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Voronoi Diagrams of Real Algebraic Sets

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Abstract. A collection of n (possibly singular) semi-algebraic sets in $\mathbb{R}^d$ of dimension $d-1$, each defined by polynomials of maximal degree δ, has $\Theta((n\delta)^d)$ first-order Voronoi cells (for any fixed d). In the nonhypersurface case, where the maximal dimension of the semi-algebraic sets is $m \leqslant d-2$, the number of first-order Voronoi cells is bounded above by $\mathrm{O}(n^{m+1}\delta^d)$ (for nonsingular semi-algebraic sets) or by $\mathrm{O}((n\delta)^d)$ (in general). The complexity of the entire kth-order Voronoi diagram of a generic collection of n non-singular real algebraic sets in $\mathbb{R}^d$ of maximal dimension $m < d$ and maximal degree Δ is $\mathrm{O}(n^{\min(d+k,2d)}\Delta^{2(m+1)d})$.

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1. Introduction and Notation

The classical first-order Voronoi diagram of a set of points $X = \{X_1, \dots, X_n\}$ in $\mathbb{R}^d$ is a decomposition of $\mathbb{R}^d$ into connected regions, called Voronoi cells, whose points are closer to some given point in X than to any other point in X. The corresponding regions in the kth-order Voronoi diagram consist of points that are closer to some given subset of points $Y \subset X$ of cardinality k than to any point in $X \setminus Y$. More generally, one can consider Voronoi diagrams of collections X of real algebraic (or of semi-algebraic) sets $X_1, \dots, X_n$ in $\mathbb{R}^d$ (d being fixed but arbitrary) of maximal degree Δ and maximal dimension $m := \dim X < d$. If V_k denotes the union of the boundaries of the kth-order Voronoi cells associated with X then it is known that there are at most $\mathrm{O}(n^{\min(d+k,2d)}\Delta^{2(m+1)d})$ connected regions of $\mathbb{R}^d \setminus V_k$ (see Propositions 3.2 and 6.1 of [8]).

In the present paper, we present sharper bounds for the number of first-order Voronoi cells of collections of (possibly singular) real algebraic and semi-algebraic sets, which are asymptotically tight in the hypersurface case, where $m = d-1$. We also give an upper bound for the complexity of the entire kth-order Voronoi diagram of a generic collection of nonsingular real algebraic sets X.

Before stating the main results, we have to fix some notation. The Voronoi diagram of order k of a set $S := \{X_1, \dots, X_n\}$ of (semi-)algebraic sets $X_i \subset \mathbb{R}^d$ is defined as follows. Set $\mu_p(X_i) := \inf_{q \in X_i} \|q-p\|^2$ and let $\tilde{S} \subset S$ be a subset with k elements, $1 \leqslant k \leqslant n-1$. Then

$$V_k(\tilde{S}) := \{p \in \mathbb{R}^d : \mu_p(X_i) < \mu_p(X_j), \forall (X_i, X_j) \in \tilde{S} \times (S \backslash \tilde{S})\}$$

is the kth-order Voronoi cell of $\tilde{S}$ (which, in general, is not connected). The kth-order Voronoi surface V_k of S is the union of the boundaries of such Voronoi cells, i.e. $V_k := \bigcup_{\tilde{S} \subset S} \partial V_k(\tilde{S})$, and the kth-order Voronoi diagram is the arrangement $\mathcal{A}(V_k)$. In the special case of first-order Voronoi diagrams (where $\tilde{S} = \{X_i\}$) we shall simply write

$$V_1(X_i) := \{p \in \mathbb{R}^d : \mu_p(X_i) < \mu_p(X_j), \forall j \neq i\}.$$

Recall the notion of a singular stratification of a semi-algebraic set M. The singular set $S(M)$ of M has strictly lower dimension than M. Hence, setting $S^{i+1} := S(S^i(M))$ and $S^0(M) := M$, the set $S^{\dim M+1}(M)$ will be empty, and taking as strata the connected components of $S^i(M) \backslash S^{i+1}(M)$, for $0 \leqslant i \leqslant \dim M$, we obtain a stratification of M into connected submanifolds of dimension $j = 0, \ldots, \dim M$, called j-strata. This stratification is called singular stratification of M. In general, the strata of the singular stratification do not satisfy regularity conditions, such as the Whitney conditions (a) or (b). We shall see, however, that the semi-algebraic Voronoi boundaries $M = V_k$ generically do not contain nonimmersive points, so that their singular stratification trivially satisfies the Whitney conditions. Given a (possibly singular) semi-algebraic hypersurface $M \subset \mathbb{R}^d$, we denote the arrangement cut-out by M by $\mathcal{A}(M)$. We denote by $|\mathcal{A}(M)|$ the size of this arrangement, that is the number of i-cells, $0 \leqslant i \leqslant d$, in $\mathcal{A}(M)$, and by $e_i(M)$ the number of cells of dimension i. Here the i-cells are defined as follows: for $i < d$ the i-cells are the i-strata of the singular stratification of M and as d-cells we take the connected regions of $\mathbb{R}^d \backslash M$.

Using this notation, $e_d(V_k)$ denotes the number of Voronoi cells of order k and $|\mathcal{A}(V_k)|$ denotes the complexity (or size) of the kth-order Voronoi diagram. Furthermore, let n denote the number of (semi-)algebraic sets X_i, m the maximal dimension of the X_i, δ the maximal degree of the defining (in-) equations of the X_i and Δ the maximal degree of the X_i. (If $c_i = d - m_i$ is the codimension of X_i, where X_i is defined as a complete intersection, and δ_i is the maximal degree of the defining polynomials of X_i then $\Delta_i := \deg X_i \sim \mathrm{O}(\delta_i^{c_i})$.) Recall that the ambient dimension d is assumed to be fixed (but arbitrary). Also, we use the standard O-, Ω- and Θ-notation for upper, lower and asymptotically tight bounds.

