M_9-free groups

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Abstract

We give a characterisation of all finite groups whose subgroup lattice does not contain a sublattice isomorphic to M_9.

1. Introduction

For a finite group G, the subgroup lattice $L(G)$ is the partially ordered set of subgroups of G, where the partial ordering is inclusion. It can be represented by a diagram that is known as the Hasse diagram (or Hasse graph) of the group G.

An old question, going back to 1928 (Rottlaender), asks: What properties of a finite group can be deduced from its subgroup lattice? For example, the subgroup lattice does not indicate whether or not a group is abelian, as can be seen from the subgroup lattices of an elementary abelian group of order 9 and the symmetric group $\mathcal{S}_3$. Subgroup lattices have been widely studied and we refer the reader to [6] and [4] for further results and background information.

In this paper we classify all finite groups whose subgroup lattice does not contain the following lattice as a sublattice:

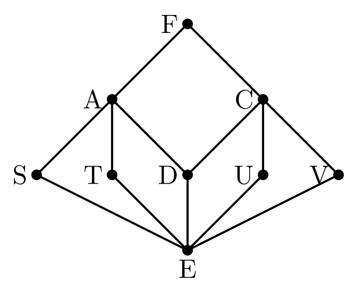

Figure 1. The lattice M_9

This is part of a programme to classify finite groups whose subgroup lattice does not contain a certain sublattice of the subgroup lattice L_{10} of the dihedral group D_8. For the sublattices L_5, L_6, L_7, L_8 and M_8 this had already been completed, but so far M_9 and L_9 were missing. There are partial results available for L_{10} (see [5]).

After some preliminaries, we look at the relationship between M_9-free groups and L_8-free groups, where L_8 is the following lattice:

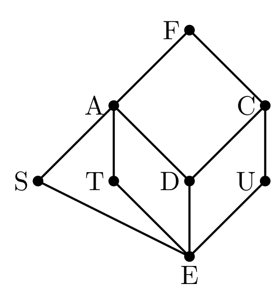

Figure 2. The lattice L_8

As L_8 is a sublattice of M_9, the classification of L_8-free groups (see [1]) is an important tool in our analysis. In Section 3 we classify M_9-free groups whose order is divisible by exactly two different primes and this turns out to be the crucial step towards the general classification in Section 4.
There are numerous places where our arguments follow those in [1].

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2. Preliminaries

Throughout this paper we use the standard notation for lattices as in [4] and standard group theory notation as, for example, in [3]. Moreover G always denotes a finite group and p, q and r are always prime numbers. Whenever the primes p and q appear, we suppose automatically that they are distinct.

Definition 2.1. We write $L(G)$ for the **subgroup lattice of** G, i.e. the set of subgroups of G with inclusion. In this lattice, the infimum of two elements is their intersection and the supremum is the subgroup that they generate.
If L is any lattice, then we say that G is **L-free** if and only if $L(G)$ does not contain a sublattice that is isomorphic to L.

Whenever we deal with a group G where $L(G)$ contains a sublattice isomorphic to M_9, we use the notation from Figure 1 for the names of the corresponding subgroups.

Remark 2.2. If $H \leq G$, then $L(H)$ is a sublattice of $L(G)$. In particular, if L is any lattice and G is L-free, then also H is L-free. Moreover, if $N \trianglelefteq G$, then $L(G/N)$ is isomorphic to a sublattice of $L(G)$ and hence, if G is L-free, then also G/N is L-free.

Definition 2.3. A lattice L is called **modular** if and only if for all $X, Y, Z \in L$, with $X \leq Z$, the following holds:

$$(X \cup Y) \cap Z = X \cup (Y \cap Z).$$

We say that G is **modular** if and only if its subgroup lattice is modular.

Lemma 2.4. *If G is abelian, then G is modular.*

Proof. If G is abelian, then all subgroups of G are normal in G, so Dedekind's modular law yields the assertion. $\square$

Lemma 2.5. [4, Lemma 2.3.2]
A p-group G is modular if and only if any two subgroups of G permute (as sets).

Theorem 2.6. [3, 8.3.3]
Suppose that A is an abelian group that acts irreducibly on an elementary abelian group V. Then $A/C_A(V)$ is cyclic.

Lemma 2.7. [4, Lemma 4.1.1(b)]
Suppose that $H \leq G$ and that $N \trianglelefteq G$ is such that $N \cap H = 1$. Then, for all $x \in N$, we have that $H \cap H^x = C_H(x)$.

We describe an example of an M_9-free group that will play a role in our analysis later on. But before, we recall what it means for a sublattice of $L(G)$ to be isomorphic to M_9. Let $M := \{E, S, T, D, U, V, A, C, F\}$ be a subset of $L(G)$ such that $|M| = 9$ and let L denote the generated sublattice of $L(G)$. Then Figure 1 indicates that L is isomorphic to M_9 if and only if the following hold:

(i) $T \cap S = T \cap D = S \cap D = U \cap D = V \cap D = U \cap V = E$
(ii) $\langle T, S\rangle = \langle D, S\rangle = \langle T, D\rangle = A$
(iii) $\langle U, V\rangle = \langle U, D\rangle = \langle V, D\rangle = C$
(iv) $A \cap C = D$
(v) $\langle U, S\rangle = \langle U, T\rangle = \langle V, S\rangle = \langle V, T\rangle = \langle A, C\rangle = F$.

Example 2.8. Suppose that $G = D_{12}$. Then G is M_9-free, but not L_8-free. Let $v, r \in G$ be such that $o(v) = 2$, $o(r) = 6$ and $G = \langle v, r\rangle$. Here is a picture of $L(G)$:

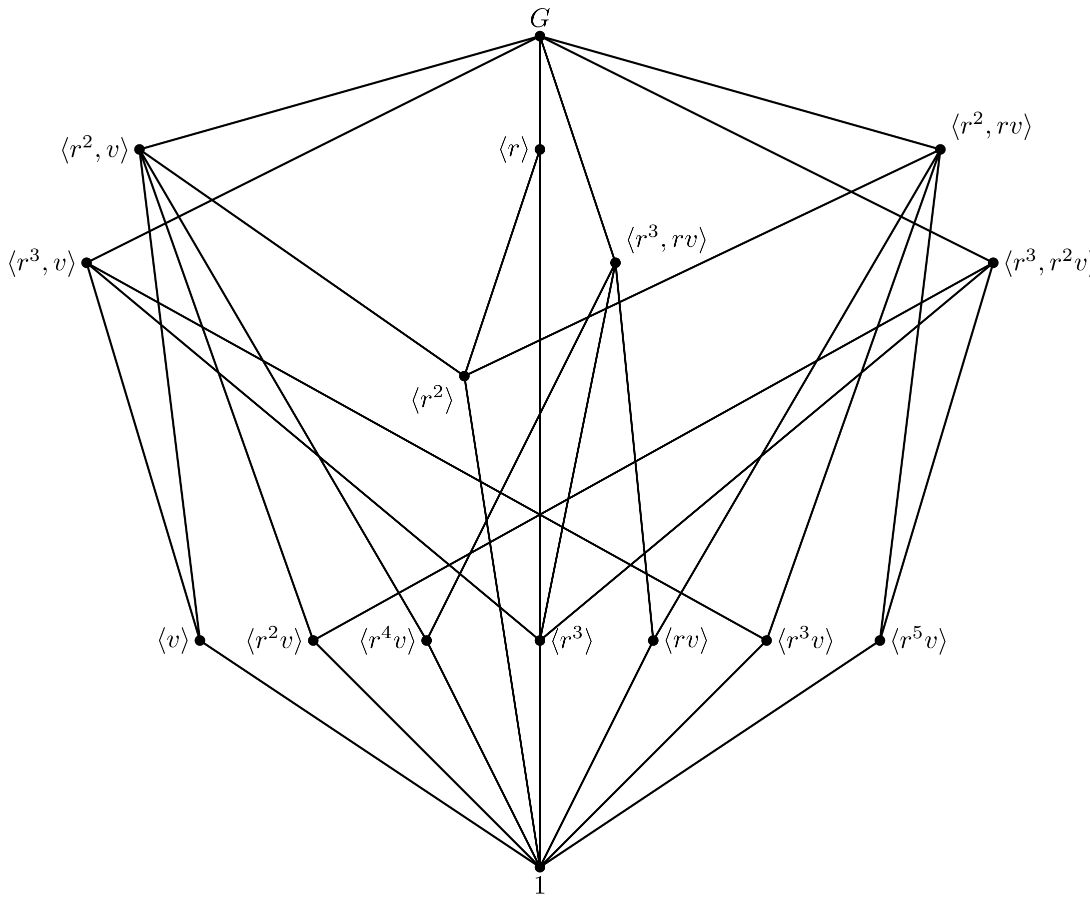

FIGURE 3. Subgroup lattice of D_{12}

The sublattice generated by $\{1, \langle v\rangle, \langle r^2v\rangle, \langle r^2\rangle, \langle rv\rangle, \langle v, r^2\rangle, \langle r^2, rv\rangle, G\}$ is isomorphic to L_8.
We explain briefly why G is M_9-free. Assume otherwise, which means that $L(G)$ has a sublattice isomorphic to M_9 with our standard notation. Then in particular G has subgroups $S, T < A < F \leq G$ and $U, V < C < F$ such that $\langle S, U\rangle = \langle S, V\rangle = \langle U, T\rangle = \langle V, T\rangle = F$. This implies that $F = G$ and that S, T, U and V are subgroups of order at most 3. The subgroup generated by r^2 and any involution of G is a proper subgroup of G, and therefore S, T, U and V have order 2. By assumption S and U generate G, but then $\langle T, U\rangle$ has order 4 or 6, which is impossible.

Theorem 2.9. ([4], Theorem 2.1.2)
$L(G)$ is modular if and only if $L(G)$ is L_5-free.

Lemma 2.10. *Suppose that L is a lattice and that $L_5 \leq L \leq L_{10}$. Let G be a p-group. Then G is modular if and only if it is L-free. In particular, for p-groups, being L_5-free and being L_{10}-free are equivalent.*

Proof. This is a combination of Theorem 2.9 and Lemma 2.1 in [5]. □

For the following lemma we recall that, if G is a p-group and $i \in \mathbb{N}$, then $\Omega_i(G) := \langle x \in G \mid x^{p^i} = 1\rangle$ and $\Omega(G) := \Omega_1(G)$.

Lemma 2.11. ([4], Lemma 2.3.5)
Suppose that G is a modular p-group. Then $\Omega(G)$ is elementary abelian and $\Omega_i(G) = \{x \in G \mid x^{p^i} = 1\}$ for all $i \in \mathbb{N}$.

Lemma 2.12. ([4], Lemma 2.3.6)
Suppose that G is a modular 2-group. Then all the subgroups of G are normalised by $\langle x \in G \mid x^4 = 1\rangle$.

Definition 2.13. We say that G is **hamiltonian** if and only if G is non-abelian and every subgroup of G is normal in G.

Theorem 2.14. ([4], Theorem 2.3.12)
Suppose that G is a p-group. All subgroups of G are normal in G if and only if G is abelian or G has an elementary abelian 2-group A such that $G \cong Q_8 \times A$.

Theorem 2.15. ([4], Theorem 2.3.8)
Suppose that G is a modular 2-group and that G involves Q_8. Then G has an elementary abelian 2-subgroup A such that $G \cong Q_8 \times A$.

Corollary 2.16. *If G is a non-hamiltonian modular p-group, then all non-trivial subgroups of G are also non-hamiltonian.*

Proof. Assume otherwise and choose G to be a counter-example. Let $U \leq G$ be hamiltonian (hence non-abelian) and non-trivial. Then Theorem 2.14 yields that U has an elementary abelian 2-subgroup A such that $U \cong Q_8 \times A$. So G involves Q_8 and Theorem 2.15 is applicable. Then G has an elementary abelian 2-subgroup B such that $G \cong Q_8 \times B$. Theorem 2.14 implies that G is hamiltonian, contrary to our hypothesis. □

Lemma 2.17. ([5], Lemma 2.2)
Suppose that G is L_{10}-free, that P is a normal p-subgroup of G and that K is a p'-subgroup of G such that $G = PK$. Then the following hold:

(i) *$[P, K]$ is elementary abelian or a hamiltonian 2-group.*
(ii) *If $[P, K]$ is elementary abelian, then $P = C_P(K) \times [P, K]$.*

Theorem 2.18. ([5], Proposition 2.7)
Suppose that G is an L_{10}-free $\{p, q\}$-group. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Then $P \trianglelefteq G$ or $Q \trianglelefteq G$.

Theorem 2.19. ([5], Proposition 2.6)
Suppose that G is an L_{10}-free $\{p, q\}$-group. Suppose that $P \in Syl_p(G)$ is normal in G and let $Q \in Syl_q(G)$. Suppose further that $[P, Q] \neq 1$. Then one of the following holds:

(i) *Q is cyclic.*
(ii) *$Q \cong Q_8$.*
(iii) *$p = 3$ and $q = 2$.*

Theorem 2.20. ([5], Theorem A)
If G is L_{10}-free, then G is soluble.

