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# M-deformations of $\mathcal{A}$-simple $\Sigma^{n-p+1}$-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, $n \geqslant p$

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*Abstract*

All $\mathcal{A}$-simple singularities of map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, where $n \geqslant p$, of minimal corank (i.e. of corank $n-p+1$) have an M-deformation, that is a deformation in which the maximal numbers of isolated stable singular points are simultaneously present in the discriminant.

## 1. *Introduction*

We study real deformations of map-germs from $\mathbb{R}^n$ into $\mathbb{R}^p$, where $n \geqslant p$, for which the maximal numbers of isolated stable singular points are simultaneously present in the discriminant which we call M-deformations for short (M as in maximal), furthermore we call the maximal numbers of isolated stable singularities 0-stable invariants. (Notice, of course, that the numbers of isolated stable singular points in a real deformation of a map-germ are no greater than the corresponding numbers appearing in a stabilization of the complexified germ, and for an M-deformation the corresponding numbers are equal.) For map-germs of target dimension greater than the source dimension we replace discriminant by image in the definition of a M-deformation. This terminology is analogous to the concept of a M-morsification of a function-germ, which, for example, exist for singularities of type $A_k$ and $D_k$ [**2**, **5**]), and also for those of type $E_6$, $E_7$ and $E_8$. For map-germs very little is known about the existence of M-deformations beyond the classical result by A'Campo [**1**] and Gusein–Zade [**8**] that plane curve-germs always have M-deformations, i.e. deformations with $\delta$ real double-points (notice that the $\delta$-number is the only 0-stable invariant in this case). For map-germs $\mathbb{R}^n \to \mathbb{R}^p$, where $n < p$, there is also the notion of a good real perturbation, due to Mond, for which the homology of the image of a stabilization of a given germ coincides with that of its complexification (which is analogous to that of an $M$-variety $X$ in real algebraic geometry for which $b_*(X_{\mathbb{R}}) = b_*(X_{\mathbb{C}})$, where $X_{\mathbb{K}}$ is the set of $\mathbb{K}$-points of $X$ and $b_*$ the sum of the Betti numbers, see e.g. [**11**]). Again there is an analogous definition for $n \geqslant p$ with discriminant in place of image. For plane curve-germs the concept of a good real perturbation coincides with that of an M-deformation, but for map-germs of higher source dimension such good perturbations exist only for a small class of map-germs – e.g. for germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ there is only one series of $\mathcal{A}$-simple corank-1 mono-germs having good real perturbations [**12**]. On the other hand, good perturbations are known to exist for all singular map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$ of $\mathcal{A}_e$-codimension 1 and minimal corank (i.e. of corank $\max(1, n-p+1)$), see [**4**] and [**10**].

The main result of this paper is that all $\mathcal{A}$-simple singularities of map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, where $n \geqslant p$, of minimal corank (i.e. of corank $n-p+1$) have an M-deformation. The proof is based on the following property $(*)$: all $\mathcal{A}$-simple singularities $f$ of minimal corank can be deformed into a germ of lower codimension whose 0-stable invariants differ from those of $f$ by at most one – one can then inductively split-off real stable singular points from 0 one by one. As a corollary we also get lower bounds for the $\mathcal{A}_e$-codimension of $f$ in terms of its 0-stable invariants. The above property does not hold for germs of non-minimal corank nor for germs of positive $\mathcal{A}$-modality. The hypothesis of minimal corank is necessary for the existence of M-deformations (below we give an example of an $\mathcal{A}$-simple corank-2 germ from the plane to the plane that does not have an M-deformation, and that violates the above property). At present we have no example of a germ of minimal corank and positive $\mathcal{A}$-modality without an M-deformation, but there are $\mathcal{A}$-unimodal germs for which the above property $(*)$ does not hold.

Finally, looking at the existing classifications of $\mathcal{A}$-simple corank-1 germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, where $n < p$, one verifies that these have M-deformations. Hence it is reasonable to conjecture that the existence of M-deformations holds for $\mathcal{A}$-simple singularities of minimal corank for any pair of source and target dimensions.

The plan of this paper is as follows. In Section 2 we introduce some notation and state the main result and in Section 3 we briefly recall from [**18**] the definition of certain map-germs $G_{k(s,n)}: \mathbb{K}^{n+s-1} \to \mathbb{K}^{n+s-1}$ associated with $f: \mathbb{K}^n \to \mathbb{K}^n$ (here $k(s,n)$ denotes a partition of $n$ with $s$ summands, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) whose local multiplicity gives the 0-stable invariants up to an overcount factor. Section 4 contains the proof of the main result and Section 5 gives lower bounds on the $\mathcal{A}_e$-codimension in terms of the 0-stable invariants and discusses some empirical evidence for the existence of M-deformations for $\mathcal{A}$-simple corank-1 germs $f: \mathbb{R}^n \to \mathbb{R}^p$, for $n < p$.

## 2. *Statement of main result and some notation*

Any $\mathcal{A}$-simple smooth map-germ $f: \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$, where $m \geqslant n$, of rank $n-1$ is given by the pre-normal form

$$(x, y, z) \longmapsto (x, g(x, y) + Q(z)),$$

where $(x, y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{m-n}$, $Q(z) = \sum_i \epsilon_i z_i^2$ ($\epsilon_i = \pm 1$) and where $(x, y) \mapsto (x, g(x, y))$ is an $\mathcal{A}$-simple equidimensional corank-1 germ (see Lemma 4·1). Let $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_s): \mathbb{R}^m, S \to \mathbb{R}^n$, $\tilde{f}(S) =: q$, $\tilde{f}_i(x, y_i) = (x, \tilde{g}_i(x, y_i) + Q_i(z))$, $i = 1, \ldots, s := |S|$, be an $s$-germ appearing in a deformation of $f$ (here $S$ is a finite set of source points being mapped to the point $q$ in the target). The rank $n-1$ $\mathcal{K}$-classes of germs $\mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ are those of $A_k$, with representatives $(x, y^{k+1} + Q(z))$, and the $\mathcal{K}$-classes of $s$-germs $A_{(k_1,\ldots,k_s)}$ have an $A_{k_i}$-singularity at the $i$th source point. The stable rank $n-1$ multi-germs are those transverse to their $\mathcal{K}$-class $A_{(k_1,\ldots,k_s)}$, and the isolated stable (or 0-stable) singularities amongst these are those with $\sum_{i=1}^{s} k_i = n$. Let $k(s,n) := (k_1, \dots, k_s)$ be such a partition of $n$ with $s$ summands.

For equidimensional germs $f: \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ the number of isolated stable $A_{k(s,n)}$-points in a generic deformation of $f$, denoted by $r_{k(s,n)}(f)$, can be calculated by dividing the local multiplicity of a certain map-germ $G_{k(s,n)}: \mathbb{C}^{n+s-1}, 0 \to \mathbb{C}^{n+s-1}$ by some overcount factor (see [**18**] and Section 3). For rank $n-1$ germs $f: \mathbb{C}^m, 0 \to \mathbb{C}^n, 0$, where $m > n$, of the form $(x, g(x,y) + Q(z))$ the invariants $r_{k(s,n)}(f)$ can simply be calculated from the associated equidimensional germ $(x, g(x,y))$.

For real germs $f$ the invariants $r_{k(s,n)}(f)$ are defined by complexifying, but clearly the above geometric interpretation no longer holds: the number $r^{\mathbb{R}}_{k(s,n)}(f_t)$ of real $A_{k(s,n)}$-points in a deformation $f_t$ of $f$ now depends on the choice of deformation. One only has the obvious inequality $r^{\mathbb{R}}_{k(s,n)}(f_t) \leqslant r_{k(s,n)}(f)$.

We call a real deformation $f_t$ of $f$ an M-deformation, if the maximal numbers $r_{k(s,n)}(f)$ of 0-stable singularities (for all partitions $k(s,n)$ of $n$) are simultaneously present in the discriminant of $f_t$.

The main result on the existence of M-deformations in the present paper is the following:

THEOREM 2·1. *All $\mathcal{A}$-simple rank $n-1$ germs $f: \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$, where $m \geqslant n$, have an M-deformation.*

*Remark* 2·2. The condition on the rank is necessary: $\mathcal{A}$-simple germs of higher corank do in general not have an M-deformation, as the following example shows. For the corank-2 germ $f = (x^2 - y^2 + x^3, xy)$ the invariants $r_{(2)}(f) = 3$ and $r_{(1,1)}(f) = 2$ are the (complex) cusp and double-fold numbers, respectively. But any real stabilization of $f$ has 3 cusps and no double-fold (see [**19**]).