The main bounds obtained in the present paper are summarized in the following statement.

THEOREM 1.1. (a) *For a collection of n semi-algebraic sets X_i^s (possibly singular, unless the contrary is stated) in $\mathbb{R}^d$ of maximal dimension m, each defined by a constant number of polynomials of degree $\leqslant \delta$ and intersecting pairwise properly, we have the following bounds for the number of first-order Voronoi cells. When all X_i^s have dimension $d-1$: $e_d(V_1) \sim \Theta((n\delta)^d)$. In higher codimension $c = d - m \geqslant 2$:*

$e_d(V_1) \sim \mathrm{O}(n^{m+1}\delta^d)$ (*for nonsingular* X_i^s),
$e_d(V_1) \sim \mathrm{O}((n\delta)^d)$,
$e_d(V_1) \sim \Omega((n\delta)^{m+1})$.

(b) *For a generic collection of nonsingular real algebraic sets of maximal degree* Δ *we have the following bound for the complexity of the* kth-*order Voronoi diagram:*

$$|\mathcal{A}(V_k)| \sim \mathrm{O}(n^{\min(d+k,2d)}\Delta^{2(m+1)d}).$$

Part (a) is a summary of the bounds in Section 2 (see 2.1 to 2.5) and part (b) is the content of Theorem 3.1.

Remarks. (i) The bound for $|\mathcal{A}(V_k)|$ was already stated in [8], but the proof there contains a gap – see Section 3.

(ii) Note that the sets in the collection of (semi-)algebraic sets studied in the present paper are allowed to intersect and to have several connected components. This is in contrast to most other studies in the literature on Voronoi diagrams of semi-algebraic sets, such as [9] and [1]. Restricting to collections of n disjoint and connected semi-algebraic sets trivially yields n first-order Voronoi regions – but, even in this restricted situation, estimates for the complexity of the entire Voronoi diagram (of any order) are highly non-trivial.

(iii) The $\Omega((n\Delta)^d)$ lower bound for $e_d(V_1)$ in the hypersurface case (where $\Delta = \delta$) is, of course, also a lower bound for $|\mathcal{A}(V_1)|$, and there is a large gap between this and our $\mathrm{O}(n^{d+1}\Delta^{2d^2})$ upper bound. For curves in $\mathbb{R}^2$ one can easily obtain an asymptotically tight $\mathrm{O}((n\Delta)^2)$-bound (see Proposition 3.3). But in higher dimensions it seems unlikely that the gap between the lower bound for $e_d(V_1)$ and the upper bound for $|\mathcal{A}(V_1)|$ can be eliminated completely: e.g. for nonintersecting convex semialgebraic sets defined by polynomials of bounded degree the best available upper bound for $|\mathcal{A}(V_1)|$ is $\mathrm{O}(n^{d+\epsilon})$, $\epsilon > 0$ arbitrarily small [9], but $e_d(V_1) = n$. And for point-sets $|\mathcal{A}(V_1) \sim \Theta(n^{\lceil d/2 \rceil})$, but $e_d(V_1) = n$.

The paper is organized as follows. Section 2 contains the bounds for the number of first-order Voronoi cells of a collection $X_1, \dots, X_n$ of (semi-)algebraic sets. It is not hard to see that each arcwise connected component C of $X_i \backslash (X_i \cap \bigcup_{j \neq i} X_j)$ contributes at most one first-order Voronoi cell, which reduces the problem to finding bounds for the number of such components C. The derivation of the bounds for the complexity of the entire kth-order Voronoi diagram in Section 3 is more technical. The kth-order Voronoi boundary V_k is a subset of the bifurcation set of the family of distance-squared functions on $X_1, \dots, X_n$. We first construct the singular stratification of a certain semi-algebraic set R_k (which is finer than the singular stratification of V_k and coarser than that of the bifurcation set) and count its strata under certain genericity conditions on $X_1, \dots, X_n$, and then show that the set of collections $X_1, \dots, X_n$ for which these conditions fail is closed.

2. The Number of First-Order Voronoi Cells

We say that the intersection of two sets X and Y is *proper* if

$$\operatorname{codim} X \cap Y > \min(\operatorname{codim} X, \operatorname{codim} Y).$$

This, rather weak, condition implies that no component of one set can lie in some component of the other. The *regular intersection* condition, given by

$$\operatorname{codim} X \cap Y = \operatorname{codim} X + \operatorname{codim} Y,$$

is in the nonhypersurface case too strong to yield interesting bounds for the number of first-order Voronoi cells, see the remark following Proposition 2.1.

The bounds in the present section are first stated and proved under the assumption that certain algebraic sets are nonsingular, this assumption will be eliminated at the end of this section (see Theorem 2.4 and Corollary 2.5). We begin with the following bound.

PROPOSITION 2.1. *A collection of nonsingular real algebraic sets*

$$X_1, \dots, X_n \subset \mathbb{R}^d$$

of maximal dimension $m < d$, having pairwise proper intersections, has at most $O(n^{m+1}\delta^d)$ connected, first-order Voronoi cells.

Proof. Let $m_i = \dim X_i$, $X_i = V(h_1, \dots, h_{d-m_i})$, $X = \bigcup_i X_i$ and $\hat{X}_i = \bigcup_{j \neq i} X_j$. The proposition follows from the following two claims.

CLAIM 1. *Let C be an arcwise connected component of $X_i \backslash X_i \cap \hat{X}_i$, then $V_1(C)$ is path-connected.*

Given any pair of points $p_1, p_2 \in V_1(C)$ let q_1, q_2 be the respective nearest points in C (if p_i has more than one nearest point $q_i \in C$, pick any one of them). Connecting p_i and q_i, $i = 1, 2$, by straight line-segments and q_1 and q_2 by any path in C yields a path in $V_1(C)$ with endpoints p_1 and p_2. Notice that all $(d-1)$-spheres passing through q_i and whose centers lie on the line segment $\overline{p_i q_i}$ are contained in the sphere through q_i with center p_i. Hence none of these spheres contains any point of $\bigcup X_i \backslash C$, which implies that $\overline{p_i q_i} \subset V_1(C)$.