We use in Section 4 that, as a consequence of Theorem 2.20, all L_{10}-free groups have a Sylow system.

Lemma 2.21. ([5], Lemma 2.5)
Suppose that G is a dihedral group. Then G is L_{10}-free if and only if there is a prime p such that $|G| = 2p$ or $|G| = 12$.

Lemma 2.22. ([5], Lemma 2.8)
Suppose that G has normal p-subgroups N_1, N_2 and a cyclic q-subgroup Q such that $G = (N_1 \times N_2)Q$. Suppose that Q acts irreducibly on N_1 and N_2 and that $C_Q(N_1) = C_Q(N_2)$. If G is L_{10}-free, then $|N_1| = |N_2| = p$ and Q induces power automorphisms on $N_1 \times N_2$.

Lemma 2.23. ([1], Lemma 2.1)
Suppose that P is a normal p-subgroup of G and that Q is a q-subgroup of G such that $[P, Q] \neq 1$ and $G = PQ$. If P is not hamiltonian, then the following are equivalent:

(i) *G is L_7-free.*
(ii) *G is L_8-free.*
(iii) *G is M_8-free.*
(iv) *One of the following holds:*

(a) *P is elementary abelian, Q is cyclic and all subgroups of Q act irreducibly on P or by inducing power automorphisms.*
(b) *P is elementary abelian of order p^2, where $p \equiv 3$ (mod 4). Moreover $Q \cong Q_8$ and Q acts faithfully on P.*

Lemma 2.24. ([1], Lemma 2.2)
Suppose that q is odd, that P is a hamiltonian normal 2-subgroup of G and that Q is a q-subgroup of G such that $[P, Q] \neq 1$ and $G = PQ$. Then the following hold:
(i) *G is L_8-free if and only if $P \cong Q_8$ and Q is cyclic with $[Q : C_Q(P)] = 3$.*
(ii) *G is not M_8-free.*

Theorem 2.25. ([1], Proposition 2.3)
Suppose that G is a $\{p, q\}$-group and that G is L_8-free or M_8-free. Then G is nilpotent or its structure is as described in Lemmas 2.23 and 2.24.

Corollary 2.26. ([1], Corollary 2.4)
Suppose that G is a $\{p, q\}$-group and that G is L_8-free or M_8-free. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$ and suppose that Q is not normal in G. Then $G = \langle Q^x \mid x \in P\rangle$.

Definition 2.27. We say that G is a Λ^*-group if and only if there are $n \in \mathbb{N}$ and pairwise distinct prime numbers $p_1, ..., p_n, q$ such that the following hold:
(i) $Q \leq G$ is a cyclic q-subgroup;
(ii) for all $i \in \{1, ..., n\}$ there is a normal p_i-subgroup P_i of G;
(iii) $G = (P_1 \times ... \times P_n)Q$;
(iv) if $i, j \in \{1, ..., n\}$ are distinct, then $C_Q(P_i) \neq C_Q(P_j)$; and
(v) if $i \in \{1, ..., n\}$, then either $P_i \cong Q_8$ or P_i is elementary abelian and all subgoups of Q act irreducibly or by inducing power automorphisms on P_i.

Whenever we formulate a hypothesis with a Λ^*-group G, then we will use all the notation that we just introduced.

Corollary 2.28. *Suppose that G is a Λ^*-group with the corresponding notation. Then $G = \langle Q^x \mid x \in P_1 \times ... \times P_n\rangle$.*

Proof. Let $i \in \{1, ..., n\}$. It follows from 2.23 and 2.24 and the hypotheses about Q and P_i that QP_i is L_8-free and has order divisible by exactly two distinct primes. Thus Corollary 2.26 is applicable and it yields that $QP_i = \langle Q^x \mid x \in P_i\rangle$. This implies that $G = \langle QP_1, ..., QP_n\rangle = \langle Q^x \mid x \in P_1 \times ... \times P_n\rangle$. □

Theorem 2.29. ([1], Theorem A)
The group G is L_8-free if and only if there exist $n \in \mathbb{N}$ and subgroups $G_1, ..., G_n$ of G with pairwise coprime orders such that the following hold:
– $G = G_1 \times ... \times G_n$;
– for all $k \in \{1, ..., n\}$, the group G_k is a modular p-group, a Λ^*-group or a $\{2, p\}$-group satisfying Lemma 2.23 (iv)(b).

3. M_9-free $\{p, q\}$-groups

As in [1], our classification of M_9-free groups begins with the special case where G is a $\{p, q\}$-group. In fact this is where most work is necessary, and then the general results follow in the next section. It turns out that it is key to characterise those groups that are M_9-free, but not L_8-free.

Lemma 3.1. *Suppose that G is a $\{p, q\}$-group and that G is M_9-free and not nilpotent. Suppose further that $P \in Syl_p(G)$ is normal in G and let $Q \in Syl_q(G)$.*
If P is not hamiltonian, then G is L_8-free or $p = 3$, $q = 2$ and G has a section isomorphic to D_{12}.

Proof. Assume otherwise and let G be a minimal counter-example. Then G is not L_8-free and hence G does not satisfy Lemma 2.23 (iv). If $p = 3$ and $q = 2$, then G does not involve D_{12}. In particular all subgroups and sections of G do not involve D_{12}, so if they are proper and satisfy the hypothesis, then the minimality of G implies that they are L_8-free.
Then Lemma 2.10 yields that P and Q are modular. Moreover P is not hamiltonian and thus Corollary 2.16 gives that $[P,Q]$ is not hamiltonian. It follows with Lemma 2.17 that $[P,Q]$ is elementary abelian and $P = C_P(Q) \times [P,Q]$. Now we proceed in a series of steps in order to obtain a contradiction.

(1) Q is not isomorphic to Q_8.
Proof. Assume otherwise. Then $\Phi(Q) = Z(Q)$, in particular $Q/\Phi(Q) \cong V_4$. If $\Phi(Q) \trianglelefteq G$, then by minimality of G the section $G/\Phi(Q)$ is L_8-free and satisfies Lemma 2.23 (iv). This is impossible because $Q/\Phi(Q)$ is neither cyclic nor isomorphic to Q_8. So $\Phi(Q)$ is not normal in G.
Let z denote the central involution in Q. Since $z \notin Z(G)$, we know that $[P,z] \neq 1$ and Q acts faithfully on P. Hence if we let Q_0 denote a subgroup of Q of order 4, then $[P,Q_0] \neq 1$. The minimal choice of G yields that PQ_0 is L_8-free and satisfies Lemma 2.23 (iv). In particular Q_0 acts irreducibly on P or by inducing power automorphisms.
The group of power automorphisms of P is abelian and Q is not, so there exists a maximal subgroup U of Q that acts irreducibly on P. Moreover U acts faithfully on P because Q does. Since p is odd, one of $p-1$ or $p+1$ is divisible by 4 and therefore $|U| = 4$ divides $p^2 - 1 = (p-1)\cdot(p+1)$. Now II.3.10 in [2] yields that $|P| = p^2$ and that 4 does not divide $p-1$. Hence $p \equiv 3$ modulo 4 and in particular Case (iv)(b) of Lemma 2.23 is satisfied. But then G is L_8-free, contrary to our assumption. ■

(2) Q is cyclic.
Proof. Assume otherwise. Then Theorem 2.19 and (1) imply that $p = 3$ and $q = 2$. The non-trivial action of Q on P yields an element $x \in Q$ such that $[P,x] \neq 1$. Since Q is neither cyclic nor isomorphic to Q_8, there exists some $y \in Q$ of order 2 and such that $y \notin \langle x \rangle$. In particular $\langle x,y \rangle$ is also neither cyclic nor isomorphic to Q_8 and therefore $P\langle x,y \rangle$ does not satisfy Lemma 2.23 (iv) and is hence not L_8-free. The minimality of G yields that $Q = \langle x,y \rangle$. Then it follows with Lemma 2.11 that $\Omega(Q)$ is elementary abelian and $\Omega(Q) = \{g \in Q | g^q = 1\}$.
Assume that $o(x) > 2$, so in particular $x \notin \Omega(Q)$. Then $\Omega(Q) \neq Q$ and $\Omega(Q)$ contains at least three involutions, so it is neither cyclic nor isomorphic to Q_8. Moreover $P\Omega(Q) < G$ and hence this subgroup is L_8-free by our minimal choice of G and Lemma 2.23 implies that $P\Omega(Q)$ is nilpotent. Now $X := \Omega(\langle x \rangle) \leq \Omega(Q)$ is centralised by $y \in \Omega(Q)$ whence $X \trianglelefteq Q$ (recall that $Q = \langle x,y \rangle$). Moreover P centralises X and therefore $X \trianglelefteq G$. But Q/X is neither cyclic nor isomorphic to Q_8, so the factor group G/X does not satisfy Lemma 2.23 (iv) and in particular it is not L_8-free. This contradicts the minimal choice of G.
We conclude that $o(x) = 2$ and $|Q| = 4$. Let $D \leq [P,Q]$ be such that Q acts irreducibly on D. Since $P = C_P(Q) \times [P,Q]$, we know that Q acts without fixed points on $[P,Q]$ and hence on D. Theorem 2.6 implies that $Q/C_Q(D)$ is cyclic and that Q acts non-faithfully on D. Let $a,b \in Q^{\#}$ be such that $Q = \langle a,b \rangle$ and $[D,a] = 1$, but $[D,b] \neq 1$. Then b acts without fixed points on D, so it inverts D and the irreducible action of Q on D forces $|D| = 3$. In particular $DQ \cong D_{12}$, contrary to our assumption. Thus Q is cyclic as stated. ■

(3) $C_P(Q) \neq 1$.
Proof. Assume that this is false. Then $P = C_P(Q) \times [P,Q] = [P,Q]$ whence P is elementary abelian. With Maschke's Theorem we let $m \in \mathbb{N}$ and $N_1, \ldots, N_m \leq [P,Q]$ be such that $[P,Q] = N_1 \times \ldots \times N_m$ and such that Q acts irreducibly on $N_1, ..., N_m$. First suppose that $m \geq 3$ and let $2 \leq i \leq m$. Then the subgroup $(N_1 \times N_i)Q$ is L_8-free, because G is a minimal counter-example. But Q is not irreducible on $N_1 \times N_i$, so by Lemma 2.23 Q induces power automorphisms on $N_1 \times N_i$. Therefore Q induces power automorphisms on P and it follows by the same lemma that G is L_8-free. This is a contradiction and therefore $m \leq 2$.
Suppose that $m = 2$. Since Q is cyclic, we may without loss suppose that $C_Q(N_1) \leq C_Q(N_2)$. If $C_Q(N_2)$ induces power automorphisms on P, then these are universal, so in particular $C_Q(N_1) = C_Q(N_2)$. But then Q induces power automorphisms on P by Lemma 2.22 and Lemma 2.23 implies that G is L_8-free,

contrary to our assumption. Therefore $C_Q(N_2)$ does not induce power automorphisms on P. It also does not act irreducibly because it leaves N_1 invariant. Hence Lemma 2.23 yields that $PC_Q(N_2)$ is not L_8-free, and by minimality we deduce that $C_Q(N_2) = Q$. However $C_P(Q) = 1$ by assumption, so this is impossible.
Therefore $m = 1$, which means that Q acts irreducibly on P. The minimal choice of G forces all subgroups of Q to act on P as in Lemma 2.23, so this lemma implies that G is L_8-free, contrary to our hypothesis. This contradiction shows that $C_P(Q) \neq 1$. ■

(4) $|C_P(Q)| = p$ and Q acts irreducibly on $[P, Q]$.
Proof. With Maschke's Theorem let $m \in \mathbb{N}$ and $N_1, \ldots, N_m \leq [P, Q]$ be such that $[P, Q] = N_1 \times \ldots \times N_m$ and that Q acts irreducibly on $N_1, ..., N_m$. Using (3) we choose $M \leq C_P(Q)$ of order p and we set $N := N_1$. Now N and M commute and are both Q-invariant.
Assume that Q induces power automorphisms on $M \times N$. Then these are universal and in particular $C_Q(M) = C_Q(N)$. It follows that Q centralises N, which is a contradiction.
Since M is Q-invariant, we also have that Q acts non-irreducibly on $M \times N$. Thus $(M \times N)Q$ is not L_8-free by Lemma 2.23. Then the minimality of G implies that $(M \times N)Q = G$, but also $G = PQ$ and therefore $M \times N = P$. In particular P is elementary abelian.
Since $M \leq C_P(Q)$, the Dedekind identity gives that $C_P(Q) = M \times C_N(Q) = M$. We also see that $N = [P, Q]$ and hence Q acts irreducibly on $[P, Q]$. ■