We now fix some notation. Let $C_n$ denote the local ring of smooth (or complex-analytic) function germs $f: \mathbb{K}^n, 0 \to \mathbb{K}, 0$ and $\mathcal{M}_n$ its maximal ideal. For the groups $\mathcal{A}$ and $\mathcal{K}$ (of left-right and of contact equivalence, respectively) acting on the space of smooth map-germs and for the tangent spaces to the $\mathcal{A}$- and $\mathcal{K}$-orbits we use the usual notation, such as $T\mathcal{A} \cdot f = tf(\mathcal{M}_n \cdot \theta_n) + wf(\mathcal{M}_p \cdot \theta_p)$ and $T\mathcal{K} \cdot f = tf(\mathcal{M}_n \cdot \theta_n) + f^*\mathcal{M}_p \cdot \theta_f$ (a basic reference for these concepts is the survey on determinacy [**21**] by Wall). For equidimensional map-germs $f: \mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ of corank 1 we use source coordinates $(x, y) = (x_1, \dots, x_{n-1}, y)$ such that $f(x,y) = (x, g(x,y))$, and target coordinates $(X_1, \dots, X_n)$. In describing elements of $T\mathcal{A} \cdot f$ we sometimes use the shorter notation $e_i$ for the target and source vector fields $\partial/\partial X_i$ and $\partial/\partial x_i$ (where $x_n = y$).

## 3. *Defining equations of the* 0-*stable invariants*

In view of Lemma 4·1 in this section we consider equidimensional corank-1 germs $f: \mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ of the form $f(x,y) = (x, g(x,y))$. For such map-germs one can embed the space of $s$-fold points in the source (whose $f$-images are a common point in the target) in $\mathbb{K}^{n+s-1}$, with coordinates $(x, y_1, \dots, y_s) = (x_1, \dots, x_{n-1}, y_1, \dots, y_s)$. Recall that $A_{k(s,m)} := A_{(k_1,\dots,k_s)}$, where $m := \sum_{i=1}^{s} k_i$, denotes the $\mathcal{K}$-class of $s$-germs having an $A_{k_i}$-singularity at the $i$th source point. In [**7**, **17**, **18**] the closures of the sets $A_{(k_1,\dots,k_s)}$ in multi-jet space $J^{\ell}_s$, $\ell := \sum_{i=1}^{s}(k_i + 1)$, were explicitly defined by iteration for any $s$ and $m \leqslant n$, and it was shown that these sets are smooth submanifolds of codimension $\sum_{i=1}^{s}(k_i)+s-1$. (Roughly speaking, the conditions for an $A_{k_j}$ singularity at the $j$th source point, with $f$-image some given point in the target, are reduced modulo the corresponding conditions at the source points 1 to $j-1$ and then divided by a suitable power of $y_j - y_{j-1}$.) Pulling back the ideal defining the closures of these sets by the multi-jet extension of $f$ we get an ideal $(j^\ell_s f)^*(\mathcal{I}(\bar{A}_{k(s,n)}))$ in $C_{n+s-1}$, and for $m = n$ the generators of this ideal define an equidimensional map-germ

$$G_{k(s,n)} = (G_1, \ldots, G_{n+s-1})\colon \mathbb{K}^{n+s-1}, 0 \to \mathbb{K}^{n+s-1}$$

whose local multiplicity $m_{G_{k(s,n)}}(0) := \dim_{\mathbb{K}} C_{n+s-1}/G^*_{k(s,n)}\mathcal{M}_{n+s-1}$ is equal to the number $r_{k(s,n)}(f)$ of complex $A_{k(s,n)}$-points appearing in a stabilization of $f$ times an overcount factor $c$ ($c$ is equal to the number of permutations mapping source points of type $A_{k_i}$ to source points of the same type).

It will turn out (see below) that we need the defining equations of the sets $\bar{A}_{k(s,n)}$ only for $s = 1$ and 2, hence we specialize the definitions in [**17**, **18**] to these particular cases. Set $g^{(i)} := \partial^i g/\partial y^i$, then $\bar{A}_{(n)} := \{g^{(1)} = \ldots g^{(n)} = 0\}$. For $s = 2$ we first apply a linear origin-preserving coordinate change $L(x, y_1, y_2) = (x, y_1, y_2 - y_1) =: (x, y, \epsilon)$, and let $g_1^{(i)} := g^{(i)}$. Setting

$$g_2^{(0)} := \sum_{\alpha \geqslant k_1+1} g_1^{(\alpha)} \epsilon^{\alpha-k_1-1}/\alpha!, \qquad g_2^{(i)} := \partial^i g_2^{(0)}/\partial\epsilon^i, \ \ i \geqslant 1,$$

we define

$$\bar{A}_{(k_1, n-k_1)} := \left\{ g_1^{(1)} = \cdots = g_1^{(k_1)} = g_2^{(0)} = \cdots = g_2^{(n-k_1)} = 0 \right\}.$$

Notice that for even $n$ the overcount factor $c$ in $r_{(n/2,n/2)}(f) = c^{-1} \cdot m_{G_{(n/2,n/2)}}(0)$ is 2. For the other 0-stable invariants in the cases $s = 1, 2$ it is one.

The following facts will be useful.

*Remark* 3·1.

(i) Given a pair of $\mathcal{A}_e$-equivalent, equidimensional corank-1 germs $f$ and $f'$, the corresponding pairs of germs $G_{k(s,n)}$ and $G'_{k(s,n)}$ are $\mathcal{K}$-equivalent (see [**18**, lemma 2·3]).

(ii) $r_{k(s,n)}(f) = 0$ for $n + s > m_f(0)$, where $m_f(0)$ denotes the local multiplicity of $f$ at the origin (this follows from the "additivity of the local multiplicities on the diagonal" in the recognition conditions for $\bar{A}_{k(s,n)}$, see [**17**, **18**]). This fact, together with the observation that the $\mathcal{A}$-modality of germs with $m_f(0) \geqslant n+3$ is positive (Lemma 4·4 below), implies that we only have to consider 0-stable invariants with $s = 1, 2$.

In the cases $k(s, n)$, where $r_{k(s,n)}(f)$ is equal to the local multiplicity of $G_{k(s,n)}$, the following will be important in our construction of M-deformations of $f$.

Lemma 3·2. *Suppose $r_{k(s,n)}(f) = m_{G_{k(s,n)}}(0) < \infty$ and that there is a $f'$ such that $[f] \to [f']$ and $r_{k(s,n)}(f) = r_{k(s,n)}(f') + 1$, where $f$ and $f'$ are mono-germs at the origin. Then there is an origin preserving deformation, $f_t$, from $f$ to $f'$ and a neighbourhood in $\mathbb{R}^{n+s-1}$ of 0 in which the s-germ of this deformation has a real $A_{k(s,n)}$-point $q \neq 0$ of multiplicity one (recall that $q = (x, y_1, \ldots, y_s)$). Furthermore, for $m_f(0) - m_{f'}(0) < n + s$ the images $f'(x, y_1) = \cdots = f'(x, y_s)$ and $f'(0)$ are distinct.*