The number of connected regions of $\mathbb{R}^d \backslash V_1$ is therefore bounded above by the number of components C of X. The desired upper bound therefore follows from

CLAIM 2. *Each $X_i \backslash X_i \cap \hat{X}_i$ has at most $O(n^{m_i}\delta^d)$ components C, hence there are n times that many components on X.*

Note that $X_i = Z(h_1^2 + \dots + h_{d-m_i}^2)$ (here Z denotes the real zero-set) and $\deg \sum_i h_i^2 \sim O(\delta)$. Furthermore $\hat{X}_i = \bigcup_{j \neq i} X_j$ is the union of $n-1$ closed real algebraic sets X_j, each defined by some polynomial of degree $O(\delta)$. A result of Basu (Theorem 1 of [2]) then implies that

$$\sum_j b_j(X_i \cap \hat{X}_i) \sim (n-1)^{m_i} \mathrm{O}(\delta^d)$$

(here b_j denotes the jth Betti number, note that the Milnor bound would merely give $\mathrm{O}((n\delta)^d)$). Let

$$X_i = \bigsqcup_r X_{i,r} \quad \text{(disjoint union)},$$

then, by Alexander duality,

$$H^{d-q-1}(X_{i,r} \cap \hat{X}_i) \cong \tilde{H}_q(X_{i,r} \backslash X_{i,r} \cap \hat{X}_i)$$

(here $\tilde{H}$ denotes the reduced homology group), hence

$$\tilde{b}_0(X_i \backslash X_i \cap \hat{X}_i) \sim \mathrm{O}(n^{m_i}\delta^d).$$

Therefore there are at most $\mathrm{O}(n^{m_i}\delta^d)$ components C on X_i, so in total $\mathrm{O}(n^{m+1}\delta^d)$ such components on X. □

Remark. The above proof shows that a collection of algebraic sets of codimension $\geqslant 2$, having pairwise *regular* intersections, has at most $\mathrm{O}(n \cdot \delta^d)$ first-order Voronoi cells (note that $X_i \cap \hat{X}_i$ has codimension $\geqslant 2$ in X_i). Hence, as in the case of non-intersecting sets X_i, we get a combinatorially trivial bound.

The next result shows that the estimate above is asymptotically tight in the hypersurface case.

PROPOSITION 2.2. *A collection of real algebraic hypersurfaces*

$$X_1, \ldots, X_n \subset \mathbb{R}^d,$$

having pairwise proper intersections, has at most $\Theta((n\delta)^d)$ *connected, first-order Voronoi cells.*

Proof. We have to show that the maximal number of connected Voronoi cells is $\Omega((n\delta)^d)$, the corresponding upper bound is a special case of the previous proposition.

Consider the following collection of $dn \sim \Theta(n)$ algebraic sets $X_{i,j}$ of degree δ (recall that $d \sim \mathrm{O}(1)$):

$$X_{i,j} := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \prod_{\ell=1}^{\delta} (x_i - \ell n - j) = 0 \right\},$$

where $1 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant n-1$. The set $X := \bigcup X_{i,j}$ has $\Theta((n\delta)^d)$ components C. (Notice that the $X_{i,j}$ consist of δ parallel hyperplanes. There are $n\delta$ hyperplanes perpendicular to each coordinate direction x_i belonging to the sets $X_{i,j}$, $0 \leqslant j \leqslant n-1$. Ordering these parallel hyperplanes according to their x_i-coordinates, the index j of the $X_{i,j}$ containing them is periodic with period n.) □

Remark. In the nonhypersurface case, where $m \leqslant d-2$, this construction yields an $\Omega((n\delta)^{m+1})$ lower bound, which is tight in terms of n but not in terms of δ.

Next, we show that the argument leading to Proposition 2.1 can be adapted to the semi-algebraic case, giving the following estimate.

COROLLARY 2.3. *Consider closed semi-algebraic sets X_i^s of (pure) dimension m_i, which are unions of a constant number of sets of the form $B_{i,j} = \{x \in A_{i,j} : k_1(x) \geqslant 0, \ldots, k_r(x) \geqslant 0\}$, where $A_{i,j} = Z(h_1, \ldots, h_{d-m_i})$ is nonsingular of dimension m_i, and where $r \sim \mathrm{O}(1)$ and the h_a, k_b are polynomials of degree at most δ. Setting $X_i = \bigcup_j A_{i,j} \supset X_i^s$, a collection of such semi-algebraic sets $X_1^s, \ldots, X_n^s \subset \mathbb{R}^d$ of maximal dimension $m < d$, such that the $X_i \supset X_i^s$ have pairwise proper intersections, has at most $\mathrm{O}(n^{m+1}\delta^d)$ connected, first-order Voronoi cells. Furthermore, when all the X_i^s have dimension $d-1$ then there are at most $\Theta((n\delta)^d)$ first-order Voronoi cells.*

Proof. Claim 1 in the proof of Proposition 2.1 also holds with X_i^s in place of X_i, hence we want to show that each $X_i^s \backslash X_i^s \cap \hat{X}_i^s$ has at most $\mathrm{O}(n^{m_i}\delta^{2d})$ components C, hence there are n times that many components on X^s (where $X^s = \bigcup_i X_i^s$ and $\hat{X}_i^s = \bigcup_{j \neq i} X_j^s$).