(5) $|Q| = q$.
Proof. We set $N := [P, Q]$ and $M := C_P(Q)$, and we recall that Q is cyclic by (2). We assume that $|Q| > q$ and that the subgroup $Q_0 := \Phi(Q)$ of Q of index q does not centralise P. If Q_0 induces power automorphisms on P, then they are universal and hence Q_0 centralises P by (3). This contradicts our assumption that $[Q_0, P] \neq 1$. Since M is Q_0-invariant, we also know that Q_0 does not act irreducibly on $N \times M$. Lemma 2.23 implies that PQ_0 is not L_8-free which, by minimality of G, forces $PQ_0 = G$. This is a contradiction.
Next we look at the case where Q_0 centralises P. Since Q is cyclic by (2), we then have that $Q_0 \trianglelefteq G$. If PQ_0/Q_0 is centralised by Q/Q_0, then $[P, Q] = 1$ (by coprime action), contrary to our hypothesis. If Q/Q_0 acts irreducibly (or by inducing power automorphisms) on PQ_0/Q_0, then Q acts irreducibly (or by inducing power automorphisms) on P. Thus if G/Q_0 is L_8-free, then by Lemma 2.23 also G is. But G is chosen to be a counter-example, so we deduce that G/Q_0 is not L_8-free. Now the minimality of G forces $Q_0 = 1$, which is false. Hence $|Q| = q$ as stated. ■

Now we choose $J_1, J_2 \leq P$ to be distinct subgroups of order p such that J_1, J_2 are neither equal to $C_P(Q)$ nor contained in $[P, Q]$. This implies that $\langle J_1, J_2 \rangle \cap [P, Q] \neq 1$ because $|P : [P, Q]| = p$ by (4). These choices are possible because $|P| \geq p^2$ and, if $p = 2$, then $[P, Q] \geq 4$ and hence $|P| \geq 8$.
Set $V := \langle J_1, J_2 \rangle$ and $I := V \cap [P, Q]$. We choose $x \in [P, Q]$ such that $Q^x \neq Q$ and prove that

$$L := \{1, Q, Q^x, I, J_1, J_2, Q[P, Q], V, G\}$$

is isomorphic to M_9.
Of course $Q \cap Q^x = 1 = J_1 \cap J_2$. Moreover Q and Q^x are cyclic of order q by (5) and I is a p-group, so it is also clear that $Q \cap I = 1 = Q^x \cap I$. Next we see that $J_1 \cap I = 1 = J_2 \cap I$ because $I \leq [P, Q]$ and $J_1, J_2 \nleq [P, Q]$. By (4) we observe that $[P, Q] \leq \langle Q^x, Q \rangle$ whence $Q[P, Q] = \langle Q^x, Q \rangle$.
We also have that Q and Q^x are maximal subgroups of $Q[P, Q]$. Since $I \neq Q, Q^x$, it follows that $\langle Q, I \rangle = \langle Q^x, I \rangle = Q[P, Q]$. As J_1 and J_2 are maximal subgroups of V and distinct from I, we further have that

$$\langle J_1, J_2 \rangle = V = \langle J_1, I \rangle = \langle J_2, I \rangle.$$

The choice of I gives that $Q[P, Q] \cap V = I$, so it remains to prove that

$$\langle Q, J_1 \rangle = \langle Q, J_2 \rangle = \langle Q^x, J_1 \rangle = \langle Q^x, J_2 \rangle = G.$$

Let $S \in \{Q, Q^x\}$ and $T \in \{J_1, J_2\}$. Assume that $\langle S, T \rangle \neq G$. Then $\langle S, T \rangle$ is contained in a maximal subgroup of G.

Now, for any $z \in [P,Q]$, we observe that $Q^z[P,Q] = Q[P,Q]$ is a maximal subgroup of G. Since $[P,Q]$ is the unique Sylow p-subgroup of $Q[P,Q]$ and $T \nleq [P,Q]$, we deduce that $T \nleq Q[P,Q]$ and therefore $\langle S,T\rangle$ cannot be contained in $Q[P,Q]$.
Also $Q^zC_P(Q)$ is maximal in G for all $z \in [P,Q]$, but $T \nleq Q^zC_P(Q)$ because $T \nleq C_P(Q)$. Hence $\langle S,T\rangle \nleq Q^zC_P(Q)$. The next maximal subgroup we consider is P itself (see (5)), but of course $\langle S,T\rangle \nleq P$. The irreducible action of Q on $[P,Q]$ (by (4)) implies that the groups that we just considered are all maximal subgroups of G. None of them contains $\langle S,T\rangle$, so $\langle S,T\rangle = G$.
This concludes the proof: G is M_9-free, so we have reached our final contradiction. □

Lemma 3.2. *Suppose that G is a $\{p,q\}$-group and that G is M_9-free, but not L_8-free. Then $p = 3$, $q = 2$ and G is not nilpotent, moreover G has a normal Sylow 3-subgroup P and a section isomorphic to D_{12}. In particular G does not have cyclic Sylow 2-subgroups.*

Proof. Assume that this is false and let G be a minimal counter-example. Our hypothesis implies that G is L_{10}-free, so by Theorem 2.18 we may suppose that G has a normal Sylow p-subgroup P. Let $Q \in \mathrm{Syl}_q(G)$. If $[Q,P] = 1$, then G is nilpotent and hence L_8-free, contrary to our hypothesis. Therefore $[Q,P] \neq 1$. If P is not hamiltonian, then Lemma 3.1 gives the assertion of the lemma and hence a contradiction.
Thus P is hamiltonian. Theorem 2.14 gives subgroups $H, K \leq P$ such that $P = H \times K$ and such that H is an elementary abelian 2-group and $K \cong Q_8$. In particular $\Phi(K) = \Phi(P)$ is central in G. Since G is not nilpotent, this implies that $\bar{G} := G/\Phi(P)$ is not nilpotent.
As $\bar{P}$ is elementary abelian, Lemma 3.1 is applicable to $\bar{G}$. First suppose that $\bar{G}$ is L_8-free. Then Lemma 2.23 yields that $\bar{Q}$ is cyclic and acts irreducibly on $\bar{P}$. Hence Q is cyclic and, since Q normalises $\Omega(P)$, it follows that $\Omega(P) = \Phi(P)$. In particular $H = 1$ and $P \cong Q_8$, and $|Q : C_Q(P)| = 3$. Now Lemma 2.24 (i) tells us that G is L_8-free, but by hypothesis it is not.
Therefore $\bar{G}$ is not L_8-free. But then the minimal choice of G forces $p = 3$ and $q = 2$, which is impossible.
So the first statement of the lemma is proved, and the fact that G involves D_{12} implies that G does not have cyclic Sylow 2-subgroups. □

Hypothesis 3.3. *Suppose that G is a non-nilpotent $\{2,3\}$-group, that G has a normal Sylow 3-subgroup P and that $Q \in Syl_2(G)$.*

Lemma 3.4. *Suppose that Hypothesis 3.3 holds and that G is M_9-free. Then $C_P(Q) = 1$.*

Proof. Assume otherwise and choose G to be a minimal counter-example. Then $C_P(Q) \neq 1$. Moreover we let $a \in Q$ be such that $[P,a] \neq 1$, but $[P,a^2] = 1$. Then $\langle a\rangle P$ is an M_9-free subgroup of G and hence the minimal choice of G implies that $G = \langle a\rangle P$. In particular $Q = \langle a\rangle$. As a^2 centralises P, it follows that $\langle a^2\rangle \trianglelefteq G$. The group $G/\langle a^2\rangle$ is also M_9-free and has a normal Sylow 3-subgroup. Then the choice of a and the minimality of G yield that $a^2 = 1$.
We recall that P is a 3-group and therefore $[P,Q]$ is not hamiltonian. Then Lemma 2.17 implies that $[P,Q]$ is elementary abelian and that $P = C_P(Q) \times [P,Q]$.
We deduce that $C_P(a) \cap [P,Q] = C_P(Q) \cap [P,Q] = 1$ which means that a acts without fixed points on $[P,Q]$. By choice a induces an automorphism of order 2 on P whence a inverts $[P,Q]$. In particular $[P,Q] = [P,a]$. We choose a subgroup N of order 3 of $[P,Q]$ and we also choose $M \leq C_P(Q)$ such that $|M| = 3$.
Then $\langle a\rangle$ normalises M, hence it acts on $M \times N$. Let $x \in M$ and $y \in N$ be such that $\langle x\rangle = M$ and $\langle y\rangle = N$. Then we see that $U := \langle xy\rangle$ and $V := \langle yx^{-1}\rangle$ are also subgroups of $M \times N$, and moreover $U^a = V$ and $V^a = U$. We set

$$L := \{1, U, V, N, \langle a\rangle, \langle a^y\rangle, (M \times N), \langle a, y\rangle, (M \times N)\langle a\rangle\}$$

and we show that this lattice is isomorphic to M_9.
First $U \cap V = U \cap N = N \cap V$ and moreover $\langle U,V\rangle = \langle N,V\rangle = \langle U,N\rangle = M \times N$.
With Lemma 2.7 we see that the subgroups N, $\langle a\rangle$ and $\langle a^y\rangle$ intersect pair-wise trivially.
Instead of presenting all the remaining calculations, we just observe that neither U nor V is normalised by a or by a^y, and therefore $\langle U,a\rangle = \langle V,a\rangle = \langle U,a^y\rangle = \langle V,a^y\rangle = (M \times N)\langle a\rangle$.
Thus L is isomorphic to M_9, contrary to our hypothesis. □

Corollary 3.5. *Suppose that Hypothesis 3.3 holds and that G is M_9-free. Then P is elementary abelian.*

Proof. By hypothesis P is a 3-group and hence non-hamiltonian. Then Corollary 2.16 implies that $[P,Q]$ is also non-hamiltonian. It follows with Lemma 2.17 that $[P,Q]$ is elementary abelian and that $P = C_P(Q) \times [P,Q]$. Now Lemma 3.4 gives the statement. $\square$

Lemma 3.6. *Suppose that Hypothesis 3.3 holds and that G is M_9-free. If $w \in Q$ induces an automorphism of order 2 on P, then w inverts P.*

Proof. We know that P is elementary abelian by Corollary 3.5 and then the coprime action of w on P gives that $P = C_P(w) \times [P,w]$. In particular w inverts $[P,w]$. Since $P\langle w\rangle$ satisfies the hypotheses of Lemma 3.4, we obtain that $C_P(w) = 1$ and hence w inverts P. $\square$

Lemma 3.7. *Suppose that Hypothesis 3.3 holds and that G is M_9-free. If Q does not induce power automorphisms on P, then $|P| = 9$.*

Proof. Assume otherwise and choose G to be a minimal counter-example. Lemma 3.6 and our hypothesis imply that some element $w \in Q$ induces an automorphism on P that is not a power automorphism, and in particular it has order at least 4. We may choose w such that w^4 centralises P.
Now $H := P\langle w\rangle$ satisfies the hypotheses of the lemma and $\langle w^4\rangle \trianglelefteq H$. The group $H/\langle w^4\rangle$ also satisfies our hypotheses and therefore the minimal choice of G forces $G = P\langle w\rangle$ and $w^4 = 1$.
We also note that w^2 inverts P by Lemma 3.6. Let $a, b \in P, a \notin \langle b\rangle$ be such that $a^w = b, b^w = a^{-1}$ and $(a^{-1})^w = b^{-1}$. Then let $P_0 := \langle a, b\rangle$. Since G is a counter-example, we know that $P \neq P_0$ and hence we may choose $c \in P\backslash P_0$. As w^2 inverts P, it follows that w does not normalise $\langle c\rangle$. Thus there is some $d \in P, d \notin \langle c\rangle$ such that $c^w = d$. We set

$$L = \{1, \langle bd, c\rangle, \langle ac, d\rangle, P_0, \langle w^a\rangle, \langle w\rangle, P_0\langle c, d\rangle, P_0\langle w\rangle, P_0\langle c, d, w\rangle\}$$

and we show that this lattice is isomorphic to M_9.
By choice $C_{\langle w\rangle}(a) = 1$, so Lemma 2.7 gives that any two of the groups $\langle bd, c\rangle, \langle ac, d\rangle, \langle a, b\rangle, \langle w^a\rangle$ and $\langle w\rangle$ intersect trivially.
Moreover $\langle bd, c, ac, d\rangle = \langle a, b, c, d\rangle = \langle a, b, bd, c\rangle = \langle a, b, ac, d\rangle$.
We also note that $b^{-1}a = w^{-1}a^{-1}wa = w^{-1}w^a \in \langle w, w^a\rangle$ and hence

$$a^{-1} = a^2 = b^{-1}aab = b^{-1}a(b^{-1}a)^w \in \langle w, w^a\rangle.$$

Since w conjugates a to b, it also follows that $b \in \langle w, w^a\rangle$.
Furthermore $w = aa^{-1}waa^{1-} = aw^a a^{-1} \in \langle w^a, a, b\rangle$ and hence $S := \langle w, a, b\rangle = \langle w^a, a, b\rangle = \langle w^a, w\rangle$ and $S \cap \langle a, b, c, d\rangle = \langle a, b\rangle$. We still need to show that one element from $\{\langle w^a\rangle, \langle w\rangle\}$ and $\{\langle bd, c\rangle, \langle ac, d\rangle\}$, respectively, generate $P_0\langle c, d, w\rangle = \langle a, b, c, d, w\rangle$.
But this is immediate because w^a and w interchange $\langle a\rangle$ and $\langle b\rangle$ and they also interchange $\langle c\rangle$ and $\langle d\rangle$. Thus L is isomorphic to M_9 as stated, and this is a contradiction. $\square$

Lemma 3.8. *Suppose that Hypothesis 3.3 holds and that G is M_9-free. Then every subgroup of Q either acts irreducibly on P or it induces power automorphisms on P.*

Proof. Assume that this is false and choose G to be a counter-example. We take $w \in Q$ such that w induces neither power automorphisms on P nor does it act irreducibly. We know from Corollary 3.5 and Lemma 3.7 that P is elementary abelian of order 9, so by Maschke's Theorem we let $P_1, P_2 \leq P$ be of order 3 that are w-invariant and such that $P = P_1 \times P_2$. Applying Lemma 3.4 to $P\langle w\rangle$ yields that $C_P(w) = 1$ whence w inverts P. But this contradicts our assumption. $\square$

Lemma 3.9. *Suppose that Hypothesis 3.3 holds and that G is M_9-free, but not L_8-free. If $Q \cong Q_8$, then Q induces power automorphisms on P.*

Proof. If this is false, then Lemma 3.8 yields that Q acts irreducibly on P. Then Corollary 3.5 and Lemma 3.7 imply that P is elementary abelian of order 9. By Lemma 3.6 all elements from Q that induce an automorphism of order 2 on P invert P. Thus there must be some $w \in Q$ inducing an automorphism of order 4 on P. Then w^2 inverts P and it follows that Q acts faithfully on P, so Lemma 2.23 implies that QP is L_8-free. However this contradicts our hypothesis. $\square$

The following corollary plays a role in the next section.