*Proof.* Let $F = (u, f_u)$ be a polynomial $\mathcal{A}$-versal unfolding of $f = f_0$ on $d$ parameters. The closure of the $[f']$ stratum in the base $\mathbb{R}^d$ of $F$ is a semi-algebraic set and, by the curve selection lemma, contains a curve given by the image of a Nash mapping (which is continuous and even $C^\infty$) $t \mapsto u(t)$, $t \in [0, T)$, such that the germ of $f_{u(t)}$ at 0 is $\mathcal{A}$-equivalent to $f'$ for all $t \in (0, T)$. The deformation $G^t_{k(s,n)}$:$[0,T) \times \mathbb{R}^{n+s-1} \to \mathbb{R}^{n+s-1}$ of $G_{k(s,n)} = G^0_{k(s,n)}$ induced by $f_{u(t)}$ is polynomial in $(Q_1, \ldots, Q_{n+s-1}) \in \mathbb{K}^{n+s-1}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $C^\infty$ in $t$. Choose a representative $f \in \mathbb{R}[x, y]^n$, whose germ at 0 belongs to $[f]$, from a Zariski open set of degree $d$ polynomial maps (where $d$ is sufficiently larger than the determinacy degree of the germ of $f$ at 0) such that $G^{-1}_{k(s,n)}(0)$ has outside 0 only isolated points (real or complex) $\tilde{q}_1, \ldots, \tilde{q}_k$ of multiplicity one (and we know that $0 \in G^{-1}_{k(s,n)}(0)$ has multiplicity $r_{k(s,n)}(f) < \infty$). Let $\pi_i(Q_1, \ldots, Q_{n+s-1}) = Q_i$ be the projection from $\mathbb{C}^{n+s-1}$ onto the $i$th coordinate, by a linear coordinate change we can suppose that $\pi_i(\tilde{q}_j) \neq 0$ and $\pi_i(\tilde{q}_j) \neq \pi_i(\tilde{q}_l)$, for all $i$, $j$ and $l \neq j$. By elimination theory, $\pi_i((G^t_{k(s,n)})^{-1}(0)) \subset \mathbb{C}$ is given by the roots of a polynomial $g \in \mathbb{R}[Q_i]$, whose coefficients are real $C^\infty$-functions in $t$. By the continuity of the roots (as functions of the coefficients) there are, for small enough $t$, $k$ simple roots $\pi_i(\tilde{q}^t_j)$, $j = 1, \ldots, k$, with $\pi_i(\tilde{q}^0_j) = \pi_i(\tilde{q}_j) \neq 0$. And, for geometric reasons, we know that $\pi_i(q)$ and $\pi_i(0)$ are roots of multiplicity one and $r_{k(s,n)}(f) - 1$ (tending to 0 for $t \to 0$), that are closer to 0 than any of the $\pi_i(\tilde{q}^t_j)$ (for small enough $t$). Hence $q_i := \pi_i(q) \in \mathbb{R}$, because $q_i$ cannot be the complex conjugate of any other root of $g \in \mathbb{R}[Q_i]$ for any $i$, hence $q = (q_1, \ldots, q_{n+s-1}) \in \mathbb{R}^{n+s-1}$. The last statement of the lemma is clear: if the fibre over $f'(0)$ contains, in addition to 0, extra $A_{k_i}$ points $(x, y_i)$, forming an $A_{k(s,n)}$ singularity, that tend to 0 as $f'$ degenerates to $f$ then $m_f(0) - m_{f'}(0) \geqslant n + s$.

The strategy for showing the existence of a M-deformation is then the following. Suppose that for every germ $f$ there is some $f'$ of lower codimension such that all 0-stable invariants of $f$ and $f'$ differ by at most one (and also suppose that $m_f(0) - m_{f'}(0) \leqslant n$ so that the last statement of the lemma holds for all $s$ – in fact, $m_f(0) - m_{f'}(0)$ will always be 0 or 1). First, consider the 0-stable invariants for which $r_{k(s,n)}(f) = m_{G_{k(s,n)}}(0)$ (for $s \leqslant 2$, this holds for all $k(s,n) \neq (n/2, n/2)$). If $r_{k(s,n)}(f) - r_{k(s,n)}(f') = 1$ then 0 is an $A_{k(s,n)}$ point of multiplicity $r_{k(s,n)}(f') = r_{k(s,n)}(f) - 1$ and we have another nearby $A_{k(s,n)}$ point $q \neq 0$ of multiplicity 1, which is also real (by the lemma). On the other hand, if $r_{k(s,n)}(f) - r_{k(s,n)}(f') = 0$ then 0 is an $A_{k(s,n)}$ point of $f'$ of multiplicity $r_{k(s,n)}(f)$. Continuing in this way, we decrease the codimension of the germ at 0 at each step (and either keep the $A_{k(s,n)}$-multiplicity at 0 the same or decrease it by one and at the same time split-off from 0 one real $A_{k(s,n)}$ point) until the germ at 0 is a stable $A_{k(s,n)}$ point of multiplicity one – at this stage there are $r_{k(s,n)}(f) - 1$ extra stable $A_{k(s,n)}$ points outside 0. Finally, for $k(s,n) = (n/2, n/2)$ we have to be more careful, because the multiplicity of $G_{(n/2,n/2)}$ is twice the number of $A_{(n/2,n/2)}$ points and the above lemma does not apply. In this case $f$ can have an $A_{(n/2,n/2)}$ point of multiplicity $r$ at 0 and $f'$ one of multiplicity $r - 1$, but the $A_{(n/2,n/2)}$ point $q$ of multiplicity one that splits-off 0 (that corresponds to a pair of points of $(G^t_{(n/2,n/2)})^{-1}(0)$) may not be real. But the argument in the conclusion of the proof of Theorem 4·3 will show that $q$ is real for some suitable choice of sign of the deformation parameter $t$.

*Remark* 3·3. This strategy for obtaining M-deformations and Lemma 3·2 hold for all $k(s,n)$ for which $r_{k(s,n)}(f) = m_{G_{k(s,n)}}(0)$ (see section 2 of [**18**] for the definition of the maps $G_{k(s,n)}$ for general $s$, in the present paper we only need the case $s \leqslant 2$). They are also valid for 0-stable invariants of corank-1 germs in dimensions $n < p$ corresponding to partitions $k(s,m)$ (where 0 summands are allowed) satisfying the equality $(m+s-1)(p-n+1) = (n+s-1)$ for which $r_{k(s,m)}(f) = m_{G_{k(s,m)}}(0)$ (see section 3 of [**18**]). This is useful in verifying that $\mathcal{A}$-simple corank-1 germs from $\mathbb{R}^3$ to $\mathbb{R}^4$ have M-deformations (see the concluding remarks below).

## 4. *M-deformations and $\mathcal{A}$-simplicity*

We begin with an outline of the proof of Theorem 2·1. The key property of $\mathcal{A}$-simple singularities of minimal corank, from which a M-deformation can then be obtained inductively (see discussion above), is that such germs $f$ can be deformed into a germ $f'$ of lower codimension whose 0-stable invariants differ from $f$ by at most one. The proof of this property $(*)$ consists of the following main steps:

(i) reduction to the equidimensional case $\mathbb{R}^n \to \mathbb{R}^n$;

(ii) for $n \geqslant 3$ there are no $\mathcal{A}$-simple orbits of local multiplicity $\geqslant n+3$;

(iii) germs $f$ of local multiplicity $n+1$ have $r_{k(1,n)}(f) = r_{(n)}(f)$ as the only non-zero 0-stable invariant, and positive $\mathcal{K}$-modality of $G_{(n)}$ implies positive $\mathcal{A}$-modality of $f$. Property $(*)$ then follows from the analogous property for the local multiplicities of $\mathcal{K}$-simple equidimensional map-germs;

(iv) germs $f$ of local multiplicity $n+2$ have $r_{k(s,n)}(f)$, where $s = 1, 2$, as the only non-zero 0-stable invariants. In this case property $(*)$ follows from a partial classification of $\mathcal{A}$-simple germs listed in Lemma 4·10 (we do not know whether all the germs in this list are $\mathcal{A}$-simple, but any $\mathcal{A}$-simple germ of multiplicity $n+2$ is equivalent to some germ in this list). This partial classification is the most unpleasant part of the proof. (Notice that the proofs of the Lemmas 4·9 and 4·10 merely describe the high-level structure and the cases to be considered, but omit all the routine details, which just require some care due to the fact that the dimension $n$ is not fixed.)

We begin with the reduction to the equidimensional case.

Lemma 4·1. *Any $\mathcal{A}$-simple smooth map-germ $f:\mathbb{R}^m, 0 \to \mathbb{R}^n, 0$, where $m \geqslant n$, of rank $n-1$ is given by the pre-normal form*

$$(x, y, z) \longmapsto (x, g(x, y) + Q(z)),$$

*where $(x, y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{m-n}$, $Q(z) = \sum_i \epsilon_i z_i^2$ $(\epsilon_i = \pm 1)$ and where $(x, y) \mapsto (x, g(x, y))$ is an $\mathcal{A}$-simple equidimensional corank-1 germ.*

*Proof.* The argument is similar to the one for $n = 2$ (see lemmas 1·1 and 1·2 in [**19**]). After a coordinate change we can assume that $f$ is for some $r \in \{0, \ldots, m-n+1\}$ given by

$$h = \left( x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1}, y_1, \ldots, y_r) + \sum_{i=1}^{m-n-r+1} \epsilon_i z_i^2 \right),$$

where $g(x, 0) = 0$ and $g(0, \ldots, 0, y_1, \ldots, y_r) \in \mathcal{M}_r^3$. Two such germs $h = (x, g, (x, y) + Q(z))$ and $h' = (x, g'(x, y) + Q(z))$ are $\mathcal{A}$-equivalent if and only if the corresponding germs $(x, g(x, y))$ and $(x, g'(x, y))$ are $\mathcal{A}$-equivalent. We claim that for $r \geqslant 2$ there are no simple $\mathcal{A}$-orbits over the 2-jet of $(x, g(x, y))$, which for $r = 2$ is $\mathcal{A}^2$-equivalent to

$$(x, a_1x_1y_1 + \cdots + a_{n-1}x_{n-1}y_1 + b_1x_1y_2 + \cdots + b_{n-1}x_{n-1}y_2).$$

