The uniqueness case

G. Stroth
Martin-Luther-Universität Halle-Wittenberg
Institut für Algebra und Geometrie
06099 Halle
Germany

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This paper is part of the revision of the classification of the finite simple groups and part of the Gorenstein–Lyons–Solomon program. The aim is to give a solution of the so called "uniqueness case problem". Originally this problem was solved by M. Aschbacher [Asch2]. We do not follow his proof. The proof given in this paper uses the amalgam method, which has been used successfully in many places for dealing with weak closure. Also our theorem looks different from the one in [Asch2] as definitions have changed over the time. In fact our hypothesis is weaker as in [Asch2]. First of all we need some precise definitions.

Let X be a finite group. We first define

$$e(X) = \text{maximal } p\text{–rank, } p \text{ odd, of a 2–local subgroup of } X.$$

$$\sigma(X) = \{p | p \text{ odd, } m_p(H) \geq \min\{e(X), 4\} \text{ for some 2-local } H \text{ of } X\}$$

Furthermore let Q be a p-subgroup of $X, k \leq m(Q)$, let

$$\Gamma_{Q,k}(X) = \langle N_X(R) \,|\, R \leq Q, \, m(R) \geq k \rangle \, .$$

In this paper we will consider groups G which satisfy the following conditions

(1) (i) $e(G) \geq 3$
  (ii) If H is a 2-local then $F^*(H) = O_2(H)E(H)$, where $E(H) = 1$ or components of $E(H)$ are in $\mathcal{C}_2$.

(2) A group M is called a uniqueness group provided $\sigma(M) \neq \emptyset$, $|G : M|$ is odd and one of the following holds

(α) M is a maximal 2-local of G with $F^*(M) = O_2(M)$.

(β) $F^*(M) = O_2(M)K$, where K is a quasisimple group of Lie type in characteristic 2, not $L_2(q)$, $U_3(q)$, $Sz(q)$, $L_3(q)$, $Sp_4(q)$ or ${}^2F_4(q)$, $Z(K) = O_2(K)$, and for every $p \in \sigma(M)$ we have $m_p(K) \geq 2$ and $m_p(C_M(K)) \leq 1$.

(3) Let M be a uniqueness subgroup of G and let $p \in \sigma(M)$, and $P \in \mathrm{Syl}_p(M)$ then one of the following holds

(i) If $x \in P, o(x) = p, m_p(C_M(x)) \geq 3,$ then $N_G(\langle x \rangle) \leq M$. (In particular for $p = 3$, and $e(G) \geq 4$ we have $N_G(\langle x \rangle) \leq M$ for any $x \in M, o(x) = 3$). Further for every subgroup Q of P of rank at least two we have that $N_G(Q) \leq M$ or $p = 3$, $P \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and Q is elementary abelian of order 9.

(ii) $F^*(M) = O_2(M)$. Set $M/O_2(M) = \bar{M}$. Then there is $\bar{Q} \trianglelefteq \bar{M}$ where $\bar{Q} = O_p(\bar{M})$ is elementary abelian of order p^n. We have $C_{\bar{M}}(\bar{Q}) = \bar{Q} \times \bar{X}$. Further $\bar{P} = (\bar{P} \cap \bar{X}) \times \bar{Q}$ and $m_p(\bar{X}) = 1$. $\bar{M}$ induces on $\bar{Q}$ a Borel subgroup of an automorphism group of $L_2(p^n)$, containing the Borel subgroup of $L_2(p^n)$. Let Q be a preimage of $\bar{Q}$ in P. Then $\Gamma_{Q,1}(G) \leq M$. Further if $\omega \in P$ is a nontrivial element with $\bar{\omega} \in \bar{X}$, then $C_{O_2(M)}(\omega) = 1$.

We say G is in the uniqueness case if G is $\mathcal{K}$-simple and satisfies (1) and (3) and the following holds.

(4) (i) For every $p \in \sigma(G)$ there is a uniqueness subgroup M_p with $p \in \sigma(M_p)$.

(ii) Let M be a uniqueness subgroup of G with $p \in \sigma(M)$. If H is any 2-local subgroup of G such that $H \cap M \geq E, E \cong E_{p^2}, p \in \sigma(M), \Gamma_{E,1}(G) \leq M$, then $H \leq M$.

If M_p is as in (3) (ii) we call M_p exceptional and p an exceptional prime. Recall that even if M_p is exceptional it might be non exceptional for some other prime. So to avoid duplicating arguments we will call a uniqueness group M exceptional if there is some prime p such that M is exceptional with respect to this prime.

Now we can state our theorem.

Theorem. *Let G be in the uniqueness case. Let M be a uniqueness subgroup. If $S \in Syl_2(M)$, then M contains every 2-local subgroup of G containing S.*

It remains to explain what $\mathcal{K}$-simple means. A group is called $\mathcal{K}$-simple if it is a minimal counterexample to the classification theorem, i.e. all simple nonabelian sections of all proper subgroups are in $\mathcal{K}$. Here $\mathcal{K}$ is the set of the groups of Lie type, the alternating groups and the 26 sporadic groups. Further the set $\mathcal{C}_2$ is descibed in [GoLyS1, 12.1]. It consists of the groups of Lie type in characteristic two, A_6, $L_2(p)$, p a Fermat or Mersenne prime, $L_3(3)$, $L_4(3)$, $U_4(3)$, $G_2(3)$, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HS, Suz, Ru, Co_1, Co_2, $M(22)$, $M(23)$, $M(24)'$, F_3, F_2 and F_1.

For the remainder of this paper we fix the following notation. Let X be a group and p be an odd prime. Assume $m_p(X) \geq 3$. We call an elementary abelian p–subgroup E good, if $m_p(C_X(x)) \geq 3$ for all $x \in E^\sharp$. In that case we also call $x \in E$ good. Hence if M is a uniqueness subgroup which is not exceptional with respect to p and E a good subgroup of M then M is the unique maximal 2–local containing E.

1 Some Simple Groups

In this chapter we collect some properties of quasisimple groups, mainly groups of Lie type in characteristic two, which will become important in the sequel of the proof of the main theorem.

*Z*3

Lemma 1.1 *Let X be a quasisimple group with $X/Z(X) \in \mathcal{K}$, $Z(X)$ a 3-group, then one of the following holds*

(i) $m_3(X) = 0$ *and* $X \cong Sz(q)$ *,* q *even.*

(ii) $m_3(X) = 1$ *and* $X \cong L_2(q), L_3(q), U_3(q), J_1$*.*

(iii) $m_3(X) = 2$ *and* $X \cong 3 \cdot A_6$, $3 \cdot A_7$, $3 \cdot M_{22}$, $SL_3(q)$, $SU_3(q)$, A_7, A_6, $L_3(3)$, $U_3(3)$, $L_3(q)$, $U_3(q)$, $PSp_4(q)$, $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, $L_4(q)$, $U_4(q)$, $L_5(q)$, $U_5(q)$, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, HiS, He, Ru, J_4*.*

(iv) $m_3(X) = 3$ *and* $X \cong 3 \cdot O'N$, A_9, A_{10}, A_{11}, $L_2(27)$, $PSp_4(3)$, $Sp_6(q)$, $\Omega_8^-(q)$, $L_4(q)$, $U_4(q)$, $L_6(q)$, $U_6(q)$, $L_7(q)$, $U_7(q)$, J_3*.*

Proof: Just inspection of the groups in $\mathcal{K}$. [] □

parsim

Lemma 1.2 *Let K be a finite simple group in the list $\mathcal{K}$, $m_p(K) \leq 3$ for any odd prime p. Then K is one of the following:*

(i) $L_2(q), Sz(q), L_3(q), U_3(q), PSp_4(q), q$ *some prime power*

(ii) $G_2(2^n)$, ${}^2F_4(2^n)'$, ${}^3D_4(2^n)$, $L_4(2^n)$, $L_5(2)$, $L_6(2)$, $L_7(2)$, $U_4(2^n)$, $Sp_6(2^n)$, $\Omega_8^-(2^n)$

(iii) A_n, $6 \leq n \leq 11$

(iv) $J_i, 1 \leq i \leq 4, M_n, n \in \{11, 12, 22, 23, 24\}, HiS, Ru, He.$

If $K \in \mathcal{K}, U/Z(U) \cong K, U' = U, 1 \neq |\,Z(U)\,|$ odd and $m_p(U) \leq 3$ for every odd prime p. Then U is isomorphic to $3 \cdot A_6, 3 \cdot A_7, 3 \cdot M_{22}, SL_3(q), SU_3(q), q$ a prime power, or $3 \cdot O'N$.

Proof: This is easily established by going over the list in 1.1. □

Cartan

Lemma 1.3 *Let $X \cong G(q)$ be a Lie group over a field of characteristic two, $q > 2$. Let C be the Cartan subgroup and $m_p(C) \leq 3$ for any prime p. Then X is one of the following: $L_n(q), n \leq 4, Sp_4(q), Sp_6(q), U_n(q), n \leq 7, \Omega_8^-(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, $G_2(q)$, or $Sz(q)$.*

Proof: Let first X be untwisted of Lie rank r. Then

$$|\,C\,| = \frac{1}{d}(q-1)^r$$

Let $r \geq 4$. Then as $m_p(C) \leq 3$, we have $q-1 = p = d$. Furthermore $r = 4$. But checking the possible values for d gives a contradiction. So we have $r \leq 3$ and then $X \cong L_2(q), L_3(q), L_4(q), Sp_4(q), Sp_6(q)$ or $G_2(q)$.

Assume now that X is twisted. Let $X \cong {}^2E_6(q)$. Then

$$|\,C\,| = \frac{1}{d}(q-1)^2(q^{\,2}-1)^{\,2}\,,\; d = \gcd(3, q+1)\,.$$

Let $p \,\Big|\, q-1$. Then C contains an elementary abelian subgroup of order $p^{\,4}$, a contradiction.

Let $X \cong U_n(q)$. If n is even then

$$|\,C\,| = \frac{1}{d}(q-1)(q^{\,2}-1)^{\,\frac{n}{2}-1}.$$

Thus $n \leq 6$.

Let n be odd. Then

$$|\,C\,| = \frac{1}{d}(q^{\,2}-1)^{\,\frac{n-1}{2}}.$$

This implies $n \leq 7$.

Finally assume that $X \cong \Omega^-_{2n}(q)$. Then

$$|\,C\,| = (q^{\,2}-1)(q-1)^{n-2}.$$

Hence $n-2 \leq 2$. We get $n = 4$, as $\Omega^-_6(q) \cong U_4(q)$. □

O2Chev

Lemma 1.4 *Let $G = G(q)$ be a group of Lie type over $GF(q)$, $q = 2^n$, $G \not\cong L_2(q)$, $L_3(q)$, $U_3(q)$, $Sz(q)$, $G_2(q)$ or ${}^2F_4(q)$. Let R be a long root group, $Q = O_2(N_G(R)/R)$ and L be a Levi complement in $N_G(R)$. Then Q has the following L–module structure*

(i) $G \cong L_n(q)$, $O^{2'}(L) \cong SL_{n-2}(q)$, $Q = V_1 \oplus V_2$, V_1 is the natural L–module and V_2 its dual.

(ii) $G \cong Sp_{2n}(q)$, $O^{2'}(L) \cong Sp_{2n-4}(q) \times L_2(q) = L_1 \times L_2$, $Q = V_1 \oplus V_2$, $[V_2, L_1] = 1$, V_1 is the natural L_2–module, $V_1 = V_1^{(1)} \oplus V_1^{(2)}$, $V_1^{(i)}$, $i = 1, 2$, are natural L_1–modules and $[L_2, V_1] = V_1$.

(iii) $G \cong \Omega^{\pm}_{2n}(q)$, $O^{2'}(L) \cong \Omega^{\pm}_{2n-4}(q) \times L_2(q) = L_1 \times L_2$, $Q = V_1 \oplus V_2$, V_i, $i = 1, 2$, are natural L_1–modules and $[Q, L_2] = Q$.

(iv) $G \cong U_n(q)$, $O^{2'}(L) \cong SU_{n-2}(q)$, *$Q$ is the natural module.*

(v) $G \cong E_6(q)$, $O^{2'}(L) \cong L_6(q)$, $Q \cong V(\lambda_3)$.