Note that if $A_{i,j}$ is the real zero-set of $P_j := h_1^2 + \cdots + h_{d-m_i}^2$ then $X_i = Z(\prod_j P_j)$. The latter, being the union of a constant number of non-singular m_i-dimensional sets $A_{i,j}$, is therefore the zero-set of a polynomial of degree $\mathrm{O}(\delta)$. Cutting each $A_{i,j}$ with the union of $\hat{X}_i$ and the sets $k_b^{-1}(0)$, $1 \leqslant b \leqslant r$, where the k_b are the polynomials appearing in the inequalities defining $B_{i,j}$, we obtain at most $\mathrm{O}(n^{m_i}\delta^d)$ components C on each $A_{i,j}$ (by the argument in the proof of Proposition 2.1) and hence at most $\mathrm{O}(1)$ times that many components C on X_i. Finally note that this decomposition of X_i into the sets C and their boundaries induces a decomposition of X_i^s which is a refinement of the decomposition into connected regions of $X_i^s \backslash X_i^s \cap \hat{X}_i^s$ and their boundaries. This implies the desired bound for the number of regions of the latter decomposition.

The last statement of the proposition follows from Proposition 2.2. $\square$

Remark. The condition that the algebraic sets X_i (in Proposition 2.1) and $A_{i,j}$ (in Corollary 2.3) be nonsingular is required for the use of the Alexander duality. We now remove this regularity condition in the hypersurface case.

Let $Y = Z(h)$ be a singular real algebraic set of dimension $d-1$ then, for all $\epsilon \in (0, c)$ and c small enough, $U := \{x : h^2(x) \leqslant \epsilon^2\}$ is a neighborhood of Y in $\mathbb{R}^d$ (see Chapter 3.8 of [3]) and the sets $Y_\epsilon = Z(h^2 - \epsilon^2)$ are non-singular real algebraic hypersurfaces (by Sard's theorem) such that $Y_0 = Y$. Now take $Y = X_i$ (the argument for the $A_{i,j}$ being analogous): the $n-1$ hypersurfaces making up $\hat{X}_i$ intersect X_i properly, hence – by restricting to a subinterval $(0, c') \subset (0, c)$ if necessary – we can assume that these hypersurfaces are transverse to the Y_ϵ and that the number of regions, N say, of $Y_\epsilon \backslash Y_\epsilon \cap \hat{X}_i$ is locally constant for $\epsilon \in (0, c')$. The corresponding number of regions for $X_i = Y_0$ is clearly bounded by N (any such region belongs to the closure of one of the N regions of $U \backslash U \cap (\hat{X}_i \cup X_i)$). Hence we have shown that

THEOREM 2.4. *The statements in Propositions* 2.1 *and* 2.2 *and in Corollary* 2.3 *hold without the regularity* (*nonsingularity*) *condition for the sets* X_i *and* $A_{i,j}$ *when these are* $(d-1)$*-dimensional: i.e. collections of algebraic or semi-algebraic sets of dimension* $d-1$ *have* $\Theta((n\delta)^d)$ *first-order Voronoi cells.*

In higher codimension we have the following, less sharp, bound in the singular case.

COROLLARY 2.5. *A collection of n algebraic or semi-algebraic sets* (*not necessarily nonsingular*)*, each defined by polynomials of degree* $\leqslant d$ *and having pairwise proper intersections, has at most* $O((n\delta)^d)$ *first-order Voronoi cells.*

Proof. Suppose $Y = Z(h_1, \ldots, h_r)$ is a singular real algebraic set of codimension r, then we define

$$U = \left\{ x : \sum_{a=1}^{r} h_a^2(x) \leqslant \epsilon^2 \right\} \quad \text{and} \quad Y_\epsilon = Z\left(\sum_a h_a^2 - \epsilon^2 \right)$$

(as in the hypersurface case above). We now argue as in the hypersurface case, except that we replace each set $\hat{X}_i$, which is the union of algebraic sets $X_j = Z(f_1, \ldots, f_k)$ $(j \neq i)$, by a union of hypersurfaces $H_j = Z(f_b) \supset X_j$, $b \in \{1, \ldots, k\}$, that do not contain X_i (such hypersurfaces exist, since X_i and X_j intersect properly). $\square$

3. The Complexity of the *k*th-Order Voronoi Diagram

In the present section we study collections $X_1, \ldots, X_n$ of non-singular algebraic sets, and we impose some extra 'genericity conditions' and show that the collections of algebraic sets for which these conditions fail form some closed subset of positive codimension in the space of all collections of some given degree. In fact, the regularity of the X_i is also 'generic' in this sense. We begin with a brief summary of the relation between Voronoi boundaries and bifurcation sets of families of distance-squared functions (see [8] for more details), the relation between such bifurcation sets and evolutes and symmetry sets – mostly for curves and surfaces in 2- and 3-space – has been studied in [4, 5, 7].

The order k Voronoi boundary V_k is a subset of the bifurcation set $\mathcal{B}$ of the family of distance-squared functions on the X_i:

$$F_i \colon \mathbb{R}^d \times X_i \to \mathbb{R}^d \times \mathbb{R}, \quad (p, q) \mapsto (p, f_i(p, q) := \|q - p\|^2.$$