The least degenerate $\mathcal{A}^2$-orbit, corresponding to $a_i \neq 0$ and $a_ib_j \neq a_jb_i$ (for some $i$ and $j \neq i$), has the representative (taking $i = 1$, $j = 2$)

$$\sigma := (x, x_1y_1 + x_2y_2).$$

A complete 3-transversal for $\sigma$ is given by

$$t := \big(0, ay_1^3 + by_1^2y_2 + cy_1y_2^2 + dy_2^3 + ex_3y_1^2 + \cdots + fx_{n-1}y_2^2\big).$$

Now we can argue as in [**19**, lemma 1·2] to show that the subspace $\mathbb{K}\{y_1^iy_2^j \cdot e_n : i + j = 3\}$ of $T\mathcal{A}^3 \cdot (\sigma + t)$ is foliated by (at least) a 1-parameter family of orbits. Notice that the more degenerate $\mathcal{A}^2$-orbits and the orbits corresponding to $r > 2$ are all adjacent to $T\mathcal{A}^2 \cdot \sigma$, which implies the claim.

*Remark* 4·2. Notice that the discriminants of the germs $(x, g(x, y) + Q(z))$ and $(x, g(x, y))$ coincide.

From now on we will therefore consider equidimensional germs of corank-1. For such germs we have the following result.

THEOREM 4·3. *All $\mathcal{A}$-simple corank-1 germs $f: \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ have an M-deformation.*

For $n = 2$ all real types of stabilizations of all simple corank-1 germs are known (see [**16**]) and amongst these there is always an M-deformation (the result also holds for functions of one variable, $n = 1$), hence we can concentrate on $n \geqslant 3$. The theorem will follow from Lemmas 4·4, 4·7–4·11 below.

LEMMA 4·4. *For $n \geqslant 3$, all $\mathcal{A}$-orbits inside $\mathcal{K}(x_1, \ldots, x_{n-1}, y^{\geqslant n+3})$ are at least unimodal.*

Lemma 4·4 will follow from Lemma 4·5 below in which we prove that $\mathcal{K}$-orbits of local multiplicity $\geqslant n + 3$ do not contain an open $\mathcal{A}$-orbit, that is one for which $T\mathcal{A} \cdot f = T\mathcal{K} \cdot f$. It then follows that $\mathcal{K} \cdot f$ contains a Zariski open subset foliated by $\mathcal{A}$-orbits of minimal codimension $c > 0$, hence all its $\mathcal{A}$-orbits are non-simple.

LEMMA 4·5. *Let $f: \mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ be a corank 1 $\mathcal{A}$-finitely determined germ. Suppose that the $\mathcal{A}$-orbit of $f$ is open in its $\mathcal{K}$-orbit. Then $m_f(0) \leqslant n + 2$.*

We shall need the following condition for the openess of an $\mathcal{A}$-orbit within its $\mathcal{K}$-orbit. A proof of this result first appeared in [**20**, theorem 5·1].

PROPOSITION 4·6. ([**20**, Theorem 5·1]). *Let $f: \mathbb{K}^n, 0 \to \mathbb{K}^p, 0$ be a $\mathcal{K}$-finitely determined germ and denote by $\{v_1, v_2, \ldots, v_r\}$ a basis for*

$$N := \frac{\theta_f}{T\mathcal{A}_e \cdot f + f^*\mathcal{M}_p \cdot \theta_f}.$$

*The $\mathcal{A}$-orbit of $f$ is open in its $\mathcal{K}$-orbit if and only if $f_iv_j \in T\mathcal{A} \cdot f$, $i = 1, \ldots, p$; $j = 1, \ldots, r$ (Mod $f^*\mathcal{M}_p^2 \cdot \theta_f$).*

*Proof (sketch).* The $\mathcal{K}$-finiteness of $f$ and the condition $N = \mathbb{K}\{v_1, \ldots, v_r\}$ imply, using the Preparation theorem, that $\theta_f / T\mathcal{A}_e \cdot f$ is a finite $C_p$-module via $f^*$. Then $\theta_f = T\mathcal{A}_e \cdot f + f^* C_p\{v_1, \ldots, v_r\}$ (i).

From the hypothesis $f_i v_j \in T\mathcal{A} \cdot f$ we then get $f^*\mathcal{M}_p \cdot v_j \subset T\mathcal{A} \cdot f$ (ii), which together with (i) implies that $\theta_f = T\mathcal{A}_e \cdot f + \mathbb{K}\{v_1, \ldots, v_r\}$. Multiplication by $f^*\mathcal{M}_p$, and using (ii), then gives $f^*\mathcal{M}_p \cdot \theta_f \subset T\mathcal{A} \cdot f$, hence $T\mathcal{A} \cdot f = T\mathcal{K} \cdot f$. And the converse is obvious.

*Proof of Lemma* 4·5. When $m_f(0) \leqslant n + 1$, the only map-germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ with the property that the $\mathcal{A}$-orbit is open in its $\mathcal{K}$-orbit are the infinitesimally stable ones. Then, we can assume that $m_f(0) = n + l$, $l \geqslant 2$.

Let $f(x, y) = (x, g(x, y))$, where $x = (x_1, \ldots, x_{n-1})$, and $g(x, y) = y^{n+l} + \phi_1(x)y + \cdots + \phi_{n-1}y^{n-1} + \sum_{i=n}^{n+l-2} \phi_i(x)y^i$.

The hypothesis that the $\mathcal{A}$-orbit of $f$ is open in the $\mathcal{K}$-orbit implies that rank $d\phi(0) = n - 1$, where $\phi : \mathbb{K}^{n-1}, 0 \to \mathbb{K}^{n+l-2}, 0$ is defined by $\phi(x) = (\phi_1(x), \ldots, \phi_{n+l-2}(x))$ (note that this rank is an $\mathcal{A}$-invariant of $f$). It also follows that after changing coordinates, we can write $f$ in the form:

$$(x, g(x, y)) = \left( x, y^{n+l} + x_1 y + \cdots + x_{n-1} y^{n-1} + \sum_{i=n}^{n+l-2} \phi_i(x) y^i \right).$$

Moreover, from Proposition 4·6 it follows that the $n(n + l - 2) + 1$ elements $(0, y^{n+l})$, $(0, g(x, y)y^j)$, $(0, x_i y^j)$, $i = 1, \ldots, n - 1$ and $j = 1, \ldots, n + l - 2$, must be in $T\mathcal{A} \cdot f + f^*\mathcal{M}_n^2 \cdot \theta_f$ (with $p = n$). The equations relating these elements are:

$$wf(X_n \cdot e_n) + f^*\mathcal{M}_n^2 \cdot \theta_f = 0, (\mathrm{Mod}\, T\mathcal{A} \cdot f)$$

$$tf(y^j \cdot e_n) + f^*\mathcal{M}_n^2 \cdot \theta_f = 0, \;\; j = 1, \ldots, n + l - 1 \,(\mathrm{Mod}\, T\mathcal{A} \cdot f)$$

$$tf(x_i \cdot e_j) + f^*\mathcal{M}_n^2 \cdot \theta_f = 0, \;\; i = 1, \ldots, n - 1, \;\; j = 1, \ldots, n - 1, \,(\mathrm{Mod}\, T\mathcal{A} \cdot f)$$

$$tf(g \cdot e_j) + f^*\mathcal{M}_n^2 \cdot \theta_f = 0, \;\; j = 1, \ldots, n - 1, \,(\mathrm{Mod}\, T\mathcal{A} \cdot f)$$

This system has $n(n + l - 2) + 1$ unknowns and $n^2 + l$ equations. Hence we must have $n^2 + l \geqslant n(n + l - 2) + 1$, which holds if and only if $l \leqslant (n/(n - 1)) + 1$. Then, when $n \geqslant 3$, it follows that $l \leqslant 2$.

For $m_f(0) \leqslant n$ all invariants $r_{k(s,n)}(f)$ are zero, hence we have to consider the cases $m_f(0) = n + 1$ and $n + 2$.