(vi) $G \cong {}^2E_6(q)$, $O^{2'}(L) \cong U_6(q)$, $Q \cong V(\lambda_3)$.

(vii) $G \cong E_7(q)$, $O^{2'}(L) \cong \Omega^+_{12}(q)$, $Q \cong V(\lambda_6)$.

(viii) $G \cong E_8(q)$, $O^{2'}(L) \cong E_7(q)$, $Q \cong V(\lambda_1)$.

(ix) $G \cong F_4(q)$, $O^{2'}(L) \cong Sp_6(q)$, *Q is an extension of the natural module by a spin module, where the natural module is contained in* $Z(O_2(N_G(R)))$.

(x) $G \cong {}^3D_4(q)$, $O^{2'}(L) \cong L_2(q^3)$, *Q is the 8–dimensional $GF(q)$–module for L.*

Proof: This can easily be checked using the Chevalley commutator formula (see also [AschSe]). □

2rang

Lemma 1.5 *Let $X \cong^3 D_4(r)$, ${}^2F_4(r)$, $G_2(r)$,or $\Omega^-(8,r)$, r even, then a maximal elementary abelian 2-subgroup A of X has order r^5, r^5, r^3, r^6, respectively.*

Proof: This is [GoLyS3, (3.3.3)] □

FO2

Lemma 1.6 *Let $G = G(q) \not\cong {}^2F_4(q)$, $q = 2^m$, be a group of Lie type and R be a long root group. Set $Q = O_2(C_G(R))$. Let $A \leq Q$ be elementary abelian with $[A,Q] \neq 1$. Then there is $U \leq Q$, $|U| = q$, with $|A : C_A(U)| \leq q$ and $C_A(U) = C_A(u)$ for all $u \in U^\sharp$.*

Proof: We have $R = Q'$ is of order q. Hence $|[A,Q]| \leq q$. Furthermore Q is generated by subgroups R_i, $|R_i| = q$, $R_i \cap Q' = 1$ and R_i is a TI–set in Q. Let $x \in A$ with $[x,r] = 1$ for some $r \in R_i$, $r \neq 1$. Then $R_i^x = R_i$. As $[R_i,x] \leq Q'$ and $Q' \cap R_i = 1$, we get $[R_i,x] = 1$. This now implies that $R_1 \cap C(A) = 1$ for some R_1. Further $C_A(R_1) = C_A(r)$ for all $r \in R_1^\sharp$ and so $|A : C_A(R_1)| = |A : C_A(r)| \leq q$. □

FO2F4

Lemma 1.7 *Let $G = F_4(q)$, $q = 2^m$, R be a root group, A be an elementary abelian subgroup of $N_G(R)$. Let $Z = Z(O_2(N_G(R)))$ and $S \in Syl_2(N_G(R))$. If $A \not\leq O_2(N_G(R))$ but $A \leq O_2(C_{N_G(R)}(Z(S)))$, then $|Z : C_Z(A)| \geq |A : A \cap O_2(N_G(R))|$.*

Proof: We have that Z/R is the natural $Sp_6(q)$–module by 1.4(ix). Furthermore by assumption about A we have that $AO_2(N_G(R))/O_2(N_G(R))$ is contained in the greatest normal 2-subgroup of the point stabilizer of $Sp_6(q)$ in the natural representation. This gives that there is a subgroup Z_1 of Z, $|Z : Z_1| = q$, with $Z_1 \geq [Z,A]$, $C_Z(A) \leq Z_1$ and $[Z_1,A] \leq Z(S)$. Now obviously $|Z_1 : C_{Z_1}(A)| \geq |A : A \cap O_2(N_G(R))|$. □

Liesmall

Lemma 1.8 *Let G be a group of Lie type over $GF(r)$, $r = 2^n$, $G \not\cong {}^2F_4(r)$, $L_2(r)$, $L_3(r)$, $Sp(4,r)$, $U_3(r)$ or $Sz(r)$. Let R be a long root group, $Q = O_2(N_G(R)/R)$. If there is an involution t in $C(R)\backslash O_2(N_G(R))$, with t central in a Sylow 2–subgroup of $C(R)/O_2(C(R))$, such that $|[Q,t]| \leq r^2$, then $G \cong L_n(r)$, $U_n(r)$, or $Sp(2n,r)$ and $|[Q,t]| = r^2$, or $G \cong G_2(r)$.*

Proof: We have that Q is a $GF(r)$–module (see 1.4) and so $|[Q,t]| = r$ or r^2. If $G \cong L_n(r)$, then we have two modules in Q and so we have the assertion. If $K \cong U_n(r)$, then Q is defined over r^2, again the assertion. Let $K \cong Sp(2n,r)$. As $O_2(N_G(R))/Z(O_2(N_G(R))$, is a direct sum of two modules, we also get the assertion in that case.

Let $G \cong \Omega^{\pm}_{2n}(r)$, then by 1.4 Q is a sum of two modules. On neither of them t can induce a transvection, so t has to move them. In particular, we get $|Q| = r^4$ and then we have $\Omega^{\pm}(6,r)$, which is the case $L_4(r)$ or $U_4(r)$ above.

If $G \cong F_4(r)$, then again by 1.4 there are two modules in Q. So t has to induce a transvection on the natural module, but then $|[V,t]| = r^4$ for the spin module V, a contradiction.

If $G \cong E_6(r)$, $E_7(r)$, $E_8(r)$, or ${}^2E_6(r)$, then by 1.4 we get $L_6(r)$ on the exterior cube, $\Omega^+(12,r)$ on the spin module, $E_7(r)$ on the 56–dimentional module, or $U_6(r)$ on $V(\lambda_3)$. Now t is in some root group of $N_G(R)/O_2(N_G(R))$ Hence $C_{N_G(R)/O_2(N_G(R))}(t)$ involves $L_4(r)$, $\Omega^+(8,r)$, $E_6(r)$, $U_4(r)$, respectively, which acts nontrivially on $[Q,t]$. Hence we see that $|[Q,t]| \geq r^6$, r^8, r^{27} or r^6, respectively.

So we are left with $G \cong {}^3D_4(r)$. But in this case Q is a tensor product of three algebraically conjugate natural modules and so $|[Q,t]| \geq r^4$. □

3cent

Lemma 1.9 *Let $G = G(2)$ be a group of Lie type over $GF(2)$. Assume $m_3(G) \geq 4$. Let x be a long root element, $t \in O_2(C_G(x))$ be an involution, $S \in Syl_2(C_G(x))$ and $[t,S] \leq \langle x \rangle$. Then $3 \big| |C_G(t)|$.*

Proof: Set $H = C_G(x)$. If $G \not\cong L_n(2)$, then $O_2(H)/Z(O_2(H))$ is an irreducible module for $H/O_2(H)$ and so $tZ(O_2(H))$ is centralized by some parabolic P in $H/O_2(H)$ by 1.4. Hence $3 \big| |P|$ or $H/O_2(H)$ is solvable. The latter just occurs for $G \cong \Omega^+_8(2)$. But in this group any involution is centralized by a 3–element.

Assume now $G \cong L_n(2)$. Then by 1.4 $O_2(H)/Z(O_2(H)) = H_1 \oplus H_2$, where H_1 is the natural $L_{n-2}(2)$–module and H_2 its dual. So $tZ(O_2(H))$ is centralized by a 3–element as $n \geq 8$, recall that $m_3(G) \geq 4$ and so $n - 2 \geq 6$, which gives that $Z_2(S)$ is centralized by some $L_4(2)$.

Hence we may assume that $tZ(O_2(H))$ is centralized by a 3–element in H. If $\langle x\rangle = Z(O_2(H))$ the same applies to t. So assume $\langle x\rangle < Z(O_2(H))$. This means $G \cong Sp_{2n}(2)$ or $F_4(2)$.

Let $G \cong Sp_{2n}(2)$ and set $\tilde{H} = C_H(Z(O_2(H)))$. Then $O_2(H)/Z(O_2(H)) \cong H_1 \oplus H_2$, where H_i, $i = 1, 2$, are natural modules for $\tilde{H}/O_2(H) \cong Sp_{2n-4}(2)$ by 1.4. Now as $2n - 4 \geq 4$, we have that $C_{O_2(H)/Z(O_2(H))}(S)$ is centralized by $Sp_2(2) \cong \Sigma_3$. So $tZ(O_2(H))$ is centralized by a 3–element in $\tilde{H}$ and then also t is centralized by a 3–element.

So we are left with $G \cong F_4(2)$. Let $t \notin Z(O_2(H))$. By 1.4 $O_2(H)/Z(O_2(H))$ is the spin module for $H/O_2(H)$ and so $tZ(O_2(H))$ is centralized by a subgroup $U \cong 2^6L_3(2)$ in $H/O_2(H)$. We have that U acts on $Z(O_2(H))\langle t\rangle$. As $[t, S] \leq \langle x\rangle$, we get that $[O_2(U), t] \leq \langle x\rangle$. As $Z(O_2(H))/\langle x\rangle$ is the natural module for $Sp_6(2)$, we get $C_{Z(O_2(H))/\langle x\rangle}(O_2(U))$ is the natural U–module. So $Z(O_2(H))\langle t\rangle \cap C(O_2(U))$ is an extension of the natural module by a trivial module. This shows $|t^U\langle x\rangle| = 1$ or 7. In both cases t is centralized by a 3–element.

So let $t \in Z(O_2(H))$. Then t is centralized by $Sp_4(q)$ and we are done again. □

*borel*1

Lemma 1.10 *Let $X = G(q)$ be a Lie group, q even, $q > 2$. Let $r = 2^n$ and x be a primitive prime divisor of $r - 1$, or $x = 9$ in case of $r = 64$. Suppose $r > q$. Let $\omega \in Aut(X), o(\omega) = x$, ω normalizes a Borel subgroup B of X. Then one of the following holds*

(i) $r = q^2$ *and* $X \cong U_n(q), \Omega^-_{2n}(q)$ *or* ${}^2E_6(q)$.

(ii) $r = q^3$ *or* $r^2 = q^3$ *and* $X \cong {}^3D_4(q)$.

(iii) $x = 3$ *or* 9 *and* $X \cong {}^3D_4(q)$, $q \leq 32$, *or* $D_4(q)$ *and* $q \leq 16$.

Proof: Suppose first that ω induces a graph automorphism on $G(q)$. Then $X \cong {}^3D_4(q)$ or $D_4(q)$ and $x = 3$ or 9. If $x = 3$, we get $r = 4$ and so by assumption $q = 2$, a contradiction. Let $x = 9$, then $r = 64$. So $q \leq 32$. If $q = 32$ and $X \cong D_4(q)$, then ω^3 induces an inner automorphism, which normalizes a Borel subgroup. But the odd part of the normalizer of a Borel subgroup in X is 31^4, a contradiction. Hence (iii) holds.

So assume now that ω induces a field automorphism. Then $q = t^x$ or $x = 9$ and $q = t^3$. Suppose the former. As $x \,\Big|\, t^{x-1} - 1 < q - 1$, we see that $r \leq q$, which contradicts the choice of x. So we have $q = t^3$ and $x = 9$. Hence $r = 64$ and so $q = 8$. This shows $r = q^2$. Now ω^3 induces an inner $\times$ diagonal automorphism. As $3 \not\big| q-1$, we see that X is a 2-fold twisted group and so we have (i).

So assume finally that ω induces an inner $\times$ diagonal automorphism. Let $x = p$, prime. Then $x \not\big| q-1$. This now shows $x \,\Big|\, q^2-1$ or $X \cong {}^3D_4(q)$ and $x \,\Big|\, q^3-1$.

In the former by the choice of x we have $r = q^2$ and this is (i). In the latter we get $r = q^3$ or $r^2 = q^3$ and so we have (ii).