Corollary 3.10. *Suppose that Hypothesis 3.3 holds and that G is M_9-free, but not L_8-free. If $Q \cong Q_8$, then Q does not act faithfully on P.*

Proof. We know from Lemma 3.9 that Q induces power automorphisms on P, in particular it normalises every subgroup of P of order 3. With P_0 denoting such a subgroup of order 3, we see that $C_Q(P_0)$ has index 2 in Q and in particular the central involution in Q centralises P_0. Hence it centralises P. □

Lemma 3.11. *Suppose that Hypothesis 3.3 holds and that G is M_9-free, but not L_8-free. Then Q induces power automorphisms on P.*

Proof. Assume otherwise and let G be a minimal counter-example. Then G is M_9-free and Corollaries 3.8 and 3.5 yield that Q acts irreducibly on P. Moreover P is elementary abelian. With Lemma 3.6 there exists some $w \in Q$ that induces an automorphism of order 4 on P and such that w^2 inverts P. This means that we find $a, b \in P$ such that $a \notin \langle b \rangle$ and such that $a^w = b, b^w = a^{-1}$ and $(a^{-1})^w = b^{-1}$. Lemma 3.7 implies that $|P| = 9$ and hence $P = \langle a, b \rangle$. Moreover Q is neither cyclic (by Lemma 3.2) nor quaternion of order 8 (by Lemma 3.9) and therefore Q contains some involution v outside $\langle w \rangle$. We note that v normalises every subgroup of Q by Lemma 2.12, in particular it normalises $H_0 := \langle w^4 \rangle$ and therefore $H_0 \trianglelefteq H := P\langle v, w \rangle$. Now $\langle v, w \rangle / H_0 \in \mathrm{Syl}_2(H/H_0)$ and hence H/H_0 has Sylow 2-subgroups that are neither cyclic nor quaternion. Lemma 2.23 then yields that H/H_0 is not L_8-free. But it is M_9-free and $\langle v, w \rangle / H_0$ does not induce power automorphisms on PH_0/H_0, so the minimality of G forces $H = G$ and $H_0 = 1$.

This means that $|Q| = 8$. We recall that Q has three involutions, namely v, w^2 and vw^2, and that w^2 inverts P. If $[P, v] \neq 1$, then v inverts P by Lemma 3.6 and hence vw^2 centralises P. The same holds the other way around, so we know that exactly one of the involutions in Q centralises P and the other two invert P. Let $t \in Q$ be an involution distinct from w^2 that inverts P. We show that the lattice

$$L = \{1, \langle w^a \rangle, \langle w \rangle, \langle b \rangle, \langle t^b \rangle, \langle t^{b^{-1}} \rangle, \langle w, a, b \rangle, \langle t, b \rangle, \langle w, t, a, b \rangle\},$$

is isomorphic to M_9.

Lemma 2.7 tells us that $\langle w \rangle \cap \langle w^a \rangle = C_{\langle w \rangle}(a) = 1$. It follows that

$$1 = \langle b \rangle \cap \langle t^x \rangle = \langle w^y \rangle \cap \langle t^x \rangle = \langle b \rangle \cap \langle w^y \rangle$$

where $x \in \{b, b^{-1}\}, y \in \{a, 1_G\}$ are suitably chosen. Then we see that $b^{-1}a = w^{-1}a^{-1}wa = w^{-1}w^a \in \langle w, w^a \rangle$ and hence $a^{-1} = a^2 = b^{-1}aab = b^{-1}a(b^{-1}a)^w \in \langle w, w^a \rangle$. We recall that w conjugates a to b, consequently $b \in \langle w, w^a \rangle$ and similarly $\langle b, w \rangle = \langle b, w^a \rangle = \langle a, b, w \rangle$.

We calculate that $b^{-1}a^{-1} = btbtb = btb^{-1}b^{-1}tb = t^{b^{-1}}t^b \in \langle t^b, t^{b^{-1}} \rangle$ and also $((ab)^{-1})((ab)^{-1})^{t^b} = ((ab)^{-1})b^{-1}a = b \in \langle t^b, t^{b^{-1}} \rangle$. We deduce that $\langle t^b, t^{b^{-1}} \rangle = \langle b, t \rangle = \langle t^b, b \rangle = \langle t^{b^{-1}}, b \rangle$ and moreover $\langle w, a, b \rangle \cap \langle t, b \rangle = (\langle w \rangle \langle a, b \rangle) \cap (\langle t \rangle \langle b \rangle) = \langle b \rangle$.

It remains to show that one element from $\{\langle w \rangle, \langle w^a \rangle\}$ and $\{\langle t^b \rangle, \langle t^{b^{-1}} \rangle\}$, respectively, generate $\langle a, b, t, w \rangle$. If there is some $c \in P$ such that $\langle w, t^b \rangle \leq Q^c$, then Lemma 2.7 forces t to centralise cb^{-1}, contrary to the fact that t inverts P.

Therefore $\langle w, t^b \rangle \cap P \neq 1$ and $\langle w, t^b \rangle \geq \langle b \rangle$. As w induces an automorphism of order 4 on P, it follows that $\langle w, t^b \rangle = \langle a, b, w, t \rangle$. We argue similarly in the other cases and thus L is a lattice isomorphic to M_9. This contradicts our assumption. □

Lemma 3.12. *Suppose that Hypothesis 3.3 holds and that $|P| = 3$. Suppose further that Q is a modular group. Then G is M_9-free.*

Proof. Assume that this is false and let G be a minimal counter-example.

Suppose that the lattice $L = \{E, S, T, D, U, V, A, C, F\}$ is isomorphic to M_9 and is a sublattice of the subgroup lattice of G. We note that this implies that F is not L_8-free. First we show that $F = G$.

All subgroups of G have modular Sylow subgroups by hypothesis. Since F is not L_8-free, Theorem 2.29 yields that F is not nilpotent. Hence F satisfies Hypothesis 3.3 and, by minimality, we deduce that $F = G$.

If $E \cap P \neq 1$, then $E \cap P = P$. But $G/P \simeq Q$ is a modular group, hence M_9-free, and this is impossible. This means that $E \cap P = 1$.

We also know that G is not nilpotent and hence it has more than one Sylow 2-subgroup. It follows that G has three Sylow 2-subgroups.
We show that one of A and C is a 2-group and assume otherwise. Then $P \leq A \cap C = D$ and therefore the previous paragraph implies that S, T, U and V do not contain P. So they are all 2-groups. Since there are only three distinct Sylow 2-subgroups in G, one of which is Q, we may suppose that $S, T \leq Q$. In particular $A = \langle S, T\rangle \leq Q$ is a 2-group, contrary to our assumption.
We may now suppose that A is a 2-group, without loss $A \leq Q$. We recall that $P \nleq E$ and hence $P \nleq U \cap V$. In particular we may suppose that U is a 2-group. Since $\langle S, U\rangle = G$, we also see that $U \nleq Q$. We choose $Q_1 \leq Q$ and $w \in P^{\#}$ such that $U = Q_1^w$. Next we argue that $A \leq \langle S, Q_1\rangle$. This follows because $G = \langle S, U\rangle \leq SQ_1P$ implies that $Q = SQ_1$. Moreover $S \leq A \leq Q$ and therefore

$$A = \langle S, Q_1\rangle \cap A = \langle S, Q_1 \cap A\rangle = S(Q_1 \cap A),$$

with Lemma 2.5.
Now we consider the group $Y := C_{A\cap Q_1}(P)$. Since $[Y, w] = 1$ and $Y \leq Q_1$, it follows that $Y \leq Q_1^w = U$. Hence $Y \leq A \cap U = E \leq S$. But then $S \cap Q_1 \cap A \geq Y = C_{A\cap Q_1}(P)$ and

$$|A| = |S| \cdot \frac{|Q_1 \cap A|}{|Q_1 \cap A \cap S|} \leq |S| \cdot \frac{|A \cap Q_1|}{|C_{A\cap Q_1}(P)|}.$$

Since Q normalises P, but does not centralise it, and since $|P| = 3$, we have that $|Q : C_Q(P)| = 2$. In particular $|(A \cap Q_1) : (C_{A\cap Q_1}(P))| = 2$ and $|A| = |S| \cdot 2$. Similarly $|A : T| = 2$ whence S and T are normal in A. Therefore $E = S \cap T \trianglelefteq A$ and A/E is a Klein fours group. We see that S, T and D are precisely the three proper subgroups of A that contain E, but are distinct from it.
Since $|G/C_Q(P)| = 6$, we have that $E = A \cap U \leq Q \cap Q^w = C_Q(P)$ and hence $E \leq C_A(P)$. Moreover $A \cap Q_1 \nleq C_A(P)$ and therefore $|A : C_A(P)| = 2$. Hence $C_A(P)$ is one of the groups S, T or D. We recall that $A = S(A \cap Q_1) = T(A \cap Q_1)$. Since $Y \leq E$ and (hence) $|E(A \cap Q_1) : E| = 2$, this implies that $E(A \cap Q_1) = D$. It follows that $C_A(P)$ is one of the subgroups S or T. If $C_A(P) = S$, then $S \leq C_Q(P) \leq Q^w$ and $\langle S, U\rangle \leq Q^w \neq G$, which is a contradiction. Hence $C_A(P) = T$, which is impossible for the same reason. □

Lemma 3.13. *Suppose that Hypothesis 3.3 holds, that P is elementary abelian and that Q is modular. If Q induces power automorphisms on P, then G is M_9-free.*

Proof. We assume that this is false and choose a minimal counter-example G.
Let $L := \{E, S, T, D, U, V, A, C, F\}$ denote a lattice that is isomorphic to M_9 and is a sublattice of the subgroup lattice of G. As $G = PQ$ and $P \trianglelefteq G$ by hypothesis, we know that all subgroups X of G have structure $X = P_X Q_X^a$, where $P_X \leq P$, $Q_X \leq Q$ and $a \in P$ are suitably chosen. We keep this notation. Moreover we note that all subgroups of P are normal in G because Q induces power automorphisms on P.
Now we collect a few general facts that follow from our choice of notation.

(i) If $Q_0 \leq Q$ and $a \in P$ are such that $Q_0^a \leq Q$, then $Q_0^a = Q_0$.

(ii) If $X \leq Y \leq G$, then $P_X \leq P_Y$ and $Q_X \leq Q_Y$.

(iii) If $X, Y \leq G$ are such that $G = \langle X, Y\rangle$, then $Q = \langle Q_X, Q_Y\rangle$.

Proof. Suppose that $Q_0 \leq Q$ and $a \in P$ are such that $Q_0^a \leq Q$. Then, since $\langle a\rangle \trianglelefteq G$, we see that $[Q_0, a] \leq Q \cap \langle a\rangle = 1$. This proves (i). If $X \leq Y \leq G$, then the fact that $P = O_3(G)$ implies that $P_X = X \cap P \leq Y \cap P = P_Y$.
For the next statement and also for (iii) we let $a, b \in P$ be such that $X = P_X Q_X^a$ and $Y = P_Y Q_Y^b$. If $X \leq Y$, then we let $c \in P_Y$ be such that $Q_X^{ac} \leq Q_Y^b$. Then $Q_X \leq Q$ and $Q_X^{acb^{-1}} \leq Q_Y \leq Q$, so (i) forces $Q_X = Q_X^{acb^{-1}} \leq Q_Y$ as stated. For (iii) we observe that

$$G = \langle X, Y\rangle = \langle P_X, P_Y, Q_X^a, Q_Y^b\rangle \leq \langle P, Q_X, Q_Y\rangle$$

because $a, b \in P$. Since P is normal in G, it follows that $G = P\langle Q_X, Q_Y\rangle$ and in particular $Q = \langle Q_X, Q_Y\rangle$. □

In the next few steps we restrict the structure of the members of L.