4·1. *The case* $m_f(0) = n + 1$

It is sufficient to consider germs (see [**15**], proposition 4·8)

$$f = (x, y^{n+1} + P_1(x)y + \cdots + P_{n-1}(x)y^{n-1}),$$

and $m = n$ implies $s = 1$, hence $G_{(n)}$, which is $\mathcal{K}$-equivalent to

$$P := (P_1(x), \ldots, P_{n-1}(x)),$$

is the only relevant germ here.

LEMMA 4·7. *If $G_{(n)}$ is not $\mathcal{K}$-simple then $f$ is not $\mathcal{A}$-simple.*

*Proof.* The hypothesis implies that there exists a 1-parameter deformation $P^t := (P_1^t(x), \ldots, P_{n-1}^t(x))$ of $P = P^0$ meeting an infinite number of distinct $\mathcal{K}$-orbits. Hence $f_t := (x, y^{n+1} + \sum_{i=1}^{n-1} P_i^t(x)y^i)$ is a deformation of $f = f_0$ meeting an infinite number of $\mathcal{A}$-orbits (by Remark 3·1(i)).

Note: if $G_{(n)}$ and $G'_{(n)}$ correspond to $f$ and $f'$, respectively, then the above argument shows that $[G_{(n)}] \to [G'_{(n)}]$ implies $[f] \to [f']$ (i.e., for corank-1 germs of local multiplicity $n+1$, adjacency of $\mathcal{K}$-orbits implies that of $\mathcal{A}$-orbits). The following claim now implies, by downward induction on $\mathcal{K}$-codimension of $P$, that all $\mathcal{A}$-simple germs of local multiplicity $n+1$ have M-deformations, because $r_{(n)}(f) = m_P(0)$ (notice that a $\mathcal{K}$-codimension decreasing deformation $P^t$ of $P = P^0$ induces a deformation $f^t$ of $f = f^0$ that decreases the $\mathcal{A}$-codimension, because the $\mathcal{A}$-types of $f^t$, $t \neq 0$, and $f$ differ by Remark 3·1(i)).

LEMMA 4·8. *Let $P\colon \mathbb{R}^{n-1}, 0 \to \mathbb{R}^{n-1}, 0$ be a $\mathcal{K}$-simple germ then there exists a germ $P'$ of lower $\mathcal{K}$-codimension, to which $P$ is $\mathcal{K}$-adjacent to, such that $m_P(0) - m_{P'}(0) \leqslant 1$.*

*Proof.* This uses the classification of $\mathcal{K}$-simple equidimensional real germs. For complex-analytic germ the classification of the $\mathcal{K}$-simple orbits and the description of their adjacencies is due to Giusti [**6**]. In the real case there is no complete published reference for the classification of the $\mathcal{K}$-simple equidimensional germs and their adjacencies, but at least the classification is well-known (the preprint version of [**19**] reviews the published and unpublished work on this subject, comparing notation for real and complex orbits and giving some partial adjacencies over the reals).

Here is the real classification and some partial adjacencies $X \to Y$ with the property that the local multiplicities of $X$ and $Y$ differ by at most one (this gives the desired result). For real orbits $X$ having the same normal form as the complex ones we use the notation in [**6**], or we use $X^{\pm}$ if a complex orbit $X$ simply has two real forms distinguished by different signs. This yields $A_k = (x, y^{k+1})$, $G_5 = (x^2, y^3)$, $G_7 = (x^2, y^4)$, $I_{2k-1} = (x^2 + y^3, y^k)$ $(k \geqslant 4)$, $I_{2k+2} = (x^2 + y^3, xy^k)$ $(k \geqslant 3)$ and $H_k^{\pm} = (x^2 \pm y^{k-3}, xy^2)$, where $\pm$ agree for even $k \geqslant 6$. The lower indices in Giusti's notation are Milnor numbers, to get the local multiplicities we simply have to add one (all the germs $P$ are weighted homogeneous, hence $\mu(P) = m_P(0) - 1$). Finally, for Giusti's complex orbits $F_{k+l-1}$ we now have 3 real forms. Here we use Mather's notation (where lower indices don't denote Milnor numbers!): $I_{k,l} = (xy, x^k + y^l)$ of local multiplicity $k+l$ $(2 \leqslant k \leqslant l)$, $II_{k,l} = (xy, x^k - y^l)$ of local multiplicity $k+l$ $(2 \leqslant k \leqslant l$ and $k, l$ both even), and $IV_k = (x^2 + y^2, x^k)$ of local multiplicity $2k$ $(k \geqslant 3)$. For the purpose of this lemma the following adjacencies are then sufficient: $A_k \to A_{k-1}$, $I_{k,l} \to I_{k,l-1}$, $I_{2,2} \to A_3$, $II_{k,l} \to I_{k,l-1}$, $II_{2,2} \to A_2$, $IV_k \to A_{2k-1}$, $G_5 \to I_{3,3}$, $G_7 \to H_7^+$, $I_l \to I_{l-1}$, $I_7 \to H_6^+$, $H_k^{\pm} \to H_{k-1}^+$ and $H_6^+ \to I_{3,4}$.

4·2. *The case $m_f(0) = n+2$*

Here we consider the prenormal form

$$f = (x, y^{n+2} + P_1(x)y + \cdots + P_n(x)y^n),$$

and $m = n$ implies $s = 1$ or 2. Hence $G_{(n)}$ and $G_{(n-l,l)}$ $(l = 1, \ldots, [n/2])$ are the only germs corresponding to non-zero 0-stable invariants of $f$.

Lemma 4·9. *Any $\mathcal{A}$-simple germ of local multiplicity $n + 2$ has one of the following prenormal forms*:

$$f_n = (x, y^{n+2} + x_1 y + \cdots + x_{n-1} y^{n-1}),$$

*or*

$$f_{n-1} = (x, y^{n+2} + x_1 y + \cdots + x_{n-2} y^{n-2} + P_{n-1}(x_{n-1}) y^{n-1} + x_{n-1} y^n),$$

*where $P_{n-1}$ belongs to the square of the maximal ideal.*

*Proof.* We divide the proof in several steps.

*Step* 1. Any $\mathcal{A}$-simple germ of local multiplicity $n + 2$ has the prenormal form

$$f_j = (x, y^{n+2} + x_1 y + \cdots + x_{j-1} y^{j-1} + P_j y^j + x_j y^{j+1} + \cdots + x_{n-1} y^n),$$

where $P_j := P_j(x_j, \ldots, x_{n-1})$, $1 \leqslant j \leqslant n$, is in the square of the maximal ideal.

Consider the prenormal form $f = (x, y^{n+2} + P_1(x)y + \ldots + P_n(x)y^n)$: suppose the differential of $(P_1, \ldots, P_n)$ has rank $r \leqslant n - 2$ at the origin. By a right change in the $x_i$ we can assume that $r$ of the $P_j$ are given by $x_1, \ldots, x_r$. All such $f$ are adjacent to some germ of the type

$$(x, y^{n+2} + x_1 y + \cdots + x_r y^r + P_{r+1} y^{r+1} + \cdots + P_n y^n),$$

where the $P_i$ are in the square of the maximal ideal and only depend on $x_{r+1}, \ldots, x_{n-1}$ (notice: if $f$ is not of this type and some $P_j$, $j < r + 1$, of $f$ is in the square of the maximal ideal and $x_j$ appears linearly in some $P_i$, $i > r$, then we can deform the last component of $f$ by $t \cdot x_j y^j$; for $t \neq 0$ we can then reduce $f$ to the desired form). Hence assume that $f$ is of this type. We can even assume that $f$ is the "best possible" germ of this type, namely that $r = n-2$ and $P_{n-1}, P_n \in \langle x_{n-1}^2 \rangle$ (in the sense that the other germs of this type are adjacent to one of this form). Then let $f' : \mathbb{R}^{n-1}, 0 \to \mathbb{R}^{n-1}, 0$ be the restriction of $f$ to $x_{n-1} = 0$, and consider $\theta_{f'} \subset \theta_f$ as a linear subspace. Clearly

$$T\mathcal{A} \cdot f' = (T\mathcal{A} \cdot f + \langle x_{n-1} \rangle \theta_f) \cap \theta_{f'},$$

hence $f$ has a modulus, because $f'$ has one (by the proof of Lemma 4·4).