So we are left with $x = 9, r = 64, q \leq 32$. As the torus of B is an abelian group, we see that $9 \,\Big|\, q-1, 9 \,\Big|\, q^2-1$ or $9 \,\Big|\, q^3-1$. This gives $q = 8, q = 4$ and $X \cong {}^3D_4(q)$, or $q = 16$ and $X \cong {}^3D_4(q)$. Hence we have (i) or (ii). □

$3syl$

Lemma 1.11 *Let $X \cong A_n$. Suppose $P \leq X, P$ contains a Sylow 2–subgroup of X and P is a $\{2,p\}$-group. Then $p = 3$.*

Proof: This can be found in [Asch1, (6.1)]. □

$normal3$

Lemma 1.12 *Let G be alternating of degree at least 5 or sporadic, S a Sylow 2–subgroup of G and ω some element of order p, p odd, in G which normalizes S. If $[\Omega_1(Z(S)), \omega] \neq 1$, then $G \cong A_5$ or J_1.*

Proof: We have that $Z(S)$ is not cyclic. Inspection of the sporadic groups in [CCNPW] shows that the only sporadic group will be J_1. So assume now that $G \cong A_n$. If n is a 2-power then $Z(S)$ is cyclic besides $n = 4$. The same applies for $n = 2^m + 1$, if $m > 2$, as A_{2^m} and A_{2^m+1}, so we would get A_5. Let now $n > 5$ and $m_1 + m_2 + \cdots + m_r$ be the 2-adic decomposition of n then $S\langle\omega\rangle \leq U$, where U is a subgroup of index two of $\Sigma_{m_1} \times \Sigma_{m_2} \times \cdots \times \Sigma_{m_r}$, which induces the full symmetric group on each A_{m_i}. This is as ω has to respect the different orbit lengths of S on $\{1, \cdots, n\}$. But each m_i is a power of two and so we get that some m_i equals 4 and $S \cap A_{m_i} \leq Z(S)$. But on A_{m_i} we have that S induces Σ_{m_i} and so acts nontrivially on a Sylow 2–subgroup of A_{m_i}. □

$field$

Lemma 1.13 *Let K be a Lie group in odd characteristic, $K \not\cong {}^2G_2(3^n)$. Let $\omega \in Aut(K), o(\omega) = p > 3, p$ prime, S a Sylow 2–subgroup of K, $[\omega, S] \leq S$. Then either ω induces a field automorphism on K or ω is a diagonal automorphism with $[S, \omega] = 1$.*

Proof: This is [Asch, (6.3)] . □

L2q1

Lemma 1.14 *Let $X \cong L_2(q)$ or $Sz(q)$, $q \geq 4, q$ even.*

(i) Let $t \in \Omega_1(S), S \in Syl_2(X)$. Then there are conjugates a, b of t such that $X = \langle a, b, t\rangle$.

(ii) Let $A \leq \Omega_1(S)$, $|A| \geq 4$. Then there is some $g \in X$ with $X = \langle A, A^g\rangle$.

Proof: Let $\langle t, a\rangle \leq A \leq \Omega_1(S)$, $|A| \geq 4$. Let $K \leq N_X(S)$, $|K| = q - 1$, and $a, b \in N_X(K^g)$, with $N_X(K^g) = \langle a, b\rangle$, $g \in X$. Then we get $\langle a, b, t\rangle \geq \langle \Omega_1(S), \Omega_1(S)^b\rangle$. Thus to prove (i) and (ii) it is enough to show $\langle \Omega_1(S) , \Omega_1(S)^b\rangle = X$.

We have that $Y = \langle \Omega_1(S), \Omega_1(S)^g\rangle$ contains at least $q + 1$ conjugates of $\Omega_1(S)$. Thus we are done if $X \cong L_2(q)$, as $\langle \Omega_1(S), \Omega_1(S)^b\rangle$ contains all conjugates.

So let $X \cong Sz(q)$. The number of conjugates of $\Omega_1(S)$ in Y is $nq + 1$. But then $nq + 1 \Big| q(q^2 + 1)$. Which gives $n = q$ and so $\Omega_1(S)^X \leq Y$, hence $X = Y$. □

Sp2nout

Lemma 1.15 *Let p be a Zsygmondi prime dividing $q - 1$, $q = 2^m$, or $p = 7$ for $q = 64$. Let $K \cong Sp(2n, r)$, $U_4(r)$, $U_3(r)$, $F_4(r)$, $G_2(r)$, $Sz(r)$ or $\Omega^{\pm}(2n, r)$, $r = q$ or q^2. Let ω be an automorphism of K of order p. Then ω is inner or $p = 3$ and $K \cong \Omega^+(8, r)$.*

Proof: Suppose that ω induces an outer automorphism. Then we have $K \cong \Omega^+(8, r)$ and $p = 3$. Suppose that there are diagonal automorphisms of order p. Then $K \cong U_3(r)$ and $p = 3$. Hence $q = 4$. But neither $q + 1$ nor $q^2 + 1$ is divisible by 3, a contradiction. So we have that ω induces a field automorphism. In particular $r = 2^t$ with $t = pu$. But we have always that p divides $2^{p-1} - 1$, which now gives that $m \leq p - 1$ if p is a Zsigmondy prime. Now $pu = t \leq 2(p - 1)$, so $u = 1$ and $m = p - 1$, a contradiction again. So we are left with $p = 7$. Now $t = 7u$ and $m = 6$. But then we cannot have $r = q$ or $r = q^2$. □

schur

Lemma 1.16 *Let $N/Z(N) \cong L_2(q)$, $L_3(q)$, $U_3(q)$, $Sz(q)$, $q = 2^n$ or $L_2(p)$, pprime. Assume further $m_3(N) \leq 1$. Let $N' = N$. If $Z(N)$ is a nontrivial 2–group then $Z(S) = Z(N)$, for S a Sylow 2–subgroup of N.*

Proof: We have that $Z(N)$ is in the Schur multiplier of N. Hence with [GoLy, 6.1] we have that $N/Z(N) \cong L_2(p)$ or $Sz(8)$. In case of $L_2(p)$ we have that S is a quaternion group and so $Z(S) = Z(N)$.

So we treat $N/Z(N) \cong Sz(8)$. Assume that $Z(N)$ is the Schur multiplier, i.e. $Z(N)$ is elementary abelian of order 4. We have $|S/\Omega_1(S)| = 8$. Further there is $\nu \in N_N(S)$ with $o(\nu) = 7$ and ν acts transitively on the nontrivial elements of $S/\Omega_1(S)$. Suppose that $Z(S) > Z(N)$, then $Z(S) = \Omega_1(S)$. Hence there are exactly 7 elements in $\Omega_1(S)$, which are squares. Let V be the permutation module for ν. Then V is a direct sum of two 3–dimensional modules by a 1-dimensional one. Hence we have that the subgroup U of $\Omega_1(S)$ generated by the squares is not equal to $\Omega_1(S)$, as ν centralizes a group of order 4 in $\Omega_1(S)$. Hence $|\Phi(S)| \leq 16$, contradicting $Z(N) \leq \Phi(S)$ and $\Phi(S)$ covers $\Omega_1(S/Z(N))$. So we have shown $Z(S) = Z(N)$. □

$O(6,2)$

Lemma 1.17 *(i) Let* $K \cong \Omega^-(6,q)$, $Sp(6,q)$ *or* $\Omega^-(8,q)$, *q even, and p be a prime which divides* $q+1$ *in the first case and* q^2-1 *in the other two cases. Then any p–element in K is centralized by an elementary abelian group of order* p^3.

(ii) Let $K \cong A_9$, $L_6(2)$ *or* $L_4(4)$. *Then any 3–element in K is centralized by an elementary abelian group of order 27.*

Proof: (i) If $p \neq 3$, then Sylow p–subgroups of K are abelian and of rank at least three. So we just have to deal with $p = 3$. Now any element of order three is conjugate into the corresponding group over $GF(2)$. As $\Omega^-(6,2) \leq Sp(6,2) \leq \Omega^-(8,2)$ and they all have a common Sylow 3–subgroup, we just have to prove the assertion for $\Omega^-(6,2)$. But by Witt any element of order three is conjugate in $\Omega^-(2,2) \times \Omega^-(2,2) \times \Omega^-(2,2)$, which is elementary of order 27.

(ii) For A_9 this is just inspection. As $\Omega^-(6,2) \leq L_6(2)$ nad both have a common Sylow 3–subgroup, the assertion follows with (i). So let $K \cong L_4(4)$. Now $\Omega^-(6,2) \cong U_4(2) \leq L_4(4)$ and they have common Sylow 3–subgroup, again (ii) follows from (i). □

2 Small Groups

dihed

Lemma 2.1 *Let R be a p–group , p odd, and E be an elementary abelian 2-group, acting faithfully on R. Then there is a subgroup U in RE, such that U is a direct product of dihedral groups of order $2p$ and E is a Sylow 2-subgroup of U.*

Proof: [GoLyS2, (24.1)] □

cl2p

Lemma 2.2 *Let X be a p-group, p odd, $X' \leq Z(X)$, $X = \Omega_1(X)$ and $m_p(X) \leq 3$. Then X is elementary abelian, extraspecial of width 1, a direct product of a cyclic group of order p with an extraspecial group of width 1, or an extraspecial group of width 2.*

Proof: We may assume $X' \neq 1$. We have $X = \{x \big| x^p = 1\}$. Let $|Z(X)| = p$. Then we have that $X' = Z(X) = \Phi(X)$ and so X is extraspecial. As $m_p(X) \leq 3$, we see that $|X| \leq p^5$.

So assume $|\,Z(X)| = p^2$. Then $m_p(X) = 3$. Choose $\omega \in X \setminus Z(X)$. Then $|\,X : C_X(\omega)| \leq p^2$ and as $C_X(\omega) = \langle \omega, Z(X)\rangle$, we get $|X| \leq p^5$.

Let $|X| = p^4$. Then $|X : Z(X)| = p^2$ and so $|X'| = p$. Thus X is a direct product of a cyclic group of order p with an extraspecial group of width 1.

Let $|X| = p^5$. Choose $C_X(\omega) \leq Y < X, |Y| = p^4$. Then as just seen Y is a direct product of a cyclic group of order p by an extraspecial group of width 1. Now choose $\varphi \in X \setminus Y$. Then $[\omega, \varphi] = t \notin Y'$. Let $\nu \in Y, 1 \neq [\omega, \nu] = s \in Y'$. We have

$$[\varphi, \nu] = s^i\, t^j\,, \text{ for some } 0 \leq i, j \leq p-1\,.$$

But then

$$[\varphi, \nu\omega^{\,j}] = s^{\,i}$$

and so we may assume

$$[\varphi, \nu] = s^{\,i}\,.$$

Hence $[X, \nu] \leq \langle s\rangle$ and then $|X : C_X(\nu)| \leq p$, a contradiction. □

3gr

Lemma 2.3 *a) Let R be a 3–group of rank at most three. Then Sylow p–subgroups for $p > 3$ of $Aut(R)$ are cyclic.*

b) If R is a 5–group of rank at most 2, then Sylow p–subgroups of $Aut(R)$ for odd $p \neq 5$ are cyclic. If the rank is three the same applies for $p > 5$, while for $p = 3$ Sylow 3–subgroups have rank at most two.