(1) $F = G$.

Proof. Lemma 2.4 and Theorem 2.9 yield that Q and P are M_9-free, so $|F|$ is divisible by 2 and by 3. We let $Q_0 \in \mathrm{Syl}_2(F)$ and $P_0 \in \mathrm{Syl}_3(F)$ and we assume that Q_0 centralises P_0. Then F is nilpotent and hence L_8-free (Lemma 2.29), but this is a contradiction. Hence Q_0 induces non-trivial power automorphisms on P_0 and P_0 is an elementary abelian normal subgroup of Q_0P_0. As F is not M_9-free, the minimal choice of G forces $G = F$. ∎

(2) $E \cap P = 1$.

Proof. Assume otherwise and let $N := E \cap P$. Then $1 \neq N \neq P$ and $N \trianglelefteq G$, so we see that G/N inherits the hypotheses and is therefore M_9-free. But this is false. ∎

(3) There exist $g, h \in G$ such that $E^g \leq Q$, $E^h \leq Q$, $Q \cap C^g = Q_{C^g}$ and $Q \cap A^h = Q_{A^h}$.

Proof. Let $a \in P$ be such that $C = P_C Q_C^a$. We know that E is a 2-group by (2) and $E \leq C$, so we let $c \in P_C$ be such that $E \leq (Q_C)^{ac}$. Let $g := (ac)^{-1}$. Then $E^g \leq Q_C \leq Q$, moreover $C^g = C^{a^{-1}} = P_C Q_C$ and consequently $Q_{C^g} = Q \cap C^g$.
The statement for A follows in the same way. ∎

(4) If $X \in \{S, T, U, V\}$, then $|X : E|$ is even.

Proof. We assume otherwise, without loss $X = S$ and $|S : E|$ is odd. Then $P_S \neq 1$ and $S = EP_S$. As $P_S \trianglelefteq G$, it follows from (1) that $G = \langle S, U\rangle = \langle E, P_S, U\rangle = P_S\langle E, U\rangle = P_S U$. But now $A = A \cap G = A \cap P_S U = P_S(A \cap U) = P_S E = S$, which is a contradiction. ∎

(5) Suppose that $E \leq Q$, that $Q \cap C = Q_C$ and that $c \in P_C$ is such that $S = P_S Q_S^c$ or $T = P_T Q_T^c$. Then $Q_C = Q_U = Q_V$.
If $E \leq Q$, $Q \cap A = Q_A$ and moreover if $a \in P_A$ is such that $U = P_U Q_U^a$ or $V = P_V Q_V^a$, then $Q_A = Q_S = Q_T$.

Proof. It is sufficient to prove the first statement in the case where $S = P_S Q_S^c$. Then the assertion with hypothesis for T follows in the same way and similar arguments imply the remaining statements. Since $C \cap S = E$ and $c \in C$, it follows that $(Q_S \cap C)^c = Q_S^c \cap C \leq S \cap C = E \leq Q$. Therefore $(Q_S \cap C)^c$ as well as $Q_S \cap C$ are contained in Q and (i) forces $Q_S \cap C = (Q_S \cap C)^c \leq E$. Using (ii) and (iii) we deduce that

$$Q_C = C \cap Q = C \cap \langle Q_S, Q_U\rangle = \langle Q_S \cap C, Q_U\rangle \leq \langle E, Q_U\rangle = Q_U$$

and similarly $Q_C = Q_V$. ∎

(6) At least one of S, T, U, V is a 2-group.

Proof. Assume otherwise. We set up some notation and write $S = P_S Q_S^s, T = P_T Q_T^t, U = P_U Q_U^u$ and $V = P_V Q_V^v$ with suitable elements $u, v, s, t \in P$. Then the subgroups P_S, P_T, P_U and P_V are all non-trivial and by (4) we also have that Q_S, Q_T, Q_U and Q_V are non-trivial. Assume that $P \leq A$. Then $P_U = P \cap U \leq A \cap U \cap P = E \cap P = 1$ by (2). This contradicts our assumption. So $P \nleq A$.
Next assume that $\langle P_S, P_U\rangle = P$. As P is modular and $P_A \cap P_U = 1$ by (2), we see that $\langle P_S, P_U\rangle \cap P_A = P_A = \langle P_S, P_U \cap P_A\rangle = P_S$. Therefore $P_T = P_T \cap P_A = P_S \cap P_T$ and (2) yields that $P_T = 1$, again contrary to our assumption.
Now $P \leq \langle U, S\rangle = \langle P_S, P_U, Q_S^s, Q_U^u\rangle$, hence $P = \langle P_S, P_U, s^{-1}u\rangle$. This means that $|P| = |P_S||P_U| \cdot 3$ and similarly $|P| = |P_T||P_U| \cdot 3 = |P_S||P_V| \cdot 3| = |P_T||P_V| \cdot 3$. Without loss we suppose that $|P_T| \geq |P_U|$. Then either $|P| = |P_T||P_S| \cdot 3$ or $|P| = |P_T||P_S|$. The second case yields that $P = \langle P_S, P_T\rangle = P_A$, which we already showed to be false. Therefore $|P| = |P_T||P_S| \cdot 3$.
Then $|P_S| = |P_T| = |P_U| = |P_V|$. Moreover, since $P_A \neq P$, we also have that $P_A = \langle P_S, P_T\rangle$ and so $|P_A| \cdot 3 = |P|$. A similar argument holds for P_C. Therefore P_A and P_C are maximal subgroups of P, which means that $|P_D| \cdot 9 = |P|$.
If $|P_S||P_D| \cdot 3 = |P_A| = \frac{|P|}{3}$, then $|P_S| = 1$, contradicting our assumption. Thus $|P_S||P_D| = |P_A| = \frac{|P|}{3}$ which yields that $|P_S| = |P_D| = |P_T| = |P_U| = |P_V| = 3$. We conclude that $|P| = 27$ and $|P_A| = |P_C| = 9$.

So we let $P = \langle a, b, c\rangle$ and we choose notation such that $P_C = \langle b, c\rangle$ and $P_S = \langle a\rangle$. We let $x \in P$ be such that $S = \langle a\rangle Q_S^x$. Then, since $a \in S$, we may choose $x \in \langle b, c\rangle = P_C$. In particular $|P_C| = 9$.
In light of (3) we suppose that $E \leq Q$ and $Q \cap C = Q_C$. Then (5) yields that $Q_C = Q_U = Q_V$.

If C is nilpotent, then U and V are also nilpotent and then the fact that $E \leq Q_C = Q_U = Q_V$ immediately gives that $E \in \mathrm{Syl}_2(C)$. So we suppose that C is not nilpotent. In particular $[Q_C, P_C] \neq 1$ and therefore $Q_0 := C_{Q_C}(P_C) < Q_C$. Then $Q_0 \trianglelefteq C$ and the factor group C/Q_0 has order 18 and a modular subgroup lattice (see for example Theorem 2.2.3 in [4]). Since $Q_C = Q_U$, we know that U contains a C-conjugate of Q_C, so in particular $Q_0 \leq U$. Similarly $Q_0 \leq V$ and the subgroups U/Q_0 and V/Q_0 of C/Q_0 have order 6. So they do not intersect trivially. If they intersect in a group of order 3, then 3 divides $|U \cap V| = |E|$, contrary to (2). Therefore $|U/Q_0 \cap V/Q_0| = 2$, and in particular $U \cap V \in \mathrm{Syl}_2(C)$. But $U \cap V = E$ and now this contradicts (4). ■

Suppose that S is a 2-group. Conjugating L in G if necessary (see (3)), we may suppose that $E \leq Q$ and $Q \cap C = Q_C$.
Let $s, u, v \in P$ be such that $S = Q_S^s$, $U = P_U Q_U^u$ and $V = P_V Q_V^v$. Then (1) implies that

$$G = \langle S, U\rangle = P_U \langle Q_S^s, Q_U^u\rangle \leq P_U \langle s^{-1}u\rangle Q^s.$$

Therefore $P = P_U \langle s^{-1}u\rangle$ whence $|P| \leq |P_U| \cdot 3$. The same argument with V instead of U shows that also $|P| \leq |P_V| \cdot 3$.
We know from Lemma 3.12 and from (2) that P_U and P_V are distinct from 1 and from each other, so P_U and P_V are maximal subgroups of P. It follows that $P = P_U P_V \leq \langle U, V\rangle = C$, so in particular $P_C = P$ and $s \in P_C$. Thus (5) is applicable and we have that $Q_C = Q_U = Q_V$. We recall that E is a 2-group, therefore $P_U \cap P_V = 1$ and it follows that $|P| = 9$. In particular the arguments from the last paragraph of the proof of (6) apply and they yield that $E = U \cap V \in \mathrm{Syl}_2(C)$, contrary to (4).
With similar arguments we exclude the case where T is a 2-group.
Then (6) yields that one of U or V is a 2-group.
Now we argue in the same way as in the previous paragraph. We let $u \in P$ be such that $U = Q_U^u$ and show first that $P = P_S P_T \leq \langle S, T\rangle = A$, so that $u \in P_A$. Then we use (3) to see that we may without loss suppose that $E \leq Q$ and $Q \cap A = Q_A$, and then (5) is applicable to A, S and T. It follows that $Q_A = Q_S = Q_T$ and that $E = S \cap T \in \mathrm{Syl}_2(A)$, which is impossible by (4).
This is our final contradiction. □

Definition 3.14. We say that G is a $\{2,3\}^*$-group if and only if the following holds:

(i) G is a non-nilpotent $\{2,3\}$-group;
(ii) $P \in \mathrm{Syl}_3(G)$ is normal in G and elementary abelian;
(iii) Q is a modular Sylow 2-subgroup of G;
(iv) Q induces power automorphisms on P; and
(v) G has a section isomorphic to D_{12}.

Theorem 3.15. *Suppose that G is a $\{p,q\}$-group. Then G is M_9-free, but not L_8-free if and only if G is a $\{2,3\}^*$-group.*

Proof. First suppose that G is a $\{2,3\}^*$-group. Then in particular Hypothesis 3.3 holds. Now Q induces power automorphisms on P and P is elementary abelian. Thus Lemma 3.13 is applicable and it yields that G is M_9-free. Moreover G involves D_{12} and hence it is not L_8-free (see Example 2.8).
Now we suppose, conversely, that G is an M_9-free $\{p,q\}$-group, but not L_8-free. Then G is not nilpotent and Theorem 2.18 implies that G has a normal p-subgroup P and a q-subgroup Q such that $G = PQ$ and such that Q acts non-trivially on P. With Lemma 3.2 it follows that $p = 3$ and $q = 2$ and that G involves D_{12}. Moreover P is elementary abelian by Corollary 3.5. Now Lemmas 3.11 and 3.8 give the result. □

Corollary 3.16. *Suppose that G is a $\{p,q\}$-group and that G is M_9-free, but not L_8-free. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. If Q is not normal in G, then $G = \langle Q^x \mid x \in P\rangle$.*

Proof. By Theorem 3.15, G is a $\{2,3\}^*$-group. This means that Q induces power automorphisms on P and that Q is a 2-group and P is an elementary abelian 3-group. Let $x \in P$. Lemma 3.4 gives that $C_P(Q) = 1$, hence there exists some $a \in Q$ such that $[x^{-1}, a] \neq 1$. But a normalises $\langle x\rangle$, so we must have that $(x^{-1})^a = x$. In particular $a^{-1}a^x \in \langle Q, Q^x\rangle$. But we also see that $a^{-1}a^x = a^{-1}x^{-1}ax = (x^{-1})^a x = x^2 = x^{-1}$ and hence $x \in \langle Q, Q^x\rangle \leq \langle Q^y | y \in P\rangle$. We chose $x \in P$ arbitrarily, so it follows that $P \leq \langle Q^x | x \in P\rangle$. Thus $G = \langle Q^x | x \in P\rangle$ as stated. □

Now we combine Corollaries 3.16 and 2.26:

Corollary 3.17. *Suppose that G is an M_9-free $\{p,q\}$-group, let $P \in Syl_p(G)$ and let $Q \in Syl_q(G)$. If Q is not normal in G, then $G = \langle Q^x \mid x \in P\rangle$.*

4. M_9-free groups

Based on our results about M_9-free $\{p,q\}$-groups we will now classify all finite M_9-free groups. The last class of groups for us to study before we proceed is the class of what we call "Q_8^*-groups".

Definition 4.1. We say that G is a Q_8^*-group if and only if G has subgroups Q, N and M satisfying the following:

(i) $G = (N \times M)Q$;
(ii) $Q \cong Q_8$;
(iii) N is an elementary abelian normal 3-subgroup of G;
(iv) M is an elementary abelian normal r-subgroup of G for some odd prime r distinct from 3; and
(v) MQ is L_8-free and not nilpotent, and NQ is M_9-free, but not L_8-free.