Hence, for simple germs $f$, the differential of $(P_1, \ldots, P_n)$ has rank $n - 1$ at the origin. Let $P_j$ be the first $P_\ell$ such that the rank of the differential of $(P_1, \ldots, P_\ell)$, for increasing $\ell \geqslant 1$, is less than $\ell$. We then claim that by direct coordinate changes we can assume that $P_i = x_i$, for $i < j$, $P_j = P_j(x_j, \ldots, x_{n-1}) \in \mathcal{M}_{n-j}^2$, and $P_{i+1} = x_i$, for $i \geqslant j$, as required.

(Sketch of proof of this claim: the rank of the differential of $(P_1, \ldots, P_{j-1})$ is $j - 1$, by permuting the $x_i$ we can assume that $(\partial P_\ell / \partial x_i)$, $1 \leqslant i, \ell \leqslant j - 1$, has rank $j - 1$. By a right coordinate change $h = (k(x), y)$ and a subsequent left coordinate change $(k^{-1}(X_1, \ldots, X_{n-1}), X_n)$ we get $P_i = x_i$, for $i < j$, without changing the first $n - 1$ component functions of $f$. Now we can remove terms $x_i h(x)$, $i < j$, from $P_j$ by successive coordinate changes, for example $y \mapsto y - i^{-1} h(x) y^{j-i+1}$ removes $x_i h(x)$ from $P_j$ (but this introduces higher order terms in the $P_i = x_i$, $i < j$, as well as in $P_j$, which can be pushed to higher and higher order by subsequent coordinate changes), hence $(P_1, \ldots, P_j) = (x_1, \ldots, x_{j-1}, P_j(x_j, \ldots, x_{n-1}))$ with $P_j$ in the square of the maximal ideal. In the same way one can also remove $x_i$, $i < j$, from the $P_\ell$, $\ell > j$, so that $(P_j, \ldots, P_n)$ depends only on $x_j, \ldots, x_{n-1}$. From the facts that the differential of $(P_1, \ldots, P_n)$ has rank $n-1$ and that $P_j$ is in the square of the maximal ideal it then follows that $(\partial P_\ell/\partial x_i)$, $j \leqslant i \leqslant n-1$, $j+1 \leqslant \ell \leqslant n$ has maximal rank, and that $P_{i+1} = x_i$, $i \geqslant j$, after some right change of the form $h = (x_1, \ldots, x_{j-1}, k(x_j, \ldots, x_{n-1}), y)$, where $k \in \mathrm{Diff}(n-j)$, and a subsequent left coordinate change $(X_1, \ldots, X_{j-1}, k^{-1}(X_j, \ldots, X_{n-1}), X_n)$.)

*Step* 2. Any germ of type $f_j$ is non-simple if all germs of type $f_{j+1}$ are non-simple. We will show that any $\mathcal{A}$-orbit in $\mathcal{K}(x, y^{n+2})$ of type $f_j$ is adjacent to some orbit of type $f_{j+1}$. Take a deformation of

$$f_j = (x, y^{n+2} + x_1 y + \cdots + x_l y^l + P(x_{l+1}, \ldots, x_{n-1})y^{l+1}$$
$$+ x_{l+2}y^{l+2} + \cdots + x_{n-1}y^{n-1} + x_{l+1}y^n),$$

with $l = j-1$ and $P$ in the square of the maximal ideal (recall prenormal form from Step 1), by $t.(0, x_{l+2}y^{l+1})$. For non-zero $t$ we apply successive coordinate changes

$$x_{l-2} \mapsto t^{-1}(x_{l-2} - Q(x_{l+1}, \ldots, x_{n-1})), \quad Q \in \mathcal{M}^2$$

$$x_{l-2} \longmapsto x_{l-2} - t^{-1}x_{l+2}y, \quad \text{etc.}$$

and obtain

$$(x, y^{n+2} + x_1 y + \cdots + x_l y^l + x_{l+2}y^{l+1} + Q'(x_{l+1}, \ldots, x_{n-1})y^{l+2}$$
$$+ x_{l+3}y^{l+3} + \cdots + x_{n-1}y^{n-1} + x_{l+1}y^n),$$

where $Q' \in \mathcal{M}^2$, which is of type $f_{l+2} = f_{j+1}$.

*Step* 3. All $\mathcal{A}$-orbits in $\mathcal{K}(x, y^{n+2})$ of type $f_{n-2}$ have modality at least one.

Set $s := (x, x_1 y + \cdots + x_{n-3}y^{n-3})$ and consider a general $n$-jet

$$f = s + \left(0, \left(ax_{n-2}^2 + bx_{n-2}x_{n-1} + cx_{n-1}^2\right)y^{n-2} + (dx_{n-2} + ex_{n-1})y^{n-1}\right)$$

over $s$. We have the following three cases:

*Case* 1. $d = e = 0$: not all $x_i$ appear linearly in some $P_j$, where $(x, y^{n+2} + \sum P_j(x)y^j)$. This leads to non-simple orbits (see Step 1).

*Case* 2. $e$ and $d$ are not both 0, hence we can take (after a suitable coordinate change) $e = 1$, $d = 0$ in the $n$-jet $f$ above. The least degenerate $\mathcal{A}^n$-orbit is then given by $a \neq 0$ (for $a = 0$ see Case 3. below) with representative $f = s + (0, x_{n-2}^2 y^{n-2} + x_{n-1}y^{n-1})$. A complete $(n+1)$-transversal for this $f$ is given by $(0, a'x_{n-2}y^n + b'y^{n+1})$.

There are three cases to be considered (at the $(n+1)$-jet level):

(2·1) $b' \neq 0$: leads to $\mathcal{A}$-orbits in $\mathcal{K}(x, y^{n+1})$, see earlier Section 4·1.
(2·2) $b' = 0$, $a' \neq 0$: $s' \sim s + (0, x_{n-2}^2 y^{n-2} + x_{n-1}y^{n-1} + x_{n-2}y^n)$.
(2·3) $a' = b' = 0$: $(n+1)$-jet $f$.

We have to consider the last two cases further.

*Case* 2·2: an $(n+2)$-transversal in this case is $(0, ay^{n+2})$. For $f := s' + (0, ay^{n+2})$ we have 3 generators for the $a$-subspace of $T\mathcal{A}^{n+1} \cdot f$ (suppressing terms that are obviously in $T\mathcal{A}^{n+1} \cdot f$):

$$wf(X_n \cdot e_n) = \left(0, x_{n-2}^2 y^{n-2} + x_{n-2}y^n + ay^{n+2}\right),$$

$$tf(y\cdot e_n) = \big(0, (n-2)x_{n-2}^2 y^{n-2} + n x_{n-2} y^n + (n+2) a y^{n+2}\big),$$

$$tf(x_{n-2}\cdot e_{n-2}) = \big(0, 2x_{n-2}^2 + x_{n-2} y^n\big).$$

The resulting 3 by 3 matrix has rank $<3$, hence $a$ is a modulus:

$$f = \big(x, a y^{n+2} + x_1 y + \cdots + x_{n-3} y^{n-3} + x_{n-2}^2 y^{n-2} + x_{n-1} y^{n-1} + x_{n-2} y^n\big).$$

The least degenerate orbit in Case 2·2 is therefore non-simple.

*Case* 2·3: the germs with $(n+1)$-jet

$$f = s + \big(0, x_{n-2}^2 y^{n-2} + x_{n-1} y^{n-1}\big)$$

are adjacent to those in Case 2·2, hence non-simple.

This concludes 2·1 to 2·3 in Case 2. We now come to the last case concerning the general $n$-jet $f$ at the beginning of Step 3.

*Case* 3. $e = 1$ , $a = d = 0$: for $b \neq 0$ the $n$-jet

$$f = s + (0, (b x_{n-2} + c x_{n-1}) x_{n-1} y^{n-2} + x_{n-1} y^{n-1})$$

is equivalent to $s' \coloneqq s + (0, x_{n-2} x_{n-1} y^{n-2} + x_{n-1} y^{n-1})$. A complete $(n+1)$-transversal for $s'$ is given by

$$t \coloneqq \big(0, a' x_{n-2}^3 y^{n-2} + b' x_{n-2} y^n + c' y^{n+1}\big).$$

Setting $f \coloneqq s' + t$, the subspace of $T\mathcal{A}^{n+1}\cdot f$ spanned by

$$x_{n-2} x_{n-1} y^{n-2}, x_{n-2}^3 y^{n-2}, x_{n-1} y^{n-1}, y^{n+1}, x_{n-2}^2 y^{n-1}, x_{n-2} y^n$$

in the $e_n$-component has the following generators

$$tf(x_{n-1}\cdot e_{n-1}), tf(x_{n-2}\cdot e_{n-2}), wf(X_n\cdot e_n), tf(y\cdot e_n), tf(x_{n-1}\cdot e_n), tf\big(x_{n-1}^2\cdot e_{n-2}\big).$$

The resulting 6 by 6 matrix has rank $\leqslant 5$ (here we work modulo monomials that are outside the subspace in question and are obviously in $T\mathcal{A}^{n+1}\cdot f$). Hence all orbits over the $(n+1)$-jet $f$, and all orbits corresponding to $b = 0$ above (being adjacent to these), are non-simple.