Locally (i.e. composing with a local parametrization of the m_i-manifold X_i at q) this is a d-parameter family of functions in m_i variables, and the simplest singularities that such a function can have are the A_k-singularities, given by $x_1^{k+1} + \sum_{j=1}^{m_i} \epsilon_j x_j^2$, $\epsilon_j = \pm 1$, at the origin (an A_1 point is a Morse critical point). Let $A_{(k_1,\ldots,k_s)}$ denote the s-local singularity of a function f, where f has s critical points q_1 to q_s, of type A_{k_1} to A_{k_s}, having the same critical values $f(q_j)$. In order to avoid redundancy we suppose the sequence $k_1, \ldots, k_s$ to be nonincreasing. Given some type of singularity W, we denote by $\bar{W}$ the closure of W (in the space of function-germs) – hence type $\bar{W}$ means a type W singularity or something more degenerate. We can now define the bifurcation set $\mathcal{B}$ of the family of distance-squared functions on a collection $X_1, \ldots, X_n$ of algebraic sets in $\mathbb{R}^d$. Let $\mathcal{E}_i$ be the evolute of X_i (the set of point $p \in \mathbb{R}^d$ for which $f_i(p, .)$ has an $\bar{A}_2$ singularity at some $q \in X_i$), $\mathcal{S}_i$ the intra-set level-bifurcation set (the points p for which $f_i(p, .)$ has an $\bar{A}_{(1,1)}$ singularity at some point pair $q_1, q_2 \in X_i$) and $\mathcal{S}_{i,j}$ the inter-set level-bifurcation set (the points p for which $f_i(p, .)$ and $f_j(p, .)$ have $\bar{A}_1$ singularities at $q_1 \in X_i$ and $q_2 \in X_j$ such that $f_i(p, q_1) = f_j(p, q_2)$ – this is simply an $\bar{A}_{(1,1)}$ singularity of the distance-squared function on $X = \bigcup_i X_i$ with critical points in different sets X_i, X_j). Now $\mathcal{B} = \mathcal{E} \cup \mathcal{S} \cup Y$, where

$$\mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}_i, \quad \mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i \quad \text{and} \quad Y = \bigcup_{1 \leqslant i < j \leqslant n} \mathcal{S}_{i,j}.$$

The sets V_k are subsets of the component Y of the bifurcation set.

For a collection of real algebraic sets $X_1, \ldots, X^n$ in $\mathbb{R}^d$ of maximal dimension m defined by polynomials of degree δ, the degree of smallest real algebraic set $\hat{\mathcal{B}}$ containing the semi-algebraic bifurcation set $\mathcal{B}$ is at most $O(n^2 \Delta^{2(m+1)})$ (see Section 3.2 of [8]). Using Milnor's bound for the Betti numbers [6] and Alexander duality one finds that $\mathbb{R}^d \backslash \hat{\mathcal{B}}$ and, hence, $\mathbb{R}^d \backslash \mathcal{B}$ and $\mathbb{R}^d \backslash V_k$, have at most $O((\deg \hat{\mathcal{B}})^d) \sim O(n^{2d} \Delta^{2(m+1)d}))$ connected components (Proposition 3.2 (iii) of [8]). Furthermore, it is known that the singular stratification of V_k has at most $O(n^{\min(d+k, 2d)})$ strata when Δ is bounded by some constant (Proposition 6.1 of [8]). Finally, Proposition 3.2 (iii) in [8] also states that the number of strata of the singular stratification of $\mathcal{B}$ has at most $O((\deg \hat{\mathcal{B}})^d) \sim O(n^{2d} \Delta^{2(m+1)d})$ strata (recall that $|\mathcal{A}(\mathcal{B})|$, the size of the arrangement defined by $\mathcal{B}$, is number of strata of the singular stratification plus the number of regions in the complement of $\mathcal{B}$) – but the proof of this statement uses the unproven assertion that the singular stratification of a singular hypersurface $V \subset \mathbb{R}^d$ has at most $O((\deg X)^d)$ strata and, hence, contains a gap.

A singular stratification of V_k can be obtained by deleting certain strata of a singular stratification of $\mathcal{B}$ – combining Propositions 6.1 and 3.2 (iii) from [8] would therefore give an $O(n^{\min(d+k, 2k)} \Delta^{2(m+1)d)})$ bound for the complexity of the kth-order Voronoi diagram. Rather than filling the gap in the proof of 3.2 (iii) by proving the above-mentioned assertion, we will use the construction in the proof of 6.1, without the assumption that $\Delta \sim O(1)$, together with the observation that V_k generically does not contain nonimmersive points p (the curvature sphere with such a centre p generically violates the Voronoi property of any order k) to prove this bound.

THEOREM 3.1. *Let $X_1, \ldots, X_n \subset \mathbb{R}^d$ be a generic collection of real algebraic sets of maximal dimension $m < d$, then*

$$|\mathcal{A}(V_k)| \sim \mathrm{O}(n^{\min(d+k,2k)}\Delta^{2(m+1)d}).$$

Proof. Define the following subsets of the inter-set level-bifurcation set $Y = \bigcup_{1 \leqslant i<j \leqslant n} \mathcal{S}_{i,j}$ and of $\bigcup_i X_i$, respectively: let $Y_{(r_1,\dots,r_s)}$ be the locus of centres of (exactly) s concentric (exactly) r_i-tangent spheres, $r_i \geqslant 2$, to $\bigcup_i X_i$ (i.e. spheres tangent to $\bigcup_i X_i$ at r_i distinct points) and let

$$S_{(c_1,\dots,c_s)} = \bigcap_{i\in\Lambda} X_i, \quad \Lambda \subset \{1,\dots,n\}, \quad |\Lambda| = t \geqslant 2,\ c_i = d - m_i$$

(with nonincreasing index sequences $r_1,\dots,r_s$ and $c_1,\dots,c_t$ in order to avoid redundancy). Furthermore, we use the notation $\bar{Y}_{(r_1,\dots,r_s)}$ (and analogously $\bar{S}_{(c_1,\dots,c_s)}$) for the closure of this set, which is the locus of centres of at least s concentric $\geqslant r_i$-tangent spheres.

We can now state the genericity conditions on the collections of algebraic sets.

GENERICITY CONDITIONS. Let $\mathcal{X}$ be the space of collections X of n algebraic sets X_i of dimension m_i (and codimension $c_i = d - m_i$) of maximal degree Δ, $\mathcal{X}$ can be identified with some semi-algebraic subset of the finite-dimensional space of coefficients of $\sum_{i=1}^n c_i$ polynomials in d variables of degree $\leqslant \Delta$. (Note that not all choices of coefficients yield m_i-dimensional real algebraic sets X_i.) We now define three subsets U, V and W of $\mathcal{X}$, which will be shown to be closed, and call any collection $X \in \mathcal{X}\backslash U \cup V \cup W$ generic.