Proof: a) Let C be a critical subgroup of R and $D = \Omega_1(C)$. Then we have that either D is elementary abelian or extraspecial. Let P be a Sylow p–subgroup of $\mathrm{Aut}(R)$. Then P acts faithfully on D and so P is either isomorphic to a subgroup of $GL(3,3)$ or of $Sp(4,3)$. Hence in both cases P is cyclic.

b) Let C and D be as before. If the rank of R is at most two, then P is a subgroup of $GL(2,5)$ and the assertion follows. Let now D be elementary abelian of order 5^3 or extraspecial of order 5^5. Then P is a subgroup of $GL(3,5)$ or $Sp(4,5)$. As 3 is the only odd prime dividing 5^2-1, we see that Sylow p subgroups for p odd, $p > 5$, are cyclic, while for $p = 3$ they are cyclic in the first case and of rank two in the second. □

$3notsolv$

Lemma 2.4 *Let P be a 3–group, $m_3(P) \leq 3$ with nonsolvable automorphism group. Then $m_3(P) = 3$ and there is a characteristic subgroup C in P which is either elementary abelian of order 27 or extraspecial of exponent 3 and order 3^5.*

Proof: Let C be a critical subgroup of P, then also C has a nonsolvable automorphismgroup. We may even assume that $C = \Omega_1(C)$. As $SL(2,3)$ is solvable we get that C is of order 27 if C is abelian. So we may assume that C is not abelian. Then 2.2 applies. In particular $Z(C)$ is centralized by any simple factor in the automorphism group. Now we get that $|C/Z(C)| > 9$ and so with 2.2 the assertion follows. □

$goodE$

Lemma 2.5 *Let P be a p–group, p odd. Let N be a cyclic normal subgroup and $P = NQ$. Suppose that $m_p(P) = 3$. Then there is some elementary abelian subgroup U of order p^2 with $U \leq Q$ and $m_p(C_P(U)) = 3$.*

Proof: Let $Q_0 = C_Q(N)$. Then we have that P/NQ_0 is cyclic as p is odd. So we may assume that $|P/NQ_0| = p$. If $m_p(NQ_0) = 3$, we are done. So we may assume that $m_p(NQ_0) = 2$. Let V be an elementary abelian subgroup of order p^2 in NQ_0, which is normal in P. Set $P_0 = C_P(V)$. Then $P_0 = N(P_0 \cap Q)$. As $m_p(P) = 3$, we get that also $m_p(P_0) = 3$. In particular there is some $U \leq P_0 \cap Q$ with $m_p(C_{P_0}(U)) = 3$. □

par

Lemma 2.6 *Let X be some group with $O_2(X) = 1$ and $m_p(X) \leq 3$ for every odd prime p. Suppose furthermore that for $S \in Syl_2(X)$ there is exactly one maximal subgroup Y of X containing S. Then either X is solvable or $E(X)$ is one of the following:*

(i) $Sz(q)$, $L_2(q)$, $(S)L_3(q)$, $(S)U_3(q)$, $Sp_4(q)$, $L_2(q) \times L_2(q)$, $Sz(q) \times Sz(q)$, *q even*

(ii) $L_2(p)$, $L_2(p^2)$, $L_2(p^3)$, $L_3(p)$, $U_3(p)$, $PSp_4(p)$, $L_2(p) \times L_2(p)$, $p > 3$ *some odd prime, or* $L_2(27)$ *or* $L_3(3)$

(iii) A_6, A_9, $3 \cdot A_6$, $3 \cdot A_6 * 3 \cdot A_6$, $SL_3(4) * SL_3(4)$, $SU_3(8) * SU_3(8)$.

or $F(X)$ *is a* p*-group with* $\Omega_1(F(X))$ *is elementary abelian of order* p^3 *and* $E(X/F(X)) \cong L_2(p)$ *acts irreducibly on* $\Omega_1(F(X))$.

If $E(X) \cong (S)L_3(q), Sp_4(q), q$ *even,* $3 \cdot A_6, 3 \cdot A_6 * 3 \cdot A_6$*, or* $SL_3(4) * SL_3(4)$*, there is some* $x \in X$ *acting nontrivially on the corresponding Dynkin diagram.*

Proof: We may assume that X is nonsolvable. Assume first $E(X) = 1$. Set $F = F(X)$. Let $p \Big| |F(X)|$ and $P \in \mathrm{Syl}_p(F(X))$. We may choose P such that $X/C_X(P)$ is nonsolvable. Let C be a critical subgroup of P and $U = \Omega_1(C)$. Set $X_1 = C_X(U)$.

Suppose $S \cap X_1 \neq 1$. By the Frattini argument we have $X = X_1 N_X(S \cap X_1)$. As $O_2(X) = 1$ we get $N_X(S \cap X_1) \neq X$. But then $X_1 S$ and $N_X(S \cap X_1)$ are contained in different maximal subgroups containing S, a contradiction.

So we have $S \cap X_1 = 1$. Let Q be a Sylow q–subgroup of X_1, including the case of $q = p$. Then $X = X_1 N_X(Q)$. We may assume $S \leq N_X(Q)$ and as $X_1 S$ is a proper subgroup of X, we get that $X = N_X(Q)$. Hence $X_1 = F$. Assume that there is some $t \in Z(S)^\sharp$ such that $C_X(t)$ covers $(X/X_1) / O_p(X/X_1)$. As $O_2(X) = 1$, we have $C_X(t) \neq X$. Let Y be the preimage of $O_p(X/X_1)$. Then $YS \neq X$ and so $C_X(t)$ and YS are contained in different maximal subgroups, a contradiction.

By 2.2 we know the structure of U. Suppose that U is not elementary abelian. If U is not extraspecial of width two, then a nonsolvable subgroup of $SL_2(p)$ is induced. But then we get an involution in $Z((X/F)/O_p(X/F))$, a contradiction. So we have that U is extraspecial of width two. As the 2-rank of of $Sp_4(p)$ is two and $Sp_4(p)$ has a 2-central involution, we see that again $Z(/X/F)/O_p(X/F))$ contains some involution, a contradiction. This shows that U has to be elementary abelian of order p^3 and X/X_1 is a subgroup of $SL_3(p)$. Now $S\,X_1/X_1$ is contained in exactly one maximal subgroup of X/X_1 Suppose that $|Y_1|$, $Y_1 = E(X/F)|$, is not divisible by p. Let Y be a preimage of Y_1. Then $YS = PK$, with $K \cap P = 1$ and $S \leq K$. But now $X = YN_X(S \cap Y)$. As $S \cap Y$ is not normal in X, we get that $X = YS$. But neither PS nor K is equal to X, a contradiction. So we have that $|Y_1|$ is divisble by p. With [Mi1] we get that $E(X/X_1) \cong L_2(p)$. Hence X acts irreducibly on U. In particular as $U \cap Z(P) \neq 1$, we have $U \leq Z(P)$ and so $U = \Omega_1(P)$.

Suppose $F \neq P$. Then choose $Q \in \mathrm{Syl}_q(F), q \neq p$. Then we have that $QC_X(Q)/F \geq E(X/F)$. Hence Let Y be a preimage of $E(X/F)$. Set $K = Y^{(\infty)}$. Then $[Q, K] = 1$. Hence we have that $Y = O_{p'}(F)PK$. Now we have KPS and $O_{p'}(F)S$, which cannot be both proper subgroups of X. This shows $X = KPS$ and so $P = F$.

Suppose now $E(X) \neq 1$. We have $X = E(X)S$, otherwise $N_X(S \cap E(X))$ and $E(X)S$ are contained in different maximal subgroups. Also S has to act transitively on the components of $E(X)$. As $m_p(X) \leq 3$ we get that $E(X)$ contains at most two components.

Let first $E(X)$ be quasisimple. Then $E(X)$ is described in 1.2. If $E(X)/Z(E(X))$ is a Lie group over $GF(q)$, $q = 2^n$, then $S \cap E(X) \leq P, P$ a minimal parabolic. Hence either $E(X)$ is of rank 1 or $E(X)$ is of rank 2 and S induces a diagram automorphism. This is (i). Suppose next that $E(X)/Z(E(X))$ is alternating. If $n = 11$, then $S \cap E(X) \leq A_\Omega$, $\Omega = \{1, \ldots, 10\}$ and $S \cap E(X) \leq A_{\Omega_1}\langle x\rangle$, $\Omega_1 = \{1, \ldots, 9\}$, $x = (1,2)(10,11)$. But both groups generate $E(X)$. If $n = 10$, then $S \cap E(X)$ is in $A_\Omega\langle x\rangle$, $\Omega = \{1, \ldots, 8\}$, $x = (1,2)(9,10)$ and also in $N_{E(X)}\big(\langle(1,2)(3,4),(1,2)(5,6),(1,2)(7,8),(1,2)(9,10)\rangle\big)$. If $n = 8$, then $E(X) \cong L_4(2)$, a case just done. If $n = 7$, then $S \cap E(X)$ is in A_Ω, $\Omega = \{1, 2, \ldots, 6\}$ and $A_{\Omega_1}\langle x\rangle$, $\Omega_1 = \{1, \ldots, 5\}$, $x = (1,2)(6,7)$. This finishes the case of an alternating group, as A_9, A_6 and $3 \cdot A_6$ are in (iii).

Let now $K/Z(K)$ be sporadic. Application of [RoStr] shows that K is generated by minimal parabolics up to $K \cong J_1$ or M_{11}. But the latter contains M_{10} and $GL_2(3)$. The group J_1 contains $\mathbb{Z}_2 \times A_5$ and the normalizer of a Sylow 2-subgroup. Hence in any such group there are at least two maximal subgroups containing $S \cap E(X)$.

We are left with $K/Z(K) \cong G(r), r$ odd, $r = p^f$. As $m_p(K) \leq 3$, we get from 1.2 $K/Z(K) \cong L_2(p), L_2(p^2), L_2(p^3), L_3(p), U_3(p), PSp_4(p)$. This is (ii), as for $p = 3$ we have $L_2(9) \cong A_6$, $U_3(3) \cong G_2(2)'$ and $PSp_4(3) \cong U_4(2)$.

Let now $E(X) = X_1X_2$. Then $m_p(X_1) = 1$ for any prime p which does not divide $|Z(E(X))|$ and $m_p(X_1) = 2$ for p which divides $|Z(E(X))|$. Application of 1.1 and 1.2 show that $X_1 \cong Sz(q)$, $L_2(q)$, $(S)U_3(q)$, $(S)L_3(q)$, J_1, $3 \cdot A_6$, $3 \cdot A_7$ or $3 \cdot M_{22}$.

Let $X_1 \cong (S)U_3(q)$ or $(S)L_3(q)$. If $p \,\big|\, q+1$ or $p \,\big|\, q-1, p \neq 3$, then X_1 contains an elementary abelian group of order p^2 intersecting the center trivially. So we have that either $q+1$ or $q-1$ has to be a 3–power. Then we get $X_1 \cong SL_3(4), L_3(2)$ or $SU_3(8)$.

Let $X_1 \cong J_1, 3\cdot A_7, 3\cdot M_{22}, 3\cdot A_6, SL_3(4), L_3(2)$. Then there are subgroups A, B of X_1 such that $\langle A, B\rangle = X_1, S \cap X_1 \leq A \cap B$. Let A, B be normal in $N_S(X_1)$. Choose $g \in S$ with $X_1^g = X_2$. Then $\langle A, A^g, S\rangle$ and $\langle B, B^g, S\rangle$ are both different from X, but $\langle X_1, S\rangle = X$. This shows $X_1 \cong 3\cdot A_6, SL_3(4)$ or $L_3(2)$ and there is some $x \in X$ acting nontrivally on the Dynkin diagram, i.e. $A^x = B$. $\square$

3 Some Small Modules

amal

Definition 3.1 Let G be a group and A, B be subgroups of G. We call (A, B) an amalgam if there is no nontrivial subgroup K in $A \cap B$ such that K is normal in $\langle A, B \rangle$.

If (A, B) is an amalgam, we can attach a graph $\Gamma = \Gamma(A, B)$ to this amalgam, whose vertices are the right cosets of A or B in $H = \langle A, B \rangle$ and edges are the right cosets of $A \cap B$. The incidence relation is by inclusion. Obviously H acts on Γ by right multiplication. Hence we see that the stabilizer H_x of a vertex $x \in \Gamma$ in H is a conjugate of A or B. Further Γ is connected.

Important for the amalgam method is a good knowledge of so called small modules. In this section we will establish the necessary results. First the definitions of the important types of modules

mod

Definition 3.2 Let G be a group and V be a nontrivial module for G over $GF(2)$. Further let A be an elementary abelian 2–subgroup of G with $A \not\leq C_G(V)$.

(1) We say that A acts quadratically on V if $[V, A, A] = 1$.

(2) We say that A acts cubic on V if $[V, A, A, A] = 1$.

(3) We call V an F–module with offender A if $|V : C_V(A)| \leq |A/C_A(V)|$.

(4) We call V a $2F$–module with offender A if $|V : C_V(A)| \leq |A/C_V(A)|^2$.

(5) We call an F–module V with offender A strong if $C_V(a) = C_V(A)$ for all $a \in A \setminus C_V(A)$.

(6) We call V a dual F–module with offender A if $[V, A, A] = 1$ and $|[V, A]| \leq |A/C_A(V)|$.

(7) We call a dual F–module V with offender A strong if $[v, A] = [V, A]$ for all $v \in V \setminus C_V(A)$.