We can now conclude that all orbits in $\mathcal{K}(x, y^{n+2})$ of type $f_{n-2}$ are non-simple: they either lie in the closure of the non-simple orbits in $\mathcal{K}(x, y^{n+2})$ considered in 2·2 or in the closure of the non-simple orbits in $\mathcal{K}(x, y^{n+1})$ considered in 3.

Steps 1 to 3 imply that the simple $\mathcal{A}$-orbits in $\mathcal{K}(x, y^{n+2})$ must be of type $f_n$ or $f_{n-1}$, and it is clear that we can take $P_n \equiv 0$ in $f_n$.

Lemma 4·10. *Any $\mathcal{A}$-simple germ of local multiplicity $n+2$ is equivalent to one of the following germs*:

$$\tilde{f} = (x, y^{n+2} + x_1 y + \cdots + x_{n-1} y^{n-1})$$

*or*

$$\tilde{f}_k = \big(x, y^{n+2} + x_1 y + \cdots + x_{n-2} y^{n-2} + x_{n-1}^k y^{n-1} + x_{n-1} y^n\big),$$

*where $2 \leqslant k < (n+3)/2$ (for odd $n$) or $2 \leqslant k$ (for even $n$), or for odd $n$*

$$\tilde{f}_\infty = (x, y^{n+2} + x_1 y + \cdots + x_{n-2} y^{n-2} + x_{n-1} y^n).$$

(Notice that we do not claim that all these germs are $\mathcal{A}$-simple, just that any $\mathcal{A}$-simple germ of multiplicity $n+2$ must be equivalent to one of these germs.)

*Proof.* From Lemma 4·9,

$$f = (x, y^{n+2} + x_1 y + \cdots + x_{n-2}y^{n-2} + p(x_{n-1})y^{n-1} + q(x_{n-1})y^n),$$

where $p$ and $q$ do not both belong to the square of the maximal ideal.

When the linear part of $p$ is non-zero, we can use the weighted version of the Complete Transversal Method, as presented in [**3**] to prove that $f$ is $\mathcal{A}$ equivalent to

$$\tilde{f} = (x, y^{n+2} + x_1 y + \cdots + x_{n-1}y^{n-1})$$

The calculations in the second case, when $p$ belongs to the square of the maximal ideal are harder. Under this assumption, and given the weights $w(x_i) = n+2-i$, for $i = 1, \ldots, n-2$, $w(x_{n-1}) = 2$ and $w(y) = 1$, the weighted homogeneous part of degree $n+2$ of such a germ is

$$f = (x, y^{n+2} + x_1 y + \cdots + x_{n-2}y^{n-2} + x_{n-1}y^n).$$

In what follows we denote by $\mathcal{M}_w^j$ the ideal in $C_n$ generated by all monomials of filtration $j$.

We divide the calculations in steps, using again the weighted Complete Transversal Method to prove that:

*Step* 1. All terms of filtration $n+2k$, $k \geqslant 1$, belong to $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+1}\theta_f$,

*Step* 2. If fil $v(x,y) = n+2k+1$, then $(0, v(x,y)) \equiv (0, x_{n-1}^{k+1}y^{n-1})$, Mod $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2}\theta_f$, $k \geqslant 1$.

*Step* 3. For $n$ odd and $k \geqslant (n+3)/2$ the term $(0, x_{n-1}^{k+1}y^{n-1})$ belongs to $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2}\theta_f$.

Notice that the following elements are in $T\mathcal{A}_1 \cdot f$:

(a) $wf(X_n e_n) = (0, y^{n+2} + x_1 y + \cdots + x_{n-2}y^{n-2} + x_{n-1}y^n)$,

(b) $tf(\alpha(x) \cdot e_n) = (0, \alpha(x)((n+2)y^{n+1} + x_1 + 2x_2 y + \cdots + nx_{n-1}y^{n-1}))$, $\forall \alpha \in \mathcal{M}_n$, and

(c) $tf(\alpha \cdot e_j) = (0, \alpha y^j), 1 \leqslant j \leqslant n-2$; $tf(\alpha \cdot e_{n-1}) = (0, \alpha y^n)$, $\forall \alpha \in \mathcal{M}_x$ or $\alpha = X_n$. Notice also that

(d) If $\eta(x,y) \in T\mathcal{A}_1 \cdot f$, then $\alpha(x)\eta(x,y) \in T\mathcal{A}_1 \cdot f$, $\forall \alpha \in \mathcal{M}_x$.

*Step* 1. We use induction. For $k = 2$ it follows easily that all terms of filtration $n+4$ are in $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+5}\theta_f$. By the induction hypothesis, $(0, y^{n+2l})$ and all terms $(0, \alpha(x)y^j)$, with fil $\alpha(x)y^j$ equal to $n+2l$, $1 \leqslant l \leqslant k$, are in $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+1}\theta_f$.

Let $v(x,y) = \alpha(x)y^j$, fil $(v) = n+2k+2$. If $0 \leqslant j \leqslant n-2$ or $j = n$, it follows from (c) and (d) that $(0, v) \in T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+1}\theta_f$. Otherwise, there are three possibilities:

(i) $v(x,y) = \alpha(x)y^{n+2l}$, $1 \leqslant l \leqslant k$,

(ii) $v(x,y) = \alpha(x)y^{n+2l+1}$, $\alpha \in \mathcal{M}_x^2$, $1 \leqslant l \leqslant k$,

(iii) $v(x,y) = x_{n-2(k-l)-1}y^{n+2l-1}$, $1 \leqslant l \leqslant k$,

Case (i) now follows easily from the induction hypothesis and equation (d).

In case (ii), we can assume $\alpha(x)$ is a monomial of filtration $2k-2l+1$. Since $n+2l+1$ and $n+2k+2$ have different parities, there is an index $j$ such that $\alpha(x) = x_j\beta(x)$ and fil $(x_j y^{n+2l+1}) = n+2k'+2$, for some $k' < k$. Then the result follows again from the induction hypothesis and (d).

In case (iii), we first use equation (b) to write:

$$(0, v) \equiv 0, \operatorname{Mod} T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2l}\theta_f,$$

where

$$v(x, y) = (n+2)y^{n+2l-1} + x_1 y^{2l-2} + \cdots + jx_j y^{j+2l-3} + \cdots + nx_{n-1}y^{n+2l-3}$$

and $\operatorname{fil}(x_j y^{j+2l-3}) = n + 2l - 1$.

If the parity of $j + 2l - 3$ is equal to the parity of $n + 2l - 1$ then $\operatorname{fil}(x_{n-2(k-l)-1}y^{j+2l-3}) = n + 2l' + 2$ for some $l' < k$, and we can apply the induction hypothesis. The other possibility is $\operatorname{fil}(y^{j+2l-3}) = n + 2l' + 2$ for some $l' < k$, and we again get the result.

*Step* 2. One can easily check the statement for $k = 1$. Let $v(x, y) = \alpha(x)y^j$, $\operatorname{fil} v = n + 2l + 1$, $1 \leqslant l \leqslant k$.

By the induction hypothesis, $(0, v(x, y)) \equiv (0, x_{n-1}^{l+1}y^{n-1}), \operatorname{Mod}\mathcal{M}_w^{n+2k+2}\theta_f$. From Step 1 it follows that, when $j$ is odd, the element $(0, x_{n-2k+j}y^{n+j-1})$ belongs to $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+l+2}\theta_f$. Then, using equation (b), we can write:

$$(0, v) \equiv 0, \operatorname{Mod} T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2}\theta_f,$$

where

$$\begin{aligned} v(x, y) = {} & (n+2)y^{n+2k+1} + (n-2k)y^{n-1} + \cdots + (n-2k+2l)x_{n-2k+2l}y^{n+2l-1} + \cdots \\ & + (n-2)x_{n-2}y^{n+2k-3} + nx_{n-1}y^{n+2k-1}, \end{aligned}$$

for $0 \leqslant l \leqslant k - 1$.