Let U be the set of collections $X \in \mathcal{X}$ containing some singular X_i, and let V be the set of $X \in \mathcal{X}$ for which the $\bar{A}_3$-stratum in $\mathcal{E}$ fails to be closed and of positive codimension. Finally define W, corresponding to degenerate X for which Y has 'excess intersection', as follows:

$$\begin{aligned} W = \Bigg\{ & X \in \mathcal{X} : \exists s \geqslant 1, \exists r_i \geqslant 2 : \dim Y_{(r_1,\dots,r_s)} > d + s - \sum_{i=1}^{s} r_i \quad \text{or} \\ & \exists t \geqslant 2, \exists c_i \geqslant 1 : \dim S_{(c_1,\dots,c_t)} > d - \sum_{i=1}^{t} c_i \quad \text{or} \\ & \exists s, c_i \geqslant 1, \exists t, r_j \geqslant 2 : \dim(S_{(c_1,\dots,c_t)} \cap Y_{(r_1,\dots,r_s)}) \\ & \quad > d + s - \sum_{i=1}^{t} c_i - \sum_{j=1}^{s} r_j \Bigg\}. \end{aligned}$$

The proof that $U \cup V \cup W$ is closed in $\mathcal{X}$ will be postponed to Lemma 3.2 below.

We want to count the strata of the singular stratification of a certain 'intermediate set' R_k, in the sense that $V_k \subset R_k \subset Y$, which contains all the strata of the singular stratification of V_k. In order to define this stratification, we need some definitions. The support set of a r-tangent sphere is the set of X_i that are tangent to this sphere. The radius of a r-tangent sphere tends to zero as its center approaches the self-intersection locus of $\bigcup_i X_i$, we call such spheres vanishing spheres. A minimal r-tangent sphere is the smallest one amongst those with given centre and support set, a minimal

r-tangent sphere does not contain any points of its support set in its interior. Note that any vanishing sphere is minimal and that any nonminimal sphere violates the Voronoi property for any order k.

We can now construct the sets R_k: roughly speaking, we are going to delete from Y certain 'branches' of dimension $d-1$ that cannot belong to V_k.

First, we decompose the inter-surface level bifurcation set Y into certain 'branches' which, for generic arrangements X, will be $(d-1)$-dimensional. Let $B(Y)$ denote the set of connected components ('branches') of $Y \setminus \bar{Y}_{(3)} \cup \bar{S}_{(1,1)}$. Note that all points of such a 'branch' lie either in $V_k \subset Y$ or in $Y \setminus V_k$, because for all these points we have a pair of critical points of the distance-squared function whose critical value is distinct from all other critical values.

Next, we decompose the self-intersection locus of Y into connected components of i-fold intersections, $i = 2, 3, \ldots, s$ and compare the radii of $\geqslant 2$-tangent spheres associated to the i branches of $B(Y)$ passing through an i-fold intersection point. If, at any point p of the self-intersection locus, the $\geqslant 2$-tangent sphere associated with some branch of $B(Y)$ does not belong to the k smallest minimal spheres with centre p (including the vanishing sphere if $p \in S_{(c_1,\ldots,c_t)}$) and distinct support sets then this branch cannot belong to V_k. Deleting all such branches from Y yields the set R_k. To be a bit more precise, let L be the set of 'strata' of the 'stratification' of the self-intersection locus of Y into connected components of $Y_{(r_1,\ldots,r_s)}$ $(s, r_i \geqslant 2)$, $S_{(c_1,\ldots,c_t)}$ $(t \geqslant 2)$ and $S_{(c_1,\ldots,c_t)} \cap Y_{(r_1,\ldots,r_s)}$ $(t, r_i \geqslant 2,\ s \geqslant 1)$. The reason for the quotes is that these 'strata' of Y can contain nonimmersive points (but we will see below that the subset of these 'strata' that also belong to R_k do not contain nonimmersive points, and hence form a genuine stratification). For any $l \in L$, let l_k denote the set of branches $b \in B(Y)$ passing through l which correspond to the k smallest minimal $\geqslant 2$-tangent spheres with centre in l, which by definition have distinct support sets (if there are fewer than k minimal spheres with distinct radius then l_k contains all branches through l that correspond to some minimal sphere). We can now define

$$R_k := \{b \in B(Y) : b \in l_k, \forall l \in L : l \subset \mathrm{cl} b\} \cup \bar{Y}_{(3)} \cup \bar{S}_{(1,1)}.$$

Now notice the following: the second of our genericity conditions implies that the proper A_2-stratum is open and dense in $\mathcal{E}$. The curvature sphere at an A_2-point $q \in X_i$ has, by the definition of an A_2-singularity, points in the neighborhood of q in X_i in its interior (and in its exterior). The set Y at a centre $p \in Y$ of an r-tangent sphere with only A_1-points q_i, $i = 1, \ldots, r$, in its support set is easily seen to be immersive, the tangent planes of the branches of Y at p being given by the hyperplanes perpendicular to the vectors $q_i - q_j$. On the other hand, any r-tangent sphere S with centre p and some non-A_1-point $q \in X_i$ in its support set contains points of X_i in its interior: if q is of type A_2 this is obvious, and if q is a more degenerate (i.e. of type A_3 or worse) then – by the density of the A_2 stratum in $\mathcal{E}_i$ – there is an A_2-sphere S' with centre p' and a support point $q' \in X_i$ with p and p' and q and q' arbitrarily close, and S has points of X_i in its interior because S' has (by the continuity of the distance-squared function to an algebraic set). It follows that the 'strata' of Y with nonimmersive points have not been selected for the subset R_k of Y.