The connection with amalgams and representation theory comes via so called 2–reduced normal subgroups which we will define now and prove then some elementary properties *pred*

Definition 3.3 Let X be a 2-local subgroup of G. Then a 2–reduced normal subgroup of X is an elementary abelian normal 2–subgroup Y of X such that $O_2(X/C_X(Y)) = 1$.

CL

Lemma 3.4 (i) *Let X be a 2-local subgroup of G then there exists a unique maximal 2–reduced normal subgroup Y_X of X*
(ii) *Let $S \leq L \leq X$, S a Sylow 2-subgroup of X, X a 2-local and R a 2–reduced normal subgroup of L, then $\langle R^X \rangle$ is a 2–reduced normal subgroup of X.*
(iii) *Let X, L be as in (ii), then $Y_L \leq Y_X$.*
(iv) *Let X be a 2-local with Sylow 2-subgroup S. Set $C_X = C_X(Y_X)$ and $X_0 = N_X(S \cap C_X)$. Then $X = X_0 C_X$ and $Y_X = Y_{X_0}$.*
(v) *Let X_0 be as in (iv). Then $S \cap C_X = O_2(X_0)$ and $Y_X = \Omega_1(Z(S \cap C_X))$.*

Proof: (i) Let Y_X be the subgroup generated by all 2–reduced normal subgroups. If $O_2(X/C_X(Y_X))$ is nontrivial, this also holds for all the generators of Y_X, a contradiction.

(ii) Let $Y = \langle R^X \rangle$ and $D = C_X(Y)$. Set $N/D = O_2(X/D)$. Then $N = (N \cap S)D = (N \cap L)D$. As R is 2–reduced for L, we have $[R, N \cap L] = 1$. Further $[D, R] = 1$, so $[N, R] = 1$. As N is normal in X, we have $[N, Y] = 1$, hence Y is 2–reduced.

(iii) Follows from (ii) with $R = Y_L$.

(iv) The first assertion is just the Frattini argument. Hence now $Y_X \leq Y_{X_0}$. By (iii) we have $Y_{X_0} \leq Y_X$.

(iv) As $O_2(X/C_X) = 1$, we have $O_2(X_0) \leq C_X$. So we get $O_2(X_0) \leq C_X \cap S$ and so $O_2(X_0) = C_X \cap S$. Set $R = \Omega_1(Z(S \cap C_X))$. Then $Y_X \leq R$. Set $Y = \langle R^X \rangle = \langle R^{C_X} \rangle$ as $X = X_0 C_X$ by (iv). Now R is 2–reduced for S and so by (ii) Y is 2–reduced for $C_X S$. Set $D = C_X(Y)$ and $N/D = O_2(X/D)$. Since $Y_X \leq R \leq Y$ and Y_X is 2–reduced for X, we get $N \leq C_X$. As Y is 2–reduced for $C_X S$, we get $[N, Y] = 1$. Hence Y is 2–reduced for X and so $Y \leq Y_X \leq R$. This shows $R = Y_X$, the assertion. □

critical

Definition 3.5 Let (A, B) be an amalgam, $H = \langle A, B \rangle$ and assume further that both A and B are of characteristic 2–type. For $x \in \Gamma$ define b_x as the shortest distance of some $y \in \Gamma$ such that $Y_{H_x} \leq H_y$ but there is some neighbor z of y such that $Y_{H_x} \not\leq H_z$. Further define $b = b_\Gamma$ as the minimum over all b_x with $x \in \Gamma$. A critical pair (x, y), where x, y are vertices of Γ, is a pair of distance b_Γ such that there is some neighbor z of y with $Y_{H_x} \not\leq H_z$.

The following lemma plays an important role in the amalgam method

Fmodule

Lemma 3.6 *Let (A, B) be an amalgam, $H = \langle A, B\rangle$ and A and B both of characteristic p–type. Let (x, y) be a critical pair. If $[Y_{H_x}, Y_{H_y}] \neq 1$, then one of both is an F–module for the corresponding stabilizer.*

Proof: By definition 3.5 we have that $[Y_{H_x}, Y_{H_y}] \leq Y_{H_x} \cap Y_{H_y}$. So by symmetry we may assume that $|Y_{H_x} : C_{Y_{H_x}}(Y_{H_y})| \leq |Y_{H_y} : C_{Y_{H_y}}(Y_{H_x})|$. Then Y_{H_x} is an F–module with offender $Y_{H_y}C_{H_x}(Y_{H_x})/C_{H_x}(Y_{H_x})$.

Next we will show that under some conditions amalgams with b odd also provide us with very special F–modules.

minpar

Lemma 3.7 *Let H be a finite group, $S \in \mathrm{Syl}_2(H)$ and S be contained in a unique maximal subgroup M of H. Let $P \leq S$ with $P \not\leq O_2(H)$. Then there are $L \leq H$ and $h \in H$ such that*

a) *$P \leq L$, $P \not\leq O_2(L)$*

b) *$O_2(L)P \leq M^h \cap L$, which is the unique maximal subgroup in L containing P*

c) *$P \leq S^h \cap L \in \mathrm{Syl}_2(L)$.*

Moreover for any such L, we have $L = \langle P^L\rangle$.

Proof: If M is the unique maximal subgroup containing P we may set $L = H$. So assume there is a maximal subgroup $K \neq M$, $P \leq K$. Among all such K we choose K with $|K \cap S|$ maximal and then $|K|$ minimal. Set $T = K \cap S$. By the minimal choice of K we know that $M \cap K$ is the unique maximal subgroup of K containing T. Set $R = \langle P^g \mid P^g \leq T, g \in H\rangle$. As $K \not\leq M$, we have $T \neq S$. So $T < N_S(T) \leq N_H(R)$. By the choice of K we now have $N_H(R) \leq M$ and so $N_K(R) \leq K \cap M$. In particular $T \in \mathrm{Syl}_2(K)$. Now $O_2(K) \leq T \leq M$. If $R \leq O_2(K)$, then $R \trianglelefteq K$ and so $K \leq M$, a contradiction. Hence there is $P^g \leq T$ with $P^g \not\leq O_2(K)$. Now we may replace H by K, P by P^g and M by $M \cap K$. By induction we get $L_1 \leq K$ with $P^g \leq L_1$, $P^g \not\leq O_2(L_1)$ and $h_1 \in K$ with $P^g \leq (M \cap K)^{h_1} \cap L_1$ and this is the unique maximal subgroup of L_1 containing P^g. Further $P^g \leq T^{h_1} \cap L_1 \in \mathrm{Syl}_2(L_1)$. Set $h = h_1 g^{-1}$ and $L = L_1^{g^{-1}}$. As $P^g \leq L_1$, we have $P \leq L$. As $P^g \not\leq O_2(L_1)$, we have $P \not\leq O_2(L)$ and so (a) holds. For (b) we have $O_2(L_1)P^g \leq (M \cap K)^{h_1} \cap L_1$. So $O_2(L)P \leq (M \cap K)^{h_1 g} \cap L \leq M^h \cap L$, which is (b). As $P^g \leq T^{h_1} \cap L_1$, we get that $P \leq T^h \cap L \leq S^h \cap L$ which is (c).

Now let $D = \langle P^L\rangle \neq L$. As $P \leq D$ we have $D \leq M^h \cap L$. The

Frattini argument shows $L = DN_L(S^h \cap D)$, so $L = N_L(S^h \cap D)$, otherwise by $P \leq N_L(S^h \cap D)$, we get $N_L(S^h \cap D) \leq M^h \cap L$ and then $L = DN_L(S^h \cap D) \leq M^h \cap L$, a contradiction. But now $P \leq O_2(L)$, a contradiction. □

quadratic

Lemma 3.8 *Let $\langle b, c\rangle$ be an elementary abelian group of order p^2 acting quadratically on a $GF(p)$–module V. Let $1 \neq v \in V$, then $\langle v^b\rangle \leq \langle v\rangle\langle v^{bc}\rangle\langle v^c\rangle$*

Proof: We have $(v^b v^{-1})^c = v^b v^{-1}$. So $v^{bc}v^{-c} = v^b v^{-1}$. Hence $v = v^{-bc}v^c v^b$, the assertion. □

action

Lemma 3.9 *Let H be a group and A be a 2–subgroup, $A \not\leq O_2(H)$ but A contained in a unique maximal subgroup M of H. Let V be a faithful $GF(2)H$–module with $[V, A, A] = 1$ such that for some $Z \leq V$ with $[Z, A] = 1$, we have $V = \langle Z^H\rangle$. Then the following hold*

a) *$O_2(H)$ is a Sylow 2–subgroup of $\bigcap_{g \in H} M^g$.*

b) *$C_V(t) = C_V(A)$ for all $t \in A \setminus O_2(H)$.*

c) *$|V : C_V(A)| \geq |A/A \cap O_2(H)|^c$, where c is the number of non trivial chief factors in V.*

d) *$[V, t] \cap C_V(H) = 1$ and $|[V, t]|^2 = |V : C_V(H)|$ for all $t \in A \setminus O_2(H)$.*

e) *$[V, H] \cap C_V(H) \leq [V, A]$*

f) *If $[Z, O_2(H)] \leq Z$, then $A \not\leq O_2(C_H([V, A \cap O_2(H)])$. Moreover if $H/O_2(H)$ is not dihedral we even have $[V, A \cap O_2(H)] \leq C_V(H)$.*

Proof: First notice that if $V = \langle Z^H\rangle$ then also $V = C_V(A)[V, H]$. Up to the proof of f) we just use this property, which is inductive. Set $N = \bigcap_{g \in H} M^g$. By the Frattini argument we have $O_2(H) \in \mathrm{Syl}_2(N)$, which is a).

Let $t \in A \setminus O_2(H)$. By a) t is not contained in N. Now choose $h \in H$ with $t \notin M^h$ and set $B = A^h$. Then as M^h is the unique maximal subgroup containing B, we see $H = \langle t, B\rangle$. This now shows

$$[V, H] = [V, t][V, B].$$

By quadratic action $[V, t] \leq C_V(A)$, hence $V = C_V(A)[V, B] = C_V(t)[V, B]$. So

$$C_V(B) = C_V(H)[V, B].$$

In the same way we see $V = C_V(B)[V,A]$. Now

$$C_V(t) = (C_V(B) \cap C_V(t))[V,A] = C_V(H)[V,A] = C_V(A),$$

which is b).

Let W be an irreducible nontrivial chief factor. Then

$$W = [W,A] \oplus [W,B].$$

Hence we get $[W,A] = C_W(t)$. So let $x \in [W,B]^\sharp$ then $|[A,x]| \geq |A/A \cap O_2(H)|$ and so $|[W,A]| \geq |A/A \cap O_2(H)|$, this is c).

Further let $b \in B \cap M \setminus O_2(H)$. Then there is some $k \in M$ such that $\langle b, A^k \rangle$ i a 2–group. Hence $C_W(b) \cap C_W(A^k) \neq 1$. By b) $C_W(b) = C_W(B)$ and $H = \langle B, A^k \rangle$, so $C_W(H) \neq 1$, a contradiction. So we have

$(*)$ $M \cap B \leq O_2(H)$

We have $[V,A] = [V,t]([V,A] \cap [V,B])$. Set $Y = [V,B] \cap C_V(t) \geq [V,A] \cap [V,B]$. We have

$$|[V,t]| = |[V,B,t]| = |[V,B]/C_{[V,B]}(t)| = |[V,A]|/|Y|.$$

So we see $|[V,t]||Y| = |[V,A]|$. This shows that $Y = [V,A] \cap [V,B]$ and so $[V,A] = [V,t] \oplus Y$. So we see $[V,t] \cap C_V(H) \leq [V,t] \cap Y = 1$ and then $|[V,H]| = |[V,t]|^2|Y|$, so $|[V,t]|^2 = |V : C_V(H)|$, which is d) and $C_{[V,H]}(A) = [V,A]Y$ so $C_{[V,H]}(H) = [V,A]Y \cap [V,B]Y = Y$, which is e).

