = 0$. Let N be $(3k+1)/2$ (for odd k) or $(3k+2)/2$ (for even k). For each of the N monomials $M = t^2u^{3k}, t^4u^{3k-2}, \ldots$ in $(-1)^{k+1}u^{3k+2} + A_1t^2u^{3k} + A_2t^4u^{3k-2} + \ldots$ there is exactly one deformation term $c_i t^{3k+2-r}H_r$ or $-b_j t^{3k+1-r}H_1H_r$ in $G_2 \circ K$ with $\pm c_i M$ or $\pm b_j M$ as the monomial with the highest degree in u (notice that none of the $r+1$ is a multiple of 3, hence the u^r coefficient in H_r is ± 1). It follows that the linear map from the $\mathbb{R}^N$ of deformation coefficients c_i, b_j to the real vector space spanned by the monomials $M = t^2u^{3k}, t^4u^{3k-2}, \ldots$ (of degree $3k+2$ and divisible by t^{2i}, $i \geq 1$) is an isomorphism. Hence there is a unique choice of coefficients c_i, b_j such that $A_1, \ldots, A_N$ vanish simultaneously, as claimed.

For the second claim, we recall that the bi-germs N_{3k+2} have non-singular component germs and that the n-dimensional tangent spaces of the images of the component germs span together a linear subspace of $\mathbb{R}^{2n}$ of dimension $2n-1$. The map K, defining the double-point algebra Q_K of these bi-germs, is $\mathcal{K}$-equivalent to $(\mathbf{x}, y^k)$ (corank-1). The pair of source points corresponding to to the non-transverse double point $(u, v) = (0, \pm t)$ is given by $y_0 = t, \bar{y}_0 = -t$, and the component germ $G_1 = h_2 - t^2$ of the induced deformation of $G_{(0,0)}$ is given by $G_1(y - t, \bar{y} + t) = t(\bar{y} - y) + h_2$, hence the germ of the the deformation of $G_{(0,0)}$ has corank-1 for $t \neq 0$. The double-point of multiplicity $3k+2$ must therefore be of type N_{3k+2}. Finally note that both component germs of the bi-germ are immersive (the 1-jets of the $y^3 - t^2y$ component at $y_0 = t$ and $\bar{y}_0 = -t$ are both given by $2t^2y$). $\square$

Remark 4·3. Using the recurrence for the H_r we can determine the solutions c_i, b_j of $A_1 = \ldots = A_N = 0$ ($N = (3k+1)/2$, for odd k, or $N = (3k+2)/2$, for even k). (Recall that these give an explicit deformation f_t from a type IX_k singularity f_0 to a type N_{3k+2} bi-germ f_t, $t \neq 0$.) For low k we find that $b_j = 0$ for all j and the following c_i:

$k = 1$: $c_1 = 3$, $c_2 = -1$
$k = 2$: $c_1 = -8$, $c_2 = 10$, $c_3 = -2$
$k = 3$: $c_1 = 36$, $c_2 = -52$, $c_3 = 21$, $c_4 = -3$
$k = 4$: $c_1 = -101$, $c_2 = 250$, $c_3 = -181$, $c_4 = 36$, $c_5 = -4$
$k = 5$: $c_1 = 777$, $c_2 = -1322$, $c_3 = 962$, $c_4 = -465$, $c_5 = 55$, $c_6 = -5$

We do not have a general formula for all the coefficients c_i and b_j, but $c_{k+1} = -k$ and $c_k = k + 2k^2$, for example.

We conclude with a remark on the relation between the $\mathcal{A}$-simplicity of map-germs f of local multiplicity two and their double-point maps $G_{(0,0)}$. We consider $G_{(0,0)}$ as a $\mathbb{Z}_2$-symmetric equidimensional map-germ under $\mathcal{K}^{\mathbb{Z}_2}$-equivalence.

PROPOSITION 4·4. *Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^{2n}, 0$ be a corank-1 germ of multiplicity $m_f(0) = 2$. Then f is $\mathcal{A}$-simple if and only if $G_{(0,0)} : \mathbb{K}^{n+1}, 0 \to \mathbb{K}^{n+1}, 0$ is $\mathcal{K}^{\mathbb{Z}_2}$-simple.*

Proof. For the contrapositive of $\Rightarrow$ we use Lemma 2.3 (i) of [**23**], which says in this special case that for $\mathcal{A}$-equivalent corank-1 germs f and f' the corresponding $G_{(0,0)}$ and $G'_{(0,0)}$ are $\mathcal{K}$-equivalent, and it is easy to see that this $\mathcal{K}$-equivalence preserves the $\mathbb{Z}_2$-symmetry and hence, in fact, is a $\mathcal{K}^{\mathbb{Z}_2}$-equivalence. Write (using the Preparation Theorem)

$$f = (\mathbf{x}, y^2, y g_{n+1}(\mathbf{x}, y^2), \ldots, y g_{2n}(\mathbf{x}, y^2)),$$

then the corresponding $G_{(0,0)}$ is $\mathcal{K}^{\mathbb{Z}_2}$-equivalent to

$$G := (\bar{y}, g_{n+1}(\mathbf{x}, y^2), \ldots, g_{2n}(\mathbf{x}, y^2)).$$

Hence we can lift a deformation G^t of G to one of f: suppose $G_{(0,0)}$ non-$\mathcal{K}^{\mathbb{Z}_2}$-simple, then there is a family of non-$\mathcal{K}^{\mathbb{Z}_2}$-equivalent $G^t_{(0,0)}$ (with $G_{(0,0)} = G^0_{(0,0)}$), and hence of non-$\mathcal{K}^{\mathbb{Z}_2}$-equivalent G^t, and the latter induces a family f^t of non-$\mathcal{A}$-equivalent germs with $f = f^0$.

For the contrapositive of $\Leftarrow$ we recall that any non-$\mathcal{A}$-simple germ f with $m_f(0) = 2$ lies in the closure of the union of the $\mathcal{A}$-orbits of the bordering germs B.1, B.2 and B.3, and notice that the $G_{(0,0)}$ associated with such an f lies in the closure of the union of the $\mathcal{K}^{\mathbb{Z}_2}$-orbits of the $G_{(0,0)}$ associated with B.1, B.2 and B.3. And it is easy to check that the $\mathcal{A}$-moduli of these B.i give rise to $\mathcal{K}^{\mathbb{Z}_2}$-moduli of the associated $G_{(0,0)}$. $\square$

Notice that $\Rightarrow$ in this proposition is an analogue of a result in [**25**] that states that for an $\mathcal{A}$-simple corank-1 germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ of multiplicity $n+1$ the corresponding map $G_{(n)}$ (whose multiplicity gives the A_n-number of f) has to be $\mathcal{K}$-simple. The classification of $\mathcal{K}$-simple equidimensional germs G is known, and one can then show that each of them has a real deformation with $m_G(0)$ real preimages over some point in the target near the origin. For $\mathcal{K}^{\mathbb{Z}_2}$-equivalence the classification of simple equidimensional germs and the analogous result about their real deformations are not available – otherwise we could conclude at once from Proposition 4·4 above that $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n}, 0$ of corank 1 and multiplicity 2 have an M-deformation (without using the $\mathcal{A}$-classification of such f).

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