We now derive a bound for the number of strata of R_k. Given a collection $X_1, \ldots, X_n$ of algebraic sets, there are $\prod_{i=1}^{s} \binom{n}{r_i}$ sets $\bar{Y}_{(r_1,\ldots,r_s)}$, and each of them is an algebraic set whose defining equations will be studied below. The number of connected components of the semi-algebraic set $Y_{(r_1,\ldots,r_s)}$ depends on the number of components of its closure and on the number of connected components of all the (lower-dimensional) sets

$$Y_{(a_1,\ldots,a_t)} \subset \bar{Y}_{(r_1,\ldots,r_s)} \setminus Y_{(r_1,\ldots,r_s)}$$

in its boundary. For large enough s and $r_1, \ldots, r_s$, namely for $\sum_{i=1}^{s} > d + s$ (by the dimensional genericity condition), the boundary of $Y_{r_1,\ldots,r_s}$ will be empty, so that $Y_{r_1,\ldots,r_s}$ has as many connected components as $\bar{Y}_{r_1,\ldots,r_s}$. We call a connected component of such a nonempty set $Y_{(r_1,\ldots,r_s)}$, whose boundary is empty, a *maximal component*, and $Y_{(r_1,\ldots,r_s)}$ a *maximal set*. The maximal sets are algebraic (and not merely semi-algebraic) sets, and we can estimate their number of components (in the worst case, in terms of combinatorial complexity, they will consist of isolated points). Likewise, the combinatorial complexity of the closures of the sets $S_{(c_1,\ldots,c_t)}$ and $S_{(c_1,\ldots,c_t)} \cap Y_{(r_1,\ldots,r_s)}$ is $\mathrm{O}(n^t)$ and $\mathrm{O}(n^{t+\sum r_i})$, respectively, and the complexity of the interiors of these sets will depend on the number of components in their boundary (for $\sum r_i$ and $\sum c_j$ sufficiently large we get, again, maximal sets with empty boundary). By inductively deleting the lower-dimensional boundary components from $Y = \bar{Y}_2$, beginning with the maximal components, whose boundary is empty, we obtain a 'stratification' of Y whose 'strata' are the connected components of the sets $Y_{(r_1,\ldots,r_s)}$, $S_{(c_1,\ldots,c_t)}$ and their intersections. The number of 'strata', as a function of n, obtained in this way is of the order of the number of maximal sets. Furthermore, by discarding the 'strata' in $Y \setminus R_k$, we get a genuine stratification of R_k (as remarked above).

For the 0-dimensional maximal sets $Y_{(r_1,\ldots,r_s)}$ we have, by the genericity of X, the relation $\sum_{i=1}^{s} r_i = d + s$. For the 0-dimensional maximal sets $S_{(c_1,\ldots,c_t)}$ and $S_{(c_1,\ldots,c_t)} \cap Y_{(r_1,\ldots,r_s)}$ we have in the worst case of hypersurface arrangements (where all $c_i = 1$) the relations $t = d$ and $t + \sum r_i = d + s$. Hence, there are at most $\prod_{i=1}^{s} \binom{n}{r_i} \sim \mathrm{O}(n^{d+s})$ such maximal sets, and each of them consists of a certain number, $N(\Delta, m, d)$, of isolated solutions to the defining equations of these algebraic sets (which will be studied below).

The relation $\sum r_i = d + s$, where all $r_i \geqslant 2$, implies that $s \leqslant d$, and for the maximal sets that belong to R_k we have that $s \leqslant \min(k, d)$. Hence, there are at most $\mathrm{O}(n^{\min(d+k,2d)} N(\Delta, m, d))$ maximal sets, or strata of dimension 0, of R_k. The projection of the zero-set defined by the equations of a bi-tangent sphere in Section 3.2 of [8]

$$\varphi_i(p, x) = \varphi_j(p, x') = 0, \quad \|x - p\|^2 = \|x' - p\|^2$$

(the φ_l are maps into d-space and the product of the components of this map has degree $\mathrm{O}(\Delta^{m+1})$) are the sets $\mathcal{S}_{i,j}$, which are algebraic hypersurfaces in $\mathbb{R}^d$ of degree

$O(\Delta^{2(m+1)})$. Combining the defining equations of $\mathcal{S}_{i,j}$ with the defining equations of the algebraic sets X_i we can define the algebraic sets $\bar{Y}_{(r_1,\ldots,r_s)}$, $\bar{S}_{(c_1,\ldots,c_t)}$ and $\overline{S_{(c_1,\ldots,c_t)} \cap Y_{(r_1,\ldots,r_s)}}$ in $\mathbb{R}^d$ by equations of maximal degree $O(\Delta^{2(m+1)})$. The number of connected components of these sets is therefore at most $O(\Delta^{2(m+1)d})$ (by [6]). The maximal sets coincide with their closures, hence

$$N(\Delta, m, d) \sim O(\Delta^{2(m+1)d}).$$

The desired bound for $|\mathcal{A}(R_k)|$, and therefore for $|\mathcal{A}(V_k)|$, now follows. □

LEMMA 3.2. *The set* $U \cup V \cup W$ *is closed in* $\mathcal{X}$. *As a consequence, the singular stratification of* V_k *generically satisfies the Whitney conditions* (a) *and* (b).

Proof. The collections of algebraic sets U containing some singular algebraic set X_i are clearly closed (simply consider the defining equations of the X_i together with the conditions of a singular point of X_i).