Moreover,

$$\begin{aligned} (0, x_{n-2k+2l}y^{n+2l-1}) & \equiv \left(0, x_{n-2k+2l}x_{n-1}^l y^{n-1}\right) \\ & \equiv \left(0, x_{n-1}^l\left(x_{n-2k+2l}y^{n-1}\right)\right) \equiv \left(0, x_{n-1}^{k+1}y^{n-1}\right). \end{aligned}$$

And this proves that $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2}\theta_f$ contains all terms of filtration $n + 2k + 1$, $k \geqslant 1$, Mod $(0, x_{n-1}^{k+1}y^{n-1})$.

Let $n$ be odd, $k = (n+3)/2$. Then $n + 2k + 1 = 2n + 4$, and from equations (b) and (a), we can write the following two linearly independent equations:

$$tf(y^{n+3} \cdot e_n) = (0, (n+2)y^{2n+4} + x_1y^{n+3} + \cdots + nx_{n-1}y^{2n+2}) \equiv 0,$$

modulo $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f$, and

$$wf\left(X_n^2 e_n\right) = \left(0, y^{2n+4} + 2x_1y^{n+3} + \cdots + 2x_{n-1}y^{2n+2} + x_1^2y^2 + \cdots\right),$$

modulo $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f$.

*Step* 3. It follows from Step 2 that the above system reduces to:

$$\left(0, (n+2)y^{2n+4} + Ax_{n-1}^{(k+1)/2}y^{n-1}\right) \equiv 0, \quad \operatorname{Mod} T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f,$$

$$\left(0, y^{2n+4} + Bx_{n-1}^{(k+1)/2}y^{n-1}\right) \equiv 0, \quad k = (n+3)/2, \quad \operatorname{Mod} T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f$$

and it is now easy to make an inductive procedure to conclude the proof of Step 3.

It is a simple calculation to verify that, in any case, for all $j \geqslant 1$, the $n+2k+1+j$-transversal over the weighted $n+2k+1$ jet of $f$ is empty and this completes the proof of Lemma 4·10.

LEMMA 4·11. *For the germs in the previous lemma we have*:

$$r_{(n)}(\tilde{f}) = r_{(n-l,l)}(\tilde{f}) = 2, \quad 1 \leqslant l < n/2$$
$$r_{(n/2,n/2)}(\tilde{f}) = 1 \text{ for even } n$$

*and*

$$r_{(n)}(\tilde{f}_k) = r_{(n-l,l)}(\tilde{f}_k) = 3, \quad 1 \leqslant l < n/2$$
$$r_{(n/2,n/2)}(\tilde{f}_k) = k \text{ for even } n$$

*and*

$$r_{(n)}(\tilde{f}_\infty) = r_{(n-l,l)}(\tilde{f}_\infty) = 3, \quad 1 \leqslant l < n/2.$$

*Proof.* For $\tilde{f}$ one calculates up to $\mathcal{K}$-equivalence:

$$G_{(n)} \sim (x_1, \ldots, x_{n-1}, (n+2)! y^2/2)$$

and

$$G_{(n-l,l)} \sim (x_1, \ldots, x_{n-1}, cy^2, \epsilon),$$

where $c = (n+2)(ln+l-1)/(l+1) > 0$ and $y \coloneqq y_1$, $\epsilon \coloneqq y_2 - y_1$ in the defining equations of $\bar{A}_{(k-l,l)}$.

For $\tilde{f}_\infty$:

$$G_{(n)} \sim (x_1, \ldots, x_{n-2}, -(n+2)! y^3/3, x_{n-1})$$

and

$$G_{(n-l,l)} \sim (x_1, \ldots, x_{n-2}, cy^3, x_{n-1}, \epsilon),$$

where $c = 2(2l-n)(1+n-l)$, which is zero for $l = n/2$ and non-zero for $l < n/2$.

For $\tilde{f}_k$ we get, except for $l = n/2$, the same $G_{(n)}$ and $G_{(n-l,l)}$ as for $\tilde{f}_\infty$ and for even $n$ in addition:

$$G_{(n/2,n/2)} \sim (x_1, \ldots, x_{n-2}, cy^{2k}, x_{n-1}, \epsilon),$$

where $c = (l-2)![(n+2)(l-n-1)/(2(l+1))]^k \neq 0$.

*Proof of Theorem* 4·3 *(conclusion)*. Now one easily constructs an M-deformation of $\tilde{f}$. Using the adjacencies $[\tilde{f}_2] \to [\tilde{f}]$, $[\tilde{f}_{k+1}] \to [\tilde{f}_k]$ and $[\tilde{f}_\infty] \to [\tilde{f}_{(n+1)/2}]$ and the fact that the corresponding 0-stable invariants differ by at most one, we see by induction that all $\tilde{f}_k$ and also $\tilde{f}_\infty$ have M-deformations, because we can split off the real $A_{k(s,n)}$-points (in the target) from the origin one by one. (For $k(s,n) = (n/2, n/2)$ an origin-preserving deformation of $\tilde{f}_{k+1}$ to $\tilde{f}_k$ induces a deformation

$$G^t_{(n/2,n/2)} \sim (x_1, \ldots, x_{n-2}, ay^{2k}(y^2+bt), x_{n-1}, \epsilon),$$

where $t$ is the deformation parameter and $a, b$ are non-zero constants. Thus, for appropriate $t$, we have a pair of real $A_{(n/2,n/2)}$ source points that are mapped to the same target point. For the other $k(s,n)$ we have a single real $A_{k(s,n)}$ point in the source, defined by a linear equation, that splits off the origin for $t \neq 0$.)

## 5. *Concluding remarks*

Looking at the proof of our main theorem on M-deformations we observe the following: given any $\mathcal{A}$-simple germ $f:\mathbb{R}^m,0 \to \mathbb{R}^n,0$ of rank $n-1$ and $m \geqslant n$, there exists a germ $f'$ of lower codimension such that $[f] \to [f']$ and $r_{k(s,n)}(f) - r_{k(s,n)}(f') \leqslant 1$ for all partitions $k(s,n)$ of $n$. From this property $(*)$ we have the following lower bound on the $\mathcal{A}_e$-codimension.

Corollary 5·1. *Let $f:\mathbb{R}^m,0 \to \mathbb{R}^n,0$ be an $\mathcal{A}$-simple germ of rank $n-1$, then*

$$\operatorname{cod}(\mathcal{A}_e, f) \geqslant r_{k(s,n)}(f) - 1.$$

For $\mathcal{A}$-simple corank-1 germs $f:\mathbb{R}^n,0 \to \mathbb{R}^p,0$, where $n<p$, it would be interesting to determine those $(n,p)$ for which an analogue of property $(*)$ holds and then try to show that there exist M-deformations using the techniques of the present paper. For particular pairs of dimensions $(n,p)$ such germs indeed have M-deformations. For $n<p$, $A_{k(s,m)}$ is an isolated and stable $s$-germ if $k(s,m) = (k_1,\ldots,k_s)$, where $m = \sum_i k_i$ and $k_i \geqslant 0$, satisfies the equality $(m+s-1)(p-n+1) = n+s-1$, and $r_{k(s,m)}(f)$ again denotes the number of these concentrated at the origin in the source of $f_{\mathbb{C}}$ (see [**18**]).

For $(n,p) = (1,2)$ it is known by classical results of A'Campo and Gusein–Zade that any germ $f$ has an M-deformation with $r_{(0,0)}(f) = \delta(f)$ real double points. For corank-1 germs in dimension $(2,3)$ Mond [**14**] has shown that there are real deformations with $r_{(1)}(f) = C(f)$ cross-caps and that all $\mathcal{A}$-simple germs, with the exception of the series $H_k$, have triple-point number $r_{(0,0,0)}(f) = T(f) = 0$. But for the series $H_k$ Marar and Mond [**12**] have constructed M-deformations (which are even good real perturbations). Hence all simple corank-1 germs in dimension $(2,3)$ have M-deformations. For corank-1 germs in dimension $(3,4)$ there are two 0-stable invariants, namely $r_{(1,0)}(f)$ and $r_{(0,0,0,0)}(f)$. The latter invariant is 0 for all $\mathcal{A}$-simple germs in the classification of Houston and Kirk [**9**], and one can easily show (by splitting-off real $A_{(1,0)}$-points from 0 one by one) that there are deformations with $r_{(1,0)}(f)$ real $A_{(1,0)}$-points for all simple corank-1 germs listed in [**9**]. And these deformations are M-deformations.

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