To prove f) let $h \in H \setminus M$. We have

$$[Z^h, A \cap O_2(H)] \leq Z^h \cap C(A) \leq C(\langle A, A^h \rangle) \leq C_V(H).$$

Set $Y = \langle Z^h \mid h \in H \setminus M \rangle$. Then $[Y, A \cap O_2(H)] \leq C_V(H)$.

Assume now that $|AO_2(H)/O_2(H)| \geq 4$. We then show that B normalizes Y. Then we have $Y = \langle Z^H \rangle = V$ and so f) holds, as $A \not\leq O_2(H)$. To prove this let $h \in H \setminus M$ and $b \in B$. If $hb \notin M$, then $Z^{hb} \leq Y$. So let $hb \in M$. As $|BO_2(H)/O_2(H)| \geq 4$, there is some $c \in B$, $c \notin O_2(H)$ such that $c \notin O_2(H)b$. If also $hc \in M$, then $c^{-1}b \in M \cap B \leq O_2(H)$ by $(*)$, a contradiction. Hence we have $hc \notin M$. Similar $hbc \notin M$. But $\langle b, c \rangle$ acts quadratically. So by 3.8 we have $Z^{hb} \leq Z^h Z^{hbc} Z^{hc}$. Hence Y is B–invariant.

So assume now $|A/O_2(H) \cap A| = 2$. Then $H/O_2(H)$ is dihedral of order $2r^k$, r an odd prime. If $k = 1$, then $M = AO_2(H)$ normalizes Z, as $[O_2(H), Z] \leq Z$, and so $V = ZY$. Now $[V, A \cap O_2(H)] = [Y, A \cap O_2(H)] \leq C_V(H)$ and then

f) holds.

Let $k > 1$. Choose H^* minimal with $A \leq H^*$ and $H^*O_2(H) = M$. Set $V^* = \langle Z^{H^*}\rangle = \langle Z^M\rangle$, as $[Z, O_2(H)] \leq Z$. Then $V = V^*Y$. We have that also $H^*/O_2(H^*)$ is dihedral. Further as $k > 1$, we have that $O_2(H^*) \leq O_2(H)$. Now by induction we have $A \not\leq O_2(C_{H^*}([V^*, A \cap O_2(H^*)]))$. As $[V, A \cap O_2(H)] = [V^*, A \cap O_2(H)][Y, A \cap O_2(H)]$ we get $[V, A \cap O_2(H), C_{H^*}([V^*, A \cap O_2(H^*)])] = 1$, recall $[Y, A \cap O_2(H)] \leq C_V(H)$. So $C_{H^*}([V^*, A \cap O_2(H^*)]) \leq C_H([V, A \cap O_2(H)])$ which gives f). □

amalgam

Lemma 3.10 *Let (G_α, G_β) be an amalgam with $S \in \mathrm{Syl}_2(\mathrm{G}_\alpha \cap \mathrm{G}_\beta)$ and $S \leq M_{\alpha\beta}$, where $M_{\alpha\beta}$ is the unique maximal subgroup of G_β which contains $G_\alpha \cap G_\beta$. Let further $b = b_\alpha$ be odd, $b \geq 3$. Fix a critical pair (α, α'), with $d(\alpha, \alpha') = d(\beta, \alpha') + 1$. Then $\langle Y_\alpha^{G_\beta}\rangle = V_\beta \not\leq O_2(G_{\alpha'})$. Set $\beta = \delta_1$ and $\alpha' = \delta_2$. Then one of the following holds.*

(1) *For $i = 1, 2$ there is $L_i \leq G_{\delta_i}$ and some $\mu_i \in \Delta(\delta_i)$, such that for $V_i = \langle Y_{\mu_i}^{L_i}\rangle$, $i = 1, 2$ we have the following*

   a) *$V_i \not\leq O_2(L_{3-i})$*

   b) *$V_i \leq G_{\delta_{3-i}}$ and $G_{\delta_i} \cap G_{\mu_i}$ contains a Sylow 2–subgroup of L_i.*

   c) *$L_i \cap M_{\delta_i\mu_i}$ is the unique maximal subgroup of L_i, which contains V_{3-i}*

   d) *$[V_i, Y_{\mu_{3-i}}] = 1$.*

(2) *There are $\mu_i \in \Delta(\delta_i)$, $i = 1, 2$, some $j \in \{1, 2\}$ and $L_j \leq G_{\delta_j}$ such that the following holds*

   a) *$V_j \leq G_{\mu_{3-j}}$, $Y_{\mu_{3-j}} \leq L_j$, $Y_{\mu_{3-j}} \not\leq O_2(L_j)$*

   b) *$Y_{\mu_{3-j}} \leq G_{\mu_j}$ and $G_{\mu_j} \cap G_{\delta_j}$ contains a Sylow 2–subgroup of L_j.*

   c) *$L_j \cap M_{\delta_j\mu_j}$ is the unique maximal subgroup in L_j which contains $Y_{\mu_{3-j}}$.*

   d) *$[Y_{\mu_1}, Y_{\mu_2}] = 1$.*

(3) *There are $\mu_i \in \Delta(\delta_i)$, such that $Y_{\mu_i} \leq G_{\mu_{3-i}}$, $i = 1, 2$ and $[Y_{\mu_1}, Y_{\mu_2}] \neq 1$.*

Proof: We will assume that (3) does not hold. Then choose $L_i \leq G_{\delta_i}$,and $\mu_i \in \Delta(\delta_i)$, $i = 1, 2$ such that for $V_i = \langle Y_{\mu_i}^{L_i}\rangle$ we have

(1) $V_{3-i} \leq L_i \cap G_{\delta_i\mu_i}$

(2) $L_i \cap G_{\delta_i\mu_i}$ contains a Sylow 2–subgroup of L_i and is contained in a unique maximal subgroup $M_{\delta_i\mu_i} \cap L_i$

(3) For at least one $j \in \{1,2\}$ we have that $V_j \not\leq O_2(L_{3-j})$

Such a setup exist. Choose for μ_1 with $d(\mu_1, \delta_2) = d(\delta - 1, \delta_2) - 1$ and μ_2 with $d(\delta_1, \mu_2) = d(\delta_1, \delta_2) - 1$ and $L_i = G_{\delta_i}$, hence there is also a minimal choice. or example $\mu^+ = \alpha + 2$, $\mu^- = \alpha' - 1$, and $L^\epsilon = G_{\beta^\epsilon}$.

We first show $[V_1, V_2] \neq 1$. So suppose $[V_1, V_2] = 1$. By (3) there is some j such that $V_j \not\leq O_2(L_{3-j})$. Now choose some $\epsilon \in \Delta(\mu_{3-j}^{L_{3-j}})$ with $V_j \not\leq M_{\delta_{3-j}\epsilon}$. Then $L_{3-j} = \langle M_{\delta_{3-j}\epsilon} V_j \rangle$ by (2). Then Y_ϵ is normal in L_{3-j} and so also $Y_{\mu_{3-j}}$ is normal in L_{3-j}. Now $Y_{\mu_{3-j}}$ is even normal in $\langle G_{\mu_{3-j}\delta_{3-j}}, L_{3-j} \rangle = G_{\delta_{3-j}}$. But then $Y_{\mu_{3-j}} \trianglelefteq \langle G_{\delta_{3-j}}, G_{\mu_{3-j}} \rangle$, a contradiction. So we have shown

$$[V_1, V_2] \neq 1.$$

We will assume that for both i that if $\mu \in \mu_i^{L_i}$ and $V_{3-i} \leq G_\mu$, then $[Y_\mu, V_{3-i}] = 1$.

If $V_i \leq O_2(L_{3-i})$ for some i, then as $[V_1, V_2] \neq 1$, there must be also some Y_μ with $[V_i, Y_\mu] \neq 1$, a contradiction. So we have $V_i \not\leq O_2(L_{3-i})$ for $i = 1, 2$. Further by (1) we have for both i that $[V_i, Y_{\mu_{3-i}}] = 1$. Now fix $i = 1$. By 3.7 with $H = L_1$, $P = V_2$ we get some $L \leq L_1$ such that $V_2 \leq L$, but $V_2 \not\leq O_2(L)$, and some $h \in L_1$ with $O_2(L)V_2 \leq (M_{\delta_1\mu_1} \cap L_1)^h \cap L$, which is the unique maximal subgroup of L containing V_2. Finally $V_2 \leq S^h \cap L \in \mathrm{Syl}_2(L)$, where S is a Sylow 2–subgroup of L_1. But as L also satisfies (1) - (3) with μ_1 replaced by μ^h. As $\langle (\mu^h)^L \rangle \leq V_1$, L_2 still also satisfies (1) - (3). By the minimal choice, we now get $L = L_1$. By the same argument we also get that V_1 is in a unique maximal subgroup of L_2. Hence we have the assertion (1) of the lemma.

So without loss we may now assume that there is some $\mu \in \mu_2^{L_2}$ with $V_1 \leq G_\mu$ and $[V_1, Y_\mu] \neq 1$.

We first show $Y_\mu \not\leq O_2(L_1)$. Otherwise as $O_2(L_1) \leq G_\rho$ for all $\rho \in (\mu_1)^{L_1}$, we may choose ρ such that $[Y_\mu, Y_\rho] \neq 1$, which contradicts the assumption that we do not have (3) of the lemma.

As $V_i \leq G_{\mu_{3-i}}$ and we do not have (3) of the lemma, we see $[Y_{\mu_1}, Y_{\mu_2}] = 1$. Now we replace μ_2 by μ. Then still (1) - (3) is satisfied. Hence we may assume that $Y_{\mu_2} \not\leq O_2(L_1)$. Again we apply 3.7. This provides us with $L \leq L_1$ and $h \in L_1$ such that $Y_{\mu_2} \leq L$, $Y_{\mu_2} \not\leq O_2(L)$, $(G_{\delta_1\mu_1} \cap L_1)^h \cap L$ contains a Sylow 2–subgroup of L and $(M_{\delta_1\mu_1} \cap L_1)^h \cap L$ is the unique maximal subgroup of L containing Y_{μ_2}. In particular $Y_{\mu_2} \leq G_{\mu_1}^h$ and as $V_1 \leq G_{\mu_2}$ we have $Y_{\mu_1}^h \leq G_{\mu_2}$. As we do not have (3) of the lemma, we have $[Y_{\mu_2}, Y_{\mu_1}^h] = 1$. With this L with μ_1 replaced by μ_1^h now (2) of the lemma is satisfied. $\square$

$dualF1$

Lemma 3.11 *Suppose that G_α, G_β are subgroups of a group G, forming an amalgam as in 3.10. Set $R = [O_2(G_\beta), O^2(G_\beta)]$ and $V = \langle Y_\alpha^{G_\beta}\rangle$. Assume $[Y_\alpha, R] \neq 1$. Then one of the following holds*

- *(1) Y_α is a dual F–module with offender R and $[R, Y_\alpha] = [y, R]$ for all $y \in Y_\alpha \setminus C_{Y_\alpha}(R)$*
- *(2) There are $O_2(G_\beta)O^2(G_\beta)$–submodules $V_1 \leq V_2 \leq V_3 \leq V_4$ of V such that V_2/V_1 and V_4/V_3 are nontrivial irreducible modules and $V_4 \cap Y_\alpha \not\leq V_3$, $V_2 \cap Y_\alpha \not\leq V_1$.*

Proof: Set $H = O_2(G_\beta)O^2(G_\beta)$ Assume further that (2) is false. Then there is at most one nontrivial chief factor V_2/V_1 for $O^2(H)$ in V with $Y_\alpha \cap V_2 \not\leq V_1$.

We first show that there is at least one such factor. Suppose false. Let $V_1 < V$ be a $O^2(H)$–modules such that V/V_1 is irreducible. Then we have that $Y_\alpha \not\leq V_1$. Hence we have that $[O^2(H), V] \leq V_1$. So we have some module V_2 such that V_1/V_2 is non trivial irreducible and $[O^2(H), V] \leq V_1$. As $S \cap O^2(H)$ normalizes Y_α , $V/V_2 = V_1/V_2(Y_\alpha V_2/V_2)$ and $[S \cap O^2(H), V] \not\leq V_2$, we get that $Y_\alpha \cap V_1 \not\leq V_2$.