Next, consider $V \subset \mathcal{X}$. The set

$$\tilde{\mathcal{E}_i} := \{(p, q) \in \mathbb{R}^d \times X_i : f_i(p, q) \text{ unstable}\}$$

has dimension $d - 1$ for any non-singular algebraic set X_i. (Note that for any given $q \in X_i$, the distance-squared function f_i is singular if p lies in the normal space N_qX_i, and the set of p for which f_i has a non-Morse critical point q has codimension 1 in N_qX_i.) Let $\tilde{C}_i \subset \tilde{\mathcal{E}_i}$ denote the proper A_2-stratum, and set $\tilde{D}_i := \tilde{\mathcal{E}_i} \backslash \tilde{C}_i$. Using the defining equations $\varphi_i = \det \mathrm{d}\varphi_i = 0$ for $\tilde{\mathcal{E}_i}$ in Section 3.2 in [8] (notice that these define $\tilde{\mathcal{E}_i}$ if we omit the existential quantifier in the definition of $\mathcal{E}_i$), together with the extra condition for an A_3-point (or something more degenerate), which is given by the vanishing of the second derivative in the kernel direction of $\mathrm{d}\varphi_i$, we see that the set $\tilde{D}_i$ of non-A_2 points in $\tilde{\mathcal{E}_i}$ is closed and of positive codimension, provided that one avoids a closed set of sets X_i.

Let π: $\mathbb{R}^d \times X_i \to \mathbb{R}^d$ denote the projection onto the first factor, and denote the images of the sets $\tilde{\mathcal{E}_i}$, $\tilde{C}_i$ and $\tilde{D}_i$ under this projection by $\mathcal{E}_i$, C_i and D_i, respectively. The restriction of π to $\tilde{C}_i \subset \tilde{\mathcal{E}_i}$ is locally one-to-one, the subset C_i of proper A_2-points in $\mathcal{E}_i$ is therefore open and dense and of dimension $d - 1$ (if one avoids a closed set of sets X_i).

Finally, consider the set $W \subset \mathcal{X}$. The sets $\bar{Y}_{(r_1,\ldots,r_s)}$ and $\bar{S}_{(c_1,\ldots,c_t)}$ (and their intersection) can be viewed as intersections of an appropriate number of hypersurfaces H_r, which are either inter-set level-bifurcation sets $\mathcal{S}_{i,j}$ or zero-sets $h_l^{-1}(0)$ of the defining equations of the X_i. Excess intersection occurs when the number of such intersecting hypersurfaces is greater than the codimension of the intersection locus. Suppose now that $A_j := \bigcap_{r=1}^{j} H_r$, $\operatorname{codim} A_j = j$, but $\operatorname{codim}(A_j \cap H_{j+1}) = j$. We will show that there exists a 1-parameter family of algebraic sets $V(t)$, with $V(0) = X_i$ and $\deg V(0) = \deg X_i$, such that the induced family $H_{j+1}(t)$ cuts A_j in a set of codimension greater than j, far all $t \in (0, \epsilon)$ (ϵ some sufficiently small positive constant).

By choosing the intersections of the H_r in the right order, we can ensure that the X_i, that we wish to deform, does not appear in the definition of A_j. When

$H_{j+1} = h_l^{-1}(0)$, where h_l is a defining equation of X_i, we can simply take as our $V(t)$ (sufficiently small) translates of H_{j+1}. When $H_{j+1} = \mathcal{S}_{i,j}$, choose a point $p \in A_j$ and global coordinates with p as the origin of $\mathbb{R}^d$, and define the family $V(t) := (1+t) \cdot X_i$ of scaled copies of $X_i = V(0)$. It is easy to see that $H_{j+1}(t)$, for all $t \in (0, \epsilon)$, cuts A_j in a closed set of codimension greater than j: $A_j \cap H_{j+1}(t)$ is closed in A_j and $p \notin H_{j+1}(t)$ for $t \neq 0$, hence $A_j \cap H_{j+1}(t)$ has positive codimension in A_j.

Hence $U \cup V \cup W$ is closed in $\mathcal{X}$. The Voronoi boundary V_k of a generic collection of algebraic sets $X \in \mathcal{X} \backslash U \cup V \cup W$ has no nonimmersive points, the singular stratification of V_k therefore trivially satisfies the Whitney conditions (a) and (b) (Note: the tangent space at a boundary point p of a stratum is simply the well-defined tangent space of the 'branch' B of $S^i(V_k)$ containing this stratum as an open subset, and it contains the tangent space of the lower dimensional stratum containing p and the limit of any sequence of secant lines through pairs of points in B tending to p.) □

In the plane (i.e. for $d = 2$) one can easily prove the following sharper bound.

PROPOSITION 3.3. *For closed algebraic sets $X_1, \ldots, X_n$ in $\mathbb{R}^2$ (i.e. algebraic curves) the complexity of the first-order Voronoi diagram is $|\mathcal{A}(V_1)| \sim \Theta((n\Delta)^2)$.*

Proof. For hypersurfaces we have $\delta = \Delta$. Proposition 2.2 yields the desired lower bound, it is therefore sufficient to show that $|\mathcal{A}(V_1)| \sim O((n\Delta)^2)$. Compactify $V_1 \subset \mathbb{R}^2$ by joining the unbounded arcs to a vertex p at infinity, and let e_i denote the number of connected i-cells in the arrangement $\mathcal{A}(V_1 \cup \{p\})$. We know already that $e_2 \sim \Theta((n\Delta)^2)$. There are two types of arcs: closed arcs and arcs bounded by two vertices – let e_1^c and e_1^b denote the number of arcs of the former and latter type, respectively. Clearly, $e_1^c \leqslant e_2$ and $3e_0 \geqslant 2e_1^b$ (because each bounded arc has two vertices and each vertex has degree at least three). Let $c(V_1)$ denote the number of connected components of V_1. The components of V_1 are boundaries of Voronoi cells, hence $c(V_1) \leqslant e_2$. The formula

$$e_0 - (e_1^c + e_1^b) + e_2 = 1 + c(V_1)$$

and the inequalities above now imply the desired upper bounds for e_0 and $e_1 = e_1^c + e_1^b$. □

Remark. For nonintersecting simple algebraic curves of degree O(1) and points in the plane it is known that $|\mathcal{A}(V_1)| \sim \Theta(n)$, see [1].

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