Theorem 4.2. *If G is a Q_8^*-group, then G is M_9-free.*

Proof. We assume that this is false and we choose all the notation for G as in Definition 4.1. Moreover we choose G to be minimal with the property that its subgroup lattice contains a lattice $L = \{E,S,T,D,U,V,A,C,F\}$ isomorphic to M_9, again with our standard notation. Let $x, y \in Q$ be such that they generate distinct subgroups of order 4 of Q and let z denote the central involution in Q.
First we look at MQ. By definition of a Q_8^*-group, this subgroup is L_8-free and not nilpotent, so Theorem 2.29 implies that Q acts faithfully on M and that $|M| = r^2$, where $r \equiv 3$ modulo 4. Moreover NQ is M_9-free, but not L_8-free, so it follows from Lemmas 3.9 and 3.10 that Q acts non-faithfully on N and by inducing power automorphisms. We also recall that N and M have coprime order.
(1) z inverts M.
Proof. This follows from the fact that Q acts faithfully on M and that, therefore, z acts as the central involution of $\mathrm{GL}_2(r)$. ■
(2) The subgroups of order 4 of Q act irreducibly on M.
Proof. Assume that, without loss, the subgroup $X := \langle x\rangle$ does not act irreducibly on M. Then let M_1 denote an X-invariant subgroup of M of order r. As $r - 1 \equiv 2$ modulo 4, it follows that z centralises some element in $M_1^\#$, contrary to (1). ■
(3) Two of the elements $x, y, xy \in Q$ invert N. In particular $[N, z] = 1$.
Proof. We recall that NQ is not nilpotent. Hence we may without loss suppose that $[N, x] \neq 1$. As Q induces power automorphisms on N, it normalises all subgroups of N of order 3, so there exists some element $g \in N$ of order 3 that is inverted by x. But x induces a universal power automorphism and hence it inverts N. If y centralises N, then xy inverts it and vice versa. ■

For the remainder of the proof we choose notation such that x and y invert N.
(4) Suppose that $g \in N \times M$.
If $g \in N^\#$, then $Q \cap Q^g = \langle xy\rangle$. If $g \in (N \times M)\backslash N$, then $Q \cap Q^g = 1$.
Proof. As $NM \trianglelefteq G$ and $Q \cap NM = 1$, we may apply Lemma 2.7 and we obtain that $Q \cap Q^g = C_Q(g)$. If $g \in N^\#$, then $C_Q(g) = \langle xy\rangle$ by (3). If $g \in (N \times M)\backslash N$, then (1) and (3) yield that g is not centralised by z, hence it is not centralised by any of x, y or xy or their inverses, as claimed. ■
(5) Suppose that $M_1 \leq M$ and that $g \in NM_1$. Then $z^g \in \langle z\rangle M_1$. If $h \in (M \times N)\backslash N$, then $z^h \notin \langle z\rangle N$.
Proof. Let $a \in N$, $b \in M_1$ be such that $g = ab$. Then $z^g = z^{ab} = (z^a)^b = z^b \in \langle z\rangle M_1$ by (3).
Next let $h \in (M \times N)\backslash N$ and assume that $z^h \in \langle z\rangle N$. Then we let $c \in N$, $d \in M^\#$ be such that $h = cd$. Then $zz^h = z(cd)^{-1}zcd = c^{-1}zd^{-1}zdc = c^{-1}(d^{-1})^z dc = c^{-1}d^2c = d^2 \in M$, using (1) and (3). We deduce that $z^h \in \langle z\rangle M \cap \langle z\rangle N = \langle z\rangle$, which contradicts (4). ■
(6) $F = G$.
Proof. Let Q_1 denote a maximal subgroup of Q. Then Q_1 is cyclic of order 4 and contains z, so we know by (3) that Q_1 does not act faithfully on N. But Q_1 acts faithfully on M (by (1)) and therefore $C_{Q_1}(M) \neq C_{Q_1}(N)$. Moreover Q_1 acts irreducibly on M and by inducing power automorphisms on

N. Hence $(N \times M)Q_1$ is a Λ^*-group as introduced in Definition 2.27, which means that this group is M_9-free (Theorem 2.29). As G is not M_9-free by assumption, we may suppose that F contains Q. But we also know that NQ and MQ are M_9-free, which implies that F contains non-trivial subgroups of N and of M, respectively. Now (2) implies that $M \leq F$. Let $N_1 := N \cap F$. Then $F = (N_1 \times M)Q$ and we look at the subgroup N_1Q. By Theorem 3.15, Q induces non-trivial power automorphisms in N and hence in N_1, therefore Theorem 2.29 yields that N_1Q is not L_8-free. It follows that F is a Q_8^*-group that is not M_9-free, so the minimal choice of G forces $F = G$. ■

(7) $E \cap N$=1 and $M \nleq E$.

Proof. Set $W := E \cap N$ and assume that $W \neq 1$. We note that $W \leq N$, so M centralises W and Q normalises W because it induces power automorphisms on N. This means that $W \trianglelefteq G$. We also note that W is contained in all members of L because $W \leq E$. Let $\bar{G} := G/W$. Then $\bar{G}$ is not M_9-free, in particular it is not isomorphic to a subgroup of QM and this forces $W \neq N$. Theorems 3.15 and 2.29 yield that $\bar{G}$ is a Q_8^*-group, and this contradicts the minimal choice of G as a counter-example. Hence $E \cap N = W = 1$.

Next we assume that $M \leq E$. As $M \trianglelefteq G$, we may look at the factor group G/M. This group is isomorphic to QN, but it is also not M_9-free, and this is impossible. ■

(8) The subgroups S, T, U, V (from the lattice L) are not contained in NM.

Proof. Assume otherwise. The arguments are similar for S, T, U and V, so we assume that $S \leq NM$. Using (6) we see that $G = F = \langle S, U\rangle = \langle S, V\rangle$, so it follows that U and V both contain a conjugate of Q. Without loss $Q \leq U$ and let $g \in G$ be such that $Q^g \leq V$. We may choose g in NM.

Assume that there exists $a \in N$ such that $Q^g = Q^a$. Then (4) implies that $xy \in Q \cap Q^a \leq U \cap V = E \leq S \leq NM$, which is false. Therefore $g \in NM \backslash N$. As $NM \trianglelefteq G$, it follows that $\langle Q, Q^g\rangle$ contains a non-trivial element of M. Thus C contains Q and some element of $M^\#$, which by (2) means that

$$M \leq C.$$

Now $S \cap M \leq S \cap C = E = U \cap V$. If $S \cap M \neq 1$, then $1 \neq S \cap M \leq U$ and $1 \neq S \cap M \leq V$. The irreducible action of $Q \leq U$ on M (and of $Q^g \leq V$ on M) forces $M \leq U \cap V = E$. This contradicts (7).

Hence $S \cap M = 1$ and it follows that $S = (S \cap N) \times (S \cap M) \leq N$. In particular $S \trianglelefteq G$ whence $G = SU$. Now Dedekind's identity gives that $A = S(A \cap U) = S$, which is false.

This final contradiction shows that $S \nleq NM$. ■

(9) The subgroups S, T, U and V are not contained in $NM\langle z\rangle$.

Proof. Assume otherwise and without loss assume that $S \leq NM\langle z\rangle$. Then we know from (8) that S has even order, so there exists some $g \in NM$ such that $z^g \in S$. We may choose $g \in M$ by (3). As $G = F = \langle S, U\rangle = \langle S, V\rangle$ by (6), it follows as in the previous step that U and V contain a conjugate of Q. Without loss $Q \leq U$ and we let $h \in G$ be such that $Q^h \leq V$. Then we may take $h \in NM$ and (4) yields that $h \notin N$. In particular $C = \langle U, V\rangle$ contains a non-trivial subgroup of M and then all of M, by (2).

We know that $M \nleq E = U \cap V$ by (7), hence (2) implies that M intersects U or V trivially. Without loss suppose that $U \cap M = 1$. Then U is an r'-subgroup of G that contains Q, so $U \leq NQ$. We note that, by (3), this implies that z is the unique involution in U. We also recall that $z^g \in S$ and that $\langle z, M\rangle \leq C$, so $z^g \in S \cap C = E \leq U$. But then $z^g = z$, so $z \in S$. We deduce that z is the unique involution in S. Hence $S \cap M = 1$ by (1) and moreover $S \leq N\langle z\rangle$. Finally $G = F = \langle S, U\rangle \leq NQ$, which is false. ■

(10) $D \cap M = 1$.

Proof. We assume that this is false and we let $M_1 := D \cap M$. It follows from (9) that A and C have order divisible by 4, so they contain a cyclic subgroup of order 4 that is conjugate to a subgroup of Q, respectively. As $M_1 \neq 1$, (2) implies that $M \leq A \cap C = D$ and hence $M_1 = M$. Let $N_1 := A \cap N$ and $N_2 := C \cap N$. We may suppose that $Q \cap A \in \mathrm{Syl}_2(A)$ and hence $A \leq QN_1M$. Let $h \in G$ be such that $C \leq Q^hN_2M$. Then we may suppose that $h \in N$. As A and C contain a conjugate of z, it follows that $z \in A \cap C$ and therefore $z \in D$.

If $M \cap S \neq 1$, then (9) implies that $M \leq S$ and hence $M \leq S \cap D = E$. This is impossible by (7) and thus $M \cap S = 1$. Similarly $M \cap T = 1$. Therefore we may suppose that $T \leq NQ$ and that there is some $a \in NM$ such that $S \leq NQ^a$. If $a \in N$, then $A = \langle S, T\rangle \leq NQ$, and this contradicts the fact that

$M \leq A$. Therefore $a \in (NM) \setminus N$. Since T and S have order divisible by 4, we deduce that $z \in T$ and $z^a \in S$. By (5) we also have that $z^a \in M\langle z\rangle \leq D$, so $z^a \in S \cap D = E \leq T$, which is a contradiction. ■

We can now finish the proof. As $A \cap C \cap M = 1$ by (10), we deduce from (9) and (2) that at most one of A and C intersects M non-trivially. We suppose that $A \cap M = 1$ and hence, without loss, that $A \leq QN$. Step (9) also yields that T and S contain conjugates of z, so it follows that $z \in S \cap T = E$. Thus $z \in U \cap V$. As $M \leq \langle S, U\rangle = \langle S, V\rangle = F = G$ (by (6)), we also see that U and V contain non-trivial subgroups of M or a conjugate z^y of z such that $y \in (M \times N) \setminus N$. Moreover U and V contain elements of order 4 from G, by (9). Hence they contain M by (2). Now $M \leq U \cap V = E$, contrary to (7). Thus G is M_9-free. □

Corollary 4.3. *Suppose that G is a Q_8^*-group with notation as in Definition 4.1. Then $G = \langle Q^x \mid x \in N \times M\rangle$.*

Proof. The subgroups QN and QM of G are M_9-free and their orders have only two different prime divisors, respectively. Therefore Corollary 3.17 yields that $QM = \langle Q^x \mid x \in M\rangle$ and that $QN = \langle Q^x \mid x \in N\rangle$. Hence $G = \langle QN, QM\rangle = \langle Q^x \mid x \in N \times M\rangle$. □

Lemma 4.4. *M_9 is subdirectly irreducible.*

Proof. We choose our usual notation $M_9 = \{E, S, T, D, U, V, A, C, F\}$. Suppose that L is a lattice and that ϕ is a lattice homomorphism from M_9 to L that is not injective. We show that $E^\phi = D^\phi$.
Assume otherwise and first assume that there is some element $X \in M_9$ such that $X^\phi = D^\phi$. Then $X \neq E$ by assumption.
If $X \in \{T, S, U, V\}$, then $E^\phi = (D \cap X)^\phi = D^\phi \cap X^\phi = D^\phi$ because ϕ is a lattice homomorphism. But this is false.
If $X = A$, then $D^\phi = A^\phi = (D \cup T)^\phi = D^\phi \cup T^\phi$ and therefore $T^\phi \leq D^\phi$. But this implies that $T^\phi = T^\phi \cap D^\phi = (T \cap D)^\phi = E^\phi$ and consequently $F^\phi = (U \cup T)^\phi = U^\phi \cup T^\phi = U^\phi \cup E^\phi = U^\phi$. Then we conclude that $A^\phi \leq F^\phi = U^\phi$ and hence $A^\phi = A^\phi \cap U^\phi = (A \cap U)^\phi = E^\phi$, so $D^\phi = E^\phi$. This is a contradiction. We argue similarly if $X = C$, so this is impossible as well.
If $X = F$, then $U^\phi \leq F^\phi = D^\phi$ and moreover $U^\phi \cap D^\phi = E^\phi$. Therefore $U^\phi = E^\phi$. With the same argument $T^\phi = E^\phi$. It follows that $E^\phi = U^\phi \cup T^\phi = (U \cup T)^\phi = F^\phi = D^\phi$ and this is another contradiction.
We deduce that D is the unique pre-image in M_9 of D^ϕ. But we chose ϕ to be non-injective, and therefore there must be an image with two distinct pre-images. We choose $X_1, X_2 \in M_9$ to be distinct from D and from each other and such that $X_1^\phi = X_2^\phi$.
Assume that $X_1 \in \{S, T, U, V\}$. Then we choose some $Y \in \{S, T, U, V\}$ such that $X_1 \cup Y = F$. We now argue by excluding all possibilities for X_2.
Assume that $X_2 = E$. Then $F^\phi = (X_1 \cup Y)^\phi = X_2^\phi \cup Y^\phi = E^\phi \cup Y^\phi = (E \cup Y)^\phi = Y^\phi$ and therefore $D^\phi \cup Y^\phi \leq F^\phi = Y^\phi$ also $D^\phi \leq Y^\phi$. This implies that $D^\phi = D^\phi \cap Y^\phi = (D \cap Y)^\phi = E^\phi$, contrary to the fact that D^ϕ has a unique pre-image. This means that $X_2 \neq E$.
Next assume that $X_2 \neq E$, but $X_1 \cap X_2 = E$. Then let $H := X_2 \cup X_1$. Then $H^\phi = (X_1 \cup X_2)^\phi = X_1^\phi \cup X_2^\phi = X_1^\phi = X_1^\phi \cap X_2^\phi = (X_1 \cap X_2)^\phi = E^\phi$. It follows that $E^\phi \leq D^\phi \leq H^\phi = E^\phi$ and thus $D^\phi = E^\phi$. Again we have a contradiction.
Now we assume that $X_1 \leq X_2$. Then $X_2 \in \{A, C, F\}$ whence $D \leq X_2$. Therefore $D^\phi \leq X_2^\phi = X_1^\phi$ and this means that $(D \cap X_1)^\phi = D^\phi \cap X_1^\phi = D^\phi$. In particular $D \cap X_1 = D$ whence $X_1 \in \{A, C\}$, contrary to our assumption.