From now on we will assume that there is exactly one such chief factor. Assume that this chief factor is contained in $[V, O_2(H)]$. Then as just seen this implies that $V = [V, O_2(H)]Y_\alpha$. But then as $O_2(H)$ is a 2–group we see $[V, O_2(H)] \leq Y_\alpha$. But then V is normal in $\langle G_\alpha, G_\beta\rangle$, a contradiction. Hence $[V, O_2(H)]$ does not contain such a chief factor. Now set $W = [Y_\alpha, O_2(H)]^H$. But then by the same argument we see that $[W, O^2(H)] = 1$, in particular $[Y_\alpha, O_2(H), O^2(H)] = 1$. As $V = \langle Y_\alpha^H\rangle$, we get $[V, O_2(H), O^2(H)] = 1$.

So we have that R acts quadratically on V. Let now $y \in Y_\alpha \setminus C_{Y_\alpha}(R)$. Set $W = C_V(O^2(H))\langle x^{O^2(H)}\rangle$. Suppose W does not contain out chief factor. Then as above we get that W has just trivial chief factors. But then $[O^2(H), W] = 1$, contradicting $[x, R] \neq 1$. So we have that our chief factor is in W and then there is no such chief factor in V/W. Again we see that $V = W + Y_\alpha$. Further we have that $[V, O^2(H)] \leq W$. As $[y, R, O^2(H)] = 1$ we get

$$[W, R] \geq [y, R] = \langle [y, R]^{O^2(H)}, \rangle = [W, R]$$

So we have $[W, R] = [y, R]$. Now as $[V, O^2(H), R] \leq [W, R] = [y, R]$ and $[R, V, O^2(H)] = 1$, we get with the 3–subgroup lemma $[R, V] = [O^2(H), R, V] \leq [W, R] = [y, R]$. This shows $[V, R] = [y, R]$ and so also $[Y_\alpha, R] = [y, R]$. Now by quadratic action we see

$$|R/C_R(Y_\alpha)| \geq |R/C_R(y)| = |[R, y]| = |[Y_\alpha, R]|.$$

□

$dualF$

Lemma 3.12 *Suppose that G_α, G_β are subgroups of a group G, forming an amalgam as in 3.10. Adopt the notation from there. Assume that we do not have 3.10(3). Assume further that there is exactly one nontrivial chief factor of L_j in V_j, where j is arbitrary if we have 3.10(1). If $[V_j, O_2(G_{\delta_j})] \neq 1$, then we have 3.11(1).*

Proof: Set $V = \langle V_j^{G_{\delta_j}} \rangle$. Let W_1/W_2 some chief factor for $O^2(G_{\delta_j})$ with $Y_{\mu_j} \cap W_1 \not\leq W_2$. Further we may assume that there is no such chief factor for $O^2(L_j)$ with this property in W_2/W_1 Then we get that $(Y_{\mu_j} \cap W_1)W_2$ is normalized by L_j. But then it is also normalized by $\langle G_{\delta_j \mu_j}, L_j \rangle = G_(\delta_j)$, a contradiction. So as $O^2(L_j)$ induces just one nontrivial chief factor in V_j, there is also exactly one nontrivial $O^2(G_{\delta_j})$–chief factor W_1/W_2 in V with $Y_{\mu_j} \cap W_1 \not\leq W_2$. This is 3.11(1). □

$strongF1$

Lemma 3.13 *Suppose that G_α, G_β are subgroups of a group G, forming an amalgam as in 3.10. Adopt the notation from there. Assume that we do have 3.10(1). Let further $Y_\alpha \leq O_2(C_G(x))$ for all $1 \neq x \in Y_\alpha$. Assume further that there are at least two nontrivial chief factors of L_i in V_i, $i = 1, 2$. Then there is $\mu \in \mu_1^{L_1}$ such that Y_μ is a strong F–module with $V_2 \cap O_2(L_1)$ as offender. In particular Y_α is a strong F–module.*

Proof: Choose $\mu \in (\mu_1)^{L_1}$ with $Y_\mu \not\leq O_2(L_2)$. We have $V_2 \cap O_2(L_1) \leq G_\mu$.

We have that V_1 acts quadratically on V_2. Further $V_2 = \langle Y_{\mu_2}^{L_2} \rangle$, where $[V_1, Y_{\mu_2}] = 1$ Hence we may apply 3.9. By 3.9b) applied to L_2 with V_1 acting on V_2, we get $C_{V_2}(Y_\mu) = C_{V_2}(V_1)$. By 3.9a) now applied to L_1 with V_2 acting on V_1 we get $C_{V_2}(V_1) \leq V_2 \cap O_2(L_1)$. Let $1 \neq x \in [Y_\mu \cap O_2(L_2), V_2 \cap O_2(L_1)]$. Then we have $Y_\mu \leq C_{L_2}(x)$. By 3.9f) we have $Y_\mu \not\leq O_2(C_{L_2}(x))$, a contradiction.

So we have
$$[Y_\mu \cap O_2(L_2), V_2 \cap O_2(L_1)] = 1.$$

Suppose now that $V_2 \cap O_2(L_1)$ is not an offender on Y_μ as an F–module. Then

$$|V_2/V_2 \cap O_2(L_1)||V_2 \cap O_2(L_1)/C_{V_2}(V_1)| = |V_2/C_{V_2}(V_1)| \underset{3.9b)}{=}$$
$$|V_2/C_{V_2}(Y_\mu)| \underset{3.9c)}{\geq} |V_1/V_1 \cap O_2(L_2)|^2 \geq |Y_\mu/Y_\mu \cap O_2(L_2)|^2 \geq$$
$$|Y_\mu/C_{Y_\mu}(V_2 \cap O_2(L_1))|^2 \geq |V_2 \cap O_2(L_1)/C_{V_2}(Y_\mu)|^2.$$

The last inequality is because $V_2 \cap O_2(L_1)$ is assumed not to be an offender on Y_μ as an F–module. Further this inequality is strict besides $V_2 \cap O_2(L_1) = C_{V_2}(Y_\mu)$. By 3.9b) we have

$$|V_2 \cap O_2(L_1)/C_{V_2}(Y_\mu)| = |V_2 \cap O_2(L_1)/C_{V_2}(V_1)|$$

so

$$|V_2/V_2 \cap O_2(L_1)| \geq |V_2 \cap O_2(L_1)/C_{V_2}(V_1)|.$$

By 3.9c) now applied to L_1 with V_2 acting we have

$$|V_1/C_{V_1}(V_2)| \geq |V_2/V_2 \cap O_2(L_1)|^2.$$

Hence

$$|V_1/C_{V_1}(V_2)| \geq |V_2/V_2 \cap O_2(L_1)||V_2 \cap O_2(L_1)/C_{V_2}(V_1)| = |V_2/C_{V_2}(V_1)|.$$

By symmetry we also have

$$|V_2/C_{V_2}(V_1)| \geq |V_1/C_{V_1}(V_2)|.$$

Hence we have equality everywhere. But this implies $V_2 \cap O_2(L_1) = C_{V_2}(Y_\mu)$ and then also $V_2 = C_{V_2}(V_1)$, a contradiction.

Hence we have that Y_μ is an F–module with offender that $V_2 \cap O_2(L_1)$. By 3.9b) we get that it is a strong F–module. $\square$

strongF2

Lemma 3.14 *Suppose that G_α, G_β are subgroups of a group G, forming an amalgam as in 3.10. Adopt the notation from there. Assume that we do have 3.10(2). Let $Y_\alpha \leq O_2(C_G(x))$ for all $1 \neq x \in Y_\alpha$. Assume further that there are at least two nontrivial chief factors of L_j in V_j. Then $Y_{\mu_{3-j}}$ is a strong F–module with offender V_j. Further we have $[V_j, a] = [V_j, Y_{\mu_{\S-J}}]$ for all $a \in Y_{\mu_{3-j}} \setminus C_{Y_{\mu_{3-j}}}(V_j)$. In particular Y_α is a strong F–module.*

Proof: We have $V_j \leq G_{\mu_{3-j}}$ and $Y_{\mu_{3-j}} \not\leq O_2(L_j)$. Further $[Y_{\mu_1}, Y_{\mu_2}] = 1$ and so we may apply 3.10 to $Y_{\mu_{3-j}}$ acting on V_j. As in 3.13 we see with 3.9f) that $[Y_{\mu_{3-j}} \cap O_2(L_j), V_j] = 1$. Suppose now that $Y_{\mu_{3-j}}$ is not an F–module with offender V_j. Again we get

$$\begin{aligned}&|V_j/C_{V_j}(Y_{\mu_{3-j}})| \underset{3.9c)}{\geq} \\ &|Y_{\mu_{3-j}}/Y_{\mu_{3-j}} \cap O_2(L_j)|^2 \geq |Y_{\mu_{3-j}}/C_{Y_{\mu_{3-j}}}(V_j)|^2 \underset{\text{not F-mod.}}{\geq} \\ &|V_j/C_{V_j}(Y_{\mu_{3-j}})|^2.\end{aligned}$$

Again this is only possible if $V_j = C_{V_j}(Y_{\mu_{3-j}})$, a contradiction.

So we have that $Y_{\mu_{3-j}}$ is an F–module with offender V_j. If $1 \neq x \in [Y_{\mu_{3-j}}, V_j] \cap Z(L_j)$, then $Y_{\mu_{3-j}} \not\leq O_2(C_G(x))$, a contradiction. So we have $[Y_{\mu_{3-j}}, V_j] \cap C_{V_j}(L_j) = 1$. By 3.9e) we have $[V_j, L_j] \cap C_{V_j}(L_j) \leq [V_j, Y_{\mu_{3-j}}]$, so $V_j = C_{V_j}(L_j) \oplus [V_j, L_j]$. Now let $a \in Y_{\mu_{3-j}} \setminus O_2(L_j)$. Then $[V_j, a] \underset{3.9d)}{=} C_{[V_j,L_j]}(a) = \underset{3.9a)}{=} C_{[V_j,L_j]}(Y_{\mu_{3-j}})$. In particular Y_α is a strong F–module. Further $[V_j, a] = [V_j, Y_{\mu_{3-j}}]$. $\square$

FFsol

Lemma 3.15 *Let V be an F–module over $GF(2)$ for PA, where P is a p–group, p odd, normalized by an offender A. If PA acts faithfully and $C_A(P) = 1$, we have that $|V : C_V(A)| = |A|$ and $p = 3$.*

Proof: By 2.1 we may assume that $PA \cong D_1 \times \cdots \times D_n$, where the D_i are dihedral of order $2p$. Now $|V : C_V(PA)| \leq |A|^2$. Hence $|[V, P]| \leq |A|^2$. But $|A| = 2^n$ and so we get immediately that we must have equality and that $p = 3$. $\square$

FF

Lemma 3.16 *Let $F^*(X)$ be quasissimple and V be an irreducible $F^*(X)$–module over $GF(2)$ which is an F–module for X. Then $F^*(X)$ is classical, $G_2(q)$, A_n, or $3A_6$ and one of the following holds*

1) *$F^*(X)$ is classical or A_n and V is the natural module*

2) *$F^*(X) \cong L_n(q)$ and V is the exterior square of the natural module or its dual. Further this is sharp, i.e. there is no offender A with $|V : C_V(A)| < |A|$*

3) *$F^*(X) \cong Sp(6, q)$ or $\Omega^+(10, q)$ and V is the spin module or half spin module, respectively. If $F^*(X) \cong \Omega^+(10, q)$, then this is sharp.*

4) *$F^*(X) \cong G_2(q)$ and V the natural module or $3A_6$ and V is the 6–dimensional module.*

5) *$X \cong A_7$ and V is the 4–dimensional module over $GF(2)$.*

Proof: [GM], [GM1] $\square$

sFF

Lemma 3.17 *Let $F^*(X)$ be quasissimple and V be an irreducible faithful $F^*(X)$–module over $GF(2)$ which is a strong F–module. Then one of the following holds*

(1) *$X \cong SL_n(q)$ or $Sp(2n, q)$, q even, and V is the natural module*

(2) *$F^*(X) \cong 3A_6$ and V is the 6-dimensional modules over $GF(2)$ or $X \cong A_6$ or A_7 and V the 4–dimensional module over $GF(2)$. In all cases an offender is of order 4.*