So we have that $X_1 \notin \{S, T, U, V\}$ and, by symmetry, that $X_2 \notin \{S, T, U, V\}$. These arguments also show that S^ϕ, T^ϕ, U^ϕ and V^ϕ have unique pre-images in M_9. We recall that $X_1 \neq D \neq X_2$ by choice, so the only possibilities are $X_1, X_2 \in \{E, A, C, F\}$.
Again we assume that $X_1 \leq X_2$. Then $X_2 \in \{A, C, F\}$ and it follows as above that $D \cap X_1 = D$. Thus $X_1 \in \{A, C\}$ and $X_2 = F$. Let $Y \in \{T, U\}$ be such that $X_1 \cup Y = F$. Then $X_1^\phi = X_2^\phi = F^\phi = (X_1 \cup Y)^\phi = X_1^\phi \cup Y^\phi$, which means that $Y^\phi \leq X_1^\phi$ and $Y^\phi = E^\phi$. This contradicts the fact that Y^ϕ has a unique pre-image in M_9, by the previous paragraph. Therefore $X_1 \not\leq X_2$ and, by symmetry, also $X_2 \not\leq X_1$. In particular X_1 and X_2 are both distinct from E and F, which means that $\{X_1, X_2\} = \{C, A\}$.

Now $F^\phi = X_1^\phi \cup X_2^\phi = X_1^\phi = X_1^\phi \cap X_2^\phi = D^\phi$, which is impossible. □

It follows from this that, if G is a direct product of two subgroups of coprime order, then G is M_9-free if and only if these subgroups are M_9-free.

Theorem 4.5. *Suppose that $n \in \mathbb{N}$ and that $G_1, ..., G_n$ are subgroups of G of pairwise coprime orders and such that $G = G_1 \times ... \times G_n$. For all $i \in \{1, ..., n\}$ suppose further that G_i is L_8-free or a Q_8^*-group or a $\{2,3\}^*$-group. Then G is M_9-free.*

Proof. This follows from Theorems 3.15 and 4.2 and the fact that M_9 is subdirectly irreducible. □

Theorem 4.6. *Suppose that G is M_9-free and let $p, q \in \pi(G)$. Then for all $P \in Syl_p(G)$ and all $Q \in Syl_q(G)$ we have that $PQ = QP$.*

Proof. Assume that this is false and let G be a minimal counter-example. We choose $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $PQ \neq QP$ and we fix this notation.
As G is M_9-free and hence L_{10}-free, Theorem 2.20 yields that G is soluble. Hence if we let N be a minimal normal subgroup of G, then there exists a prime r such that N is an elementary abelian r-subgroup. The minimal choice of G implies that any two Sylow subgroups of G/N (for distinct primes) commute. Therefore $PNQ/N = QNP/N$. Moreover $G = \langle P, Q\rangle$, again by minimal choice of G, and hence $G = QNP = PNQ$.
Next we notice that some conjugate Q_1 of Q commutes with P, because G soluble. Then also $G = Q_1PN$. In particular Q_1P is an M_9-free subgroup of G in which Q_1 or P is normal, by Theorem 2.18. Without loss we suppose that $Q_1 \leq N_G(P)$ (which means that $PN \trianglelefteq G$) and we keep all this notation.
(1) $p \neq r \neq q$. In particular $O_p(G) = 1 = O_q(G)$, and Q and P act faithfully on N.
Proof. If $r = p$, then $N \leq P$ and hence $PQ = QP$, but this is not the case. Similarly $r \neq q$. This also implies that G has no non-trivial normal p- or q-subgroup.
We recall that Q_1 normalises P and hence it normalises $C_P(N)$. Thus $C_P(N) \leq O_p(G) = 1$. In particular $C_{Q_1P}(N)$ has trivial Sylow p-subgroups and this implies that $C_{Q_1P}(N)$ is a q-group. Being normal in G, this subgroup is now contained in $O_q(G)$ and hence trivial. As Q_1 is conjugate to Q in G, we deduce that P and Q act faithfully on N. ■
(2) Q and P act irreducibly on N or by inducing power automorphisms. In the power automorphism case every element of $Q^\#$ (or $P^\#$) acts without fixed points on N.
Proof. We look at QN. This is an M_9-free $\{q, r\}$-subgroup of G and hence Theorem 3.15 implies that QN is L_8-free or a $\{2,3\}^*$-group. In the second case Q induces power automorphisms as stated. Now we consider the L_8-free case. As QN is not nilpotent by (1), we deduce from Theorem 2.25 that $Q \cong Q_8$. (Recall that N is elementary abelian.) Moreover N has order r^2 and Q acts faithfully on it, so Q is isomorphic to a subgroup of $\mathrm{GL}_2(r)$. In particular the central involution z in Q inverts N.
Assume that Q normalises a subgroup N_1 of N of order r. As $r - 1 \equiv 2$ modulo 4, it follows that z fixes some element of $N_1^\#$ rather than inverting it. So this is impossible and we conclude that Q acts irreducibly on N.
Now suppose that Q acts by inducing power automorphisms on N. Then these are universal and (1) implies that Q acts without fixed points on N. The same arguments for the group PN prove the assertion for P. ■
(3) If $x \in N^\#$, then $Q \cap Q^x = 1$ and $P^x \cap P = 1$.
Proof. We apply Lemma 2.7 to N. If $x \in N^\#$, then the lemma yields that $Q \cap Q^x = C_Q(x)$. But (2) forces $C_Q(x) = 1$. Similarly $P \cap P^x = C_P(x) = 1$, so the proof is finished. ■
(4) Suppose that $a, b \in N$ are such that Q^a and Q^b normalise P. Then $a = b$. If $x, y \in N$ are such that Q normalises P^x and P^y, then $x = y$.
Proof. Let $c := ba^{-1}$ and $P_1 := P^{a^{-1}}$. Then by hypothesis $\langle Q, Q^c\rangle \leq N_G(P_1)$. Assume that $c \neq 1$ and let $u \in Q^\#$. Then $v := u^{-1} \cdot u^c = (c^{-1})^u \cdot c \in N_G(P_1) \cap N$. If $v = 1$, then $1 \neq u = u^c \in Q \cap Q^c$ and (3) forces $c = 1$, which is a contradiction. Therefore $v \neq 1$.
We have that $[P_1, v] \leq P_1 \cap N = 1$. In particular P_1 does not act irreducibly on N, so by (2) it induces power automorphisms. Then P also induces power automorphisms on N and every element of $P^\#$ acts without fixed points on N. Then the same holds for P_1, and hence $[P_1, v] = 1$ forces $v = 1$. So c centralises u, i.e. $u \in Q \cap Q^c$. Then (3) forces $c = 1$, which is a contradiction.

Next let $x, y \in N$ be such that Q normalises P^x and P^y. Then Q and $Q^{y^{-1}x}$ normalise P^x and so the first part of the proof gives that $x = y$. ■

(5) There is some $y \in N^\#$ such that $G = \langle Q, P\rangle = \langle Q^y, P^y\rangle = \langle Q^y, P\rangle = \langle Q, P^y\rangle$.

Proof. The coprime action of Q on PN yields (see for example 8.2.3 in [3]) that there exists some $x \in N$ such that $Q \leq N_G(P^x)$. Moreover Q does not normalise P by assumption, so $x \neq 1$. Since $[P, N] \neq 1 \neq [Q, N]$, we have that $|N| > 3$ and hence there is $y \in N^\#$ such that $x \neq y \neq x^{-1}$. Then Q does not normalise P^y and Q^y normalises only P^{xy}. The choice of y and x implies that $P \neq P^{xy} \neq P^y$. Hence we see that $G = \langle Q, P\rangle = \langle Q^y, P^y\rangle = \langle Q^y, P\rangle = \langle Q, P^y\rangle$. ■

Let $y \in N^\#$ be as in (5) and let $Y := \langle y\rangle$. We know that P and Q act irreducibly or by inducing power automorphisms on N, by (2). This gives three cases to look at, two of which we can treat together. We proceed by constructing a lattice isomorphic to M_9 in $L(G)$, giving a contradiction. For these constructions we note that (5) and the choice of y imply that one of P, P^y and one of Q, Q^y generate G.

First assume that both groups induce power automorphisms. Then they normalise the group $Y \leq N$ and hence $Y \trianglelefteq G$. But we chose N to be a minimal normal subgroup of G, so $N = Y$ and hence P and Q also act irreducibly on N.

We look at $L := \{1, P, P^y, N, Q, Q^y, PN, QN, G\}$.

By (3) any two of the groups N, P, P^y, Q and Q^y intersect trivially. Moreover $PN \cap NQ = N$ and $G = \langle PN, QN\rangle$. Then (3) and the irreducible action imply that $\langle Q, Q^y\rangle = QN$ and $\langle P, P^y\rangle = PN$. Thus L is isomorphic to M_9, which is a contradiction.

Finally, without loss, Q acts irreducibly on N and P acts by inducing power automorphisms. Then we set $L := \{1, P, P^y, Y, Q, Q^y, PY, QN, G\}$.

Again by (3) any two of the groups Y, P, P^y, Q and Q^y intersect trivially. Moreover $PY \cap NQ = Y$ and $G = \langle PY, QN\rangle$. We also see that $\langle Q, Q^y\rangle = QN$ and $\langle P, P^y\rangle = PY$. Once more we conclude that L is isomorphic to M_9, which is impossible. □

Theorem 4.7. *Suppose that G is M_9-free. Then there are $n \in \mathbb{N}$ and subgroups $G_1, ..., G_n$ of G of pairwise coprime orders such that the following hold:*

(i) *$G = G_1 \times ... \times G_n$ and*
(ii) *for all $k \in \{1, ..., n\}$ the group G_k is L_8-free or a Q_8^*-group or a $\{2, 3\}^*$-group.*

Proof. Assume that this is false and choose G to be a minimal counter-example. Then G is not nilpotent because otherwise Lemma 2.10 yields that G is a direct product of L_8-free groups of pairwise coprime orders. In particular $|G|$ has at least two distinct prime divisors.

We also know that G is soluble by Theorem 2.20.

(1) Suppose that $p, q \in \pi(G)$ and let $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(G)$. Moreover let $x, y \in G$. Then P^xQ^y is a subgroup of G that is conjugate to PQ in G.

Proof. Theorem 4.6 yields that PQ and P^xQ^y are subgroups of G. In fact they are Hall $\{p, q\}$-subgroups of G. As G is soluble, it follows that these subgroups are conjugate in G. (See for example 6.4.7 in [3].) ■

(2) Suppose that $p, q \in \pi(G)$ and let $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(G)$. If Q is not normal in G, then Q normalises P.

Proof. Assume that $Q \not\leq N_G(P)$. We know from Theorem 4.6 that $PQ = QP$ and by hypothesis PQ is M_9-free. Thus Theorem 2.18 forces $Q \trianglelefteq PQ$. With Corollary 3.17 we see that $PQ = \langle P^x \mid x \in Q\rangle$. Let $g \in G$ and $x \in Q$. Then it follows from (1) that P^xQ^g is conjugate to PQ in G. In particular $Q^g \trianglelefteq P^xQ^g$ and therefore $Q \leq \langle P^x \mid x \in Q\rangle \leq N_G(Q^g)$. As $Q, Q^g \in \mathrm{Syl}_q(G)$, this is only possible if $Q = Q^g$. But this implies that $Q \trianglelefteq G$, contrary to our hypothesis. ■

Now we set up some notation for the remainder of the proof.

We already argued that G is not nilpotent, so we fix a prime $q \in \pi(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that Q is not normal in G. The solubility of G yields that we find a Hall q'-subgroup X of G. As X is M_9-free and G is a minimal counter-example to our theorem, there exist $n \in \mathbb{N}$ and subgroups $Y, X_1, ..., X_n$ of X of pairwise coprime order such that $X = Y \times X_1 \times \cdots \times X_n$ and such that Y is L_8-free and each of the groups $X_1, ..., X_n$ is a $\{2, 3\}^*$-group or a Q_8^*-group. Now, according to Theorem 2.29, let $k \in \mathbb{N}$ and $Y_1, ..., Y_k$ be subgroups of Y of pairwise coprime order such that $Y = Y_1 \times \cdots \times Y_k$ and such that

each of these subgroups is a p-group or a $\{2,p\}$-group satisfying Lemma 2.23 (iv) (b) (for a suitable prime p) or a Λ^*-group.
Using (2) we see that every Sylow subgroup of X is normalised by Q and hence $Q \le N_G(X)$. So $X \trianglelefteq G$, but we also know for all $i \in \{1,...,n\}$ and $j \in \{1,...,k\}$ that $X_i \trianglelefteq G$ and $Y_j \trianglelefteq G$. In particular all X_iQ and all Y_jQ are subgroups of G.

(3) For all $i \in \{1,...,n\}$ and $j \in \{1,...,k\}$ we have that $[X_i, Q] \neq 1$ and $[Y_j, Q] \neq 1$.
Proof. Assume otherwise and without loss assume that Q centralises X_1. Let $W := X_2 \times \cdots X_n$. As Q, Y and X_1 normalise W, it follows that QW is a normal subgroup of G. The minimal choice of G forces QW to be a direct product of subgroups of pairwise coprime orders that are L_8-free or $\{2,3\}^*$-groups or Q_8^*-groups, respectively. But $X_1 \trianglelefteq G$ and the orders of QW and X_1 are coprime, therefore $G = Y \times X_1 \times QW$. Moreover Y is L_8-free and X_1 is a $\{2,3\}^*$-group or a Q_8^*-group. This contradicts our choice of G as a counter-example.
We argue in a similar way if Q centralises Y_1: Then Q and X normalise $W_0 := Y_2 \times \cdots Y_k$, so $QW_0 \trianglelefteq G$ and $G = Y_1 \times QW_0 \times X_1 \times \cdots \times X_n$. By minimality QW_0 is a direct product of subgroups of pairwise coprime orders that are L_8-free or $\{2,3\}^*$-groups or Q_8^*-groups, respectively. Again this is a contradiction. ∎

(4) If $j \in \{1,...,k\}$, then there exists a prime number p_j such that Y_j is a Sylow p_j-subgroup of G. Moreover $n = 0$ and all subgroups Y_j are modular.
Proof. Assume that $n \neq 0$ and let $i \in \{1,...,n\}$. We recall that X_i is a $\{2,3\}^*$-group or a Q_8^*-group. In both cases (following from the definition) there exist $m \in \mathbb{N}$, pairwise distinct prime numbers $s, r_1, ..., r_m$, an s-subgroup S, an r_1-subgroup R_1,..., and an r_m-subgroup R_m of X_i such that $X_i = S(R_1 \times ... \times R_m)$ and S is not normal in X_i. Let $R := R_1 \times ... \times R_m$. Then Corollaries 4.3 or 2.28 (depending on the case) imply that $X_i = \langle S^y \mid y \in R\rangle$. We note that $|S|$ is coprime to the orders of R, Y, Q and every subgroup X_l where $l \neq i$. This implies that $S \in \mathrm{Syl}_s(G)$. If $y \in R$, then S^y is not normal in RS^y and hence it is not normal in G. So we may apply (2) to S^y and this yields that S^y normalises Q. Conversely Q normalises S^y, again using (2). But then $[S^y, Q] = 1$. It follows that $[X_i, Q] = 1$, contrary to (3). Thus $n = 0$.

Next let $j \in \{1,...,k\}$. Let $p_j \in \pi(Y_j)$ and assume that Y_j is not a p_j-group. Then Y_j is a $\{2, p_j\}$-group satisfying Lemma 2.23 (iv) (b) or a Λ^*-group, so we argue similarly to the previous paragraph. In both cases there exist $m \in \mathbb{N}$, pairwise distinct prime numbers $s, r_1, ..., r_m$, an s-subgroup S, an r_1-subgroup R_1,..., and an r_m-subgroup R_m of Y_j such that $Y_j = S(R_1 \times ... \times R_m)$ and S is not normal in Y_j. Let $R := R_1 \times ... \times R_m$. Then $Y_i = \langle S^a \mid a \in R\rangle$, $S \in \mathrm{Syl}_s(G)$ and (2) yields that $[S^a, Q] = 1$. Hence $[Y_j, Q] = 1$, contrary to (3).
Now Y_j is a p_j-group and we recall that p_j does not divide the order of Q and of the subgroups Y_l where $l \neq j$. In fact $Y_j \in \mathrm{Syl}_{p_j}(G)$ and Y_j is M_9-free, hence modular (see Theorem 2.9 and Lemma 2.10). ∎

(5) There is at most one index $j \in \{1,...,k\}$ such that Y_jQ is not L_8-free.
Proof. Assume otherwise and assume without loss that Y_1Q and Y_2Q are both not L_8-free. Then Theorem 3.15 is applicable by (3) and it yields that Y_1 and Y_2 are 3-groups. But these groups have coprime order, so this is impossible. ∎

(6) Suppose that $i \in \{1,...,k\}$ is such that Y_iQ is not L_8-free. Then $Q \cong Q_8$ and the action of Q on Y_i is not faithful.
Proof. First Lemma 3.2 implies that Q is not cyclic. Moreover Q is a 2-group by Theorem 3.15. As G is a counter-example, it follows that $G \neq Y_iQ$ and hence there exists some $j \in \{1,...,k\}$ such that $i \neq j$. Using (5) we deduce that Y_jQ is L_8-free. As Q is a 2-group, we know that Y_j is not a 2-group and hence Theorem 2.14 gives that Y_j is not hamiltonian. Then we deduce from Theorem 2.25 that $Q \cong Q_8$ (because Q is not cyclic). The remainder follows from Corollary 3.10. ∎

(7) Let $i, j \in \{1,...,k\}$ be such that $i \neq j$. Then $C_Q(Y_i) \neq C_Q(Y_j)$.
Proof. Assume that this is false, and without loss $C_Q(Y_1) = C_Q(Y_2)$. By (5) we may suppose that Y_1Q is L_8-free. If Y_2Q is not L_8-free, then (6) yields that $Q \cong Q_8$ and that Q does not act faithfully on Y_2. Then by assumption it also acts non-faithfully on Y_1. But this contradicts Lemma 2.23. Thus Y_2Q is also L_8-free.

Now we consider $C_Q(Y_1)$. This is a proper subgroup of Q by (3) and hence we may choose a subgroup Q_1 of Q such that $C_Q(Y_1) \leq Q_1$ and $|Q_1 : C_Q(Y_1)| = q$.
Assume that $Q_1 \cong Q_8$. Then Case (iv)(b) of Lemma 2.23 holds (applied to Y_1Q_1) and so Q_1 acts faithfully on Y_1, whereas $|C_Q(Y_1)| = 4$. This is impossible. Thus Theorem 2.25 yields that Q_1 is cyclic and that Case (iv)(a) of Lemma 2.23 or Case (i) of Lemma 2.24 holds. In particular all subgroups of Q_1 are normal in Q_1. With notation as in (4) we may suppose that $p_1 > p_2$, because Y_1 and Y_2 have coprime order. In particular $p_1 \geq 3$ and hence Y_1 is elementary abelian, and Y_2 is elementary abelian or isomorphic to Q_8. (Moreover in the elementary abelian case Q_1 acts irreducibly or by inducing power automorphisms.) In the power automorphism case we will use the fact that Q_1 normalises every subgroup of prime order and hence acts irreducibly on every such subgroup.

If Y_2 is elementary abelian, then we choose normal subgroups N_1 of Y_1 and N_2 of Y_2 such that Q_1 acts non-trivially and irreducibly on both of them, and we consider the group $(N_1 \times N_2)Q_1/C_{Q_1}(Y_1)$.
Then we let $M_1 := N_1C_{Q_1}(Y_1)/C_{Q_1}(Y_1)$, $M_2 := N_2C_{Q_1}(Y_1)/C_{Q_1}(Y_1)$ and $H := Q_1/C_{Q_1}(Y_1)$.
If $Y_2 \cong Q_8$, then we choose N_1 as before and we consider $(N_1 \times Y_2)Q_1/Z(Y_2)C_{Q_1}(Y_1)$. Then we let $M_1 := N_1Z(Y_2)C_{Q_1}(Y_1)/Z(Y_2)C_{Q_1}(Y_1)$, $M_2 := Y_2C_{Q_1}(Y_1)/Z(Y_2)C_{Q_1}(Y_1)$ and $H := Q_1Z(Y_2)/Z(Y_2)C_{Q_1}(Y_1)$.

In both cases we have that $|H| = q$, that M_1 is a p_1-group and M_2 is a p_2-group and that H acts non-trivially and irreducibly on M_1 and on M_2.
Since $(M_1 \times M_2)H$ is a section of G, it is M_9-free by hypothesis. Now we will find a contradiction by constructing a lattice isomorphic to M_9 in the following way:
We set $A := M_1H$, $C := M_2H$ and $D := H = A \cap C$. Since $p_1 \geq 3$ and H is not normal in A, we find subgroups S and T of A that are conjugate to H, but distinct from H and from each other. All these groups have order q and hence intersect pairwise trivially. It follows from the irreducible action of H on M_1 that any two distinct members of $\{S, T, H\}$ generate A.
Now we argue in C, with the aim to find subgroups U, V of C that are conjugate to H and distinct from it and from each other. This is clear if $p_2 \geq 3$. Otherwise $p_2 = 2$ and the non-trivial action of H on M_2 forces $|M_2| > 2$, so M_2 is elementary abelian of order at least 4 and Q_1 does not centralise any element in $M_2^{\#}$. Again we find the desired conjugates of H.
Now $\langle H, U\rangle = \langle U, V\rangle = \langle H, V\rangle = C$.
We also know that U acts irreducibly on M_1 and on M_2. Therefore M_1U and $M_2U = C$ are the unique maximal subgroups of $(M_1 \times M_2)H$ that contain U. But $M_1U \cap A = M_1$ and $M_2U \cap A = C \cap A = H$ and this means that neither S nor T is contained in one of these maximal subgroups.
In a similar way we deduce that $\langle S, V\rangle = \langle T, V\rangle = (M_1 \times M_2)H$ and this means that the lattice generated by $\{1, S, T, U, V, D, M_1H, M_2H, (M_1 \times M_2)H\}$ is isomorphic to M_9.
This is a contradiction. ■

(8) There is a unique $i \in \{1, ..., k\}$ such that Y_iQ is not L_8-free.
Proof. Assume otherwise. Then there is no such index i by (5) and in particular Y_1Q is L_8-free. We know from Theorem 2.25 that Q is cyclic or isomorphic to Q_8. If Q is cyclic, then Y_1Q is as in Lemma 2.23 (iv) (a) or as in Lemma 2.24 (i). Hence it satisfies Definition 2.27. The same holds for $Y_2Q, ..., Y_kQ$ (if they exist). But then (7) implies that G is a Λ^*-group, hence L_8-free, and then G satisfies the conclusion of the theorem. This is false.
This means that $Q \cong Q_8$. It follows from Lemma 2.23, for all $j \in \{1, ..., k\}$, that Q acts faithfully on Y_j. Then (7) forces $k = 1$, so $G = Y_1Q$ and this is again a contradiction. ■

Using (8) we suppose that Y_1Q is not L_8-free. Then it follows from (6) that $Q \cong Q_8$, and Theorem 3.15 implies that Y_1 is a normal 3-subgroup of G. As G is a counter-example, we have that $k \geq 2$. Using Theorem 2.25 we deduce that Q acts faithfully on Y_1, and (7) implies that $k = 2$. Consequently $G = Q(Y_1 \times Y_2)$, and this is the structure of a Q_8^*-group. However, this contradicts our choice of G as a counter-example. □

Theorem 4.8. *The group G is M_9-free if and only there are $n \in \mathbb{N}$ and subgroups $G_1, ..., G_n$ of G of pairwise coprime orders such that the following holds:*

(i) $G = G_1 \times ... \times G_n$;
(ii) *for all $k \in \{1, ..., n\}$ the group G_k is L_8-free or a Q_8^*-group or a $\{2, 3\}^*$-group.*

Proof. This follows from Theorems 4.5 and 4.7. □

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