(3) $X \cong O^{\pm}(2n,q)$ *or* Σ_n *and* V *is the natural module. In this case an offender has order 2.*

Proof: This immediately follows from 3.16 □

overoff

Lemma 3.18 *Let* V *be the natural module for* $G = L_2(q)$*,* $U_4(q)$ *or* $G_2(q)$*,* q *even. Then there are no over offender as an* F*–module. In case of* $G_2(q)$ *all offender have order* q^3*.*

Proof: Let A be an over offender. As V is defined over $GF(q)$, or $GF(q^2)$ in case of $U_4(q)$ we have that $|V : C_V(A)| \geq q$, q^2, respectively. This settles the case of $G \cong L_2(q)$. By 3.17 we have that V is not a strong module. Hence we get that $|V : C_V(A)| > q$, q^2, which gives $|V : C_V(A)| \geq q^2$, q^4, respectively. This now also settles the case of $U_4(q)$ as there are no elementary abelian subgroups of order greater than q^4. So we are left with $G_2(q)$. As there are no elementary abelian subgroups of order greater than q^3, we may assume that $|V : C_V(A)| = q^2$. If there are no $GF(q)$–transvections in K, we see that again A satisfies 3.17, a contradiction. Then we have, that $|V : C_V(A)| = q^3$ and so also $|A| = q^3$. So it remains to show that there are no $GF(q)$–transvections in K. Let r be such an element. Then there is a conjugate of r, with $r \notin O_2(P)$ for one of the two parabolics P containing a given Sylow 2–subgroup. Hence we may generate P by four conjugates of r. So we can generate G by five conjugates. But then $C_V(G) \neq 1$. □

We are now going to classify the irreducible dual F–modules as well.

duF

Lemma 3.19 *Let* V *be a faithful GF(2)–module for* G *and* A *be an elementary abelian subgroup of* G *with* $[V,A,A] = 1$*. Then also* $[V^*,A,A] = 1$*, where* V^* *is the dual module. Further if* $|[V,A]| \leq |A|$*, then also* $|V^* : C_{V^*}(A)| \leq |A|$*. If further* $[V,A] = [v,A]$ *for all* $v \in V \setminus C_V(A)$*, then the same is true for* V^**.*

Proof: For U a subspace of V denote by $\alpha(U)$ the annihlator of U in V^*. Then by linear algebra we get

$$|\alpha(U)| = |V/U|.$$

Now set $U = C_V(A)$. Then we get that $\alpha(U) = [V^*,A]$ and $\alpha([V,A]) = C_{V^*}(A)$. Now we see

$$[V^*,A] = \alpha(U) \leq \alpha([V,A]) = C_{V^*}(A),$$

hence A acts quadratically on V^*.

As $|[V,A]| \leq |A|$, we have that $|\alpha([V,A])| = |V|/|[V,A]| \geq |V|/|A|$. As $\alpha([V,A]) = C_{V^*}(A)$, we have $|V : C_V(A)| \leq |A|$.

The last assertion follows as $\alpha([v,A]) = \alpha([V,A])$ and so $[v^*,A] = [V^*,A]$. □

compdual

Lemma 3.20 *Let $F^*(K)$ be quasisimple and A be an elementary abelian subgroup of K. Let V be a faithful $GF(2)K$–module with $[v, A] = [V, A]$ for all $v \in V \setminus C_V(A)$. Then $[F^*(K), V]$ is quasi irreducible.*

Proof: Let W be a quasi irreducible submodule for $F^*(K)$ in V. Let $A \leq S$, S be a Sylow 2–subgroup of K and $T = S \cap F^*(K)$. We first show that $[W, A] \leq W$. Let $a \in A$ then $[C_{V/C_V(F^*(K))}(T), A] \leq C_{V/C_V(F^*(K))}(T)$. As $[T, W] \not\leq C_W(K)$, we see that $[W, A] \leq W$ as otherwise for some $w \in W$ with $[w, T] \not\leq C_W(F^*(K))$ we would get $[w, A] \not\leq [C_{W/C_W(\mathbb{F}^*(K))}(T), A]$. Now we get that $[V, A] = [W, A]$ and so $[V, A] \leq W$, which shows that also $[V, K] = W$. □

soldual

Lemma 3.21 *Let V be a faithful $GF(2)G$–module, $A \leq G$ be an elementary abelian quadratic 2–subgroup of order at least four and $[V, A] = [v, A]$ for all $v \in V \setminus C_V(A)$, or $C_V(A) = C_V(a)$ for all $a \in A^\sharp$. Then $[A, F(G)] = 1$.*

Proof: Suppose false. By 2.1 we may assume that G is a direct product of dihedral groups $D_1, \ldots, D_n$, $n \geq 2$, with A as a Sylow 2–subgroup. By quadratic action we have that $[V, D_1, D_2] = 1$. But as $[V, A] \leq [V, D_1] \cap [V, D_2] = 1$, we get a contradiction. □

sdFF

Lemma 3.22 *Let $F^*(X)$ be quasissimple and V be an irreducible faithful $F^*(X)$–module over $GF(2)$ which is a strong dual F–module. Then one of the following holds*

(1) $X \cong SL_n(q)$ or $Sp(2n, q)$, q even, and V is the natural module

(2) $X \cong A_6$ or A_7 and V the 4–dimensional module over $GF(2)$. In all cases an offender is of order 4.

(3) $X \cong O^\pm(2n, q)$ or Σ_n and V is the natural module. In this case an offender has order 2.

Proof: By 3.19 we have that V^* is an F–module with an offender, which is also a dual offender. Now with 3.16 we get the list of the lemma for V^*. As this list is closed under duality we get that V is one of these modules. □

pointstab

Lemma 3.23 *Let $F^*(X)$ quasisimple, V a faithful $GF(2)$–module for X. Let $L = O^{2'}(C_X(C_V(S)))$ for a Sylow 2–subgroup S of X and $A \leq O_2(L)$. If V is an F–module for X with offender A then $F^*(X) \cong SL_n(q)$, $Sp(2n, q)$, $G_2(q)$ or Σ_n and $[V, F^*(X)]$ is the natural module. Further $|V : C_V(A)| = |A|$.*

Proof: Set $K = F^*(X)$ and let $U = [U, K]$ be some K–submodule with $U/C_U(K)$ irreducible. Set $W = \langle U^S \rangle$. Assume first that $W/C_W(K)$ is a direct sum of modules isomorphic to $U/C_U(K)$. First of all we have that $C_W(K)C_W(S) = 1$, and so $C_W(K) = 1$ and then also $C_U(K) = 1$. Now let $1 \neq v \in C_W(S)$. Then v projects onto some $u \in U$ which is centralized by $N_S(U)$. Hence we have that U is an irreducible F-module for K which satisfies the assumptions of the lemma. Inspection of the possibilities in 3.16 shows that we have that $K \cong SL_n(q)$, $Sp(2n, q)$, $G_2(q)$ or Σ_n and U is the natural module. Further $|U : C_U(A)| = |A|$. In particular $U = [V, K]$, the assertion.

So we may assume that S induces some graph automorphism on K. Now A cannot be a sharp offender on U and so we get that $K \cong L_n(q)$ and U is the natural module. Now we also get U^*. So we have with the same argument as before that $U + U^*$ is a direct sum of the natural module and its dual. and A is in $O_2(C_K(C_{U+U^*}(N_S(K)))) = Y$. Let R be the central root group in that group and $H, H*$ the hyperplanes centralized by R in U, U^*, respectively. Let $A_1 = C_Y(H)$. Then we see that $|H : C_H(A)| \geq |AA_1/A_1|$. Further we see that $|A \cap A_1/R \cap A| \leq |H^* : C_{H^*}(A \cap A_1)|$. Hence $|U + U^* : C_{U+U^*}(A)| \geq q|A|$, so A cannot be an offender on $U + U^*$, a contradiction. □

quad

Lemma 3.24 *Let K be a component of G and V be a $GF(2)$–module for G with $[V, K] \neq 1$. Let A be a quadratic group on V, then one of the following holds*

(i) $[K, A] \leq K$

(ii) $A \neq N_A(K)$, $|A/C_A(K)| = 2$

(iii) $K \cong SL_2(2^k)$ *and* $|A/N_A(K)| = 2$

In (ii) and (iii) A is not a quadratic offender as an F-module on $[V, K]$.

Proof: [Cher2] □

strong

Lemma 3.25 *Let $X \cong G(q), q = 2^n$, be a Lie group and V an irreducible $GF(2)$-module. Let A be a fours group with $[V, A, A] = 0$. If A intersects some root group R nontrivially but $A \not\leq R$. Then one of the following holds*

(i) $X \cong (S)L_n(q), (S)U_n(q), Sp_{2n}(q)$ *or* $F_4(q)$ *and* $V = V(\lambda)$ *for some fundamental weight* λ.

(ii) $X \cong \Omega^{\pm}_{2n}(q)$ *and V is the natural or spin module.*

(iii) $X \cong E_6(q)$ *and* $V = V(\lambda_1)$ *or* $V(\lambda_6)$

(iv) $X \cong E_7(q)$ *and* $V = V(\lambda_7)$

(v) $X \cong {}^2E_6(q)$ *and* $V = V(\lambda_4)$

(vi) $X \cong G_2(q)$ *or* ${}^3D_4(q)$ *and* V *is the natural module.*

Proof: [Str] □

The modules from 3.25 will be called strong quadratic in this paper.

spor

Lemma 3.26 *Let* X *be a group such that* $F^*(X)$ *is a perfect central extension of a finite simple group. Suppose there is some elementary abelian subgroup* A *of* $X, |A| \geq 4$, *such that for some irreducible nontrivial faithful module* V *over* $GF(2)$ *we have* $[V, A, A] = 1$. *Then*

(i) If $F^*(X)/Z(F^*(X))$ *is sporadic, then* $F^*(X)/Z(F^*(X)) \cong M_{12}$, M_{22}, M_{24}, J_2, Co_1, Co_2 *or* Sz. *If* $|A| \geq 8$, *then* $F^*(X) \cong 3 \cdot M_{22}$.

(ii) If $F^*(X)/Z(F^*(X))$ *is a Lie group in odd characteristic which is not a Lie group in even characteristic too, then* $F^*(X) \cong 3 \cdot U_4(3)$. *Furthermore* V *is the* 12*–dimensional module.*

(iii) If $F^*(X)/Z(F^*(X))$ *is alternating, then either* V *is the natural module or a spin module or* $F^*(X) \cong 3 \cdot A_6$ *and* V *is the* 6*-dimensional module. If* $|A| > 8$, *then* V *is natural or* $X \cong A_8$ *and* $|V| = 16$. *If* V *is the spinmodule and* $|A| = 4$, *then* A *is conjugate to* $\langle (12)(34), (13)(24) \rangle$ *or* $\langle (12)(34)(56)(78), (13)(24)(57)(68) \rangle$. *If* $|A| = 8$ *the* A *is conjugate to* $\langle (12)(34)(56)(78), (13)(24)(57)(68), (14)(26)(37)(48) \rangle$ *under* Σ_n.

Proof: (i) This is [MeiStr1].

(ii) This is [MeiStr2].

(iii) The first assertion is [MeiStr1]. Suppose $|A| \geq 4$. Let $a \in A^\sharp$. Let k be the number of fixed points of a. Then there is $K \leq C_X(a), K \cong \Sigma_k$. Furthermore $C_{C_X(a)}(K')$ is an extension of a 2–group by Σ_m, $m = (n-k)/2$. Now choose $a \in A$ with $m > 2$ if possible. Suppose first $[A, C_{C_X(a)}(K')] \neq 1$. If $m \geq 5$, then Σ_m is nonsolvable and so $C_{C_X(a)}([V, a])$ contains an elementary abelian subgroup of $O_2(C_X(a))$ of order 2^{m-1}. But then this group contains a conjugate of (12)(34) which contradicts [MeiStr1, (4.3)].

Let $m = 4$. Then $a \sim (12)(34)(56)(78)$. Furthermore as we may assume that no $x \sim (12)(34)$ is contained in $\langle A^{C_X(a)} \rangle$ we see that A is conjugate to a subgroup of $\langle (12)(34)(56)(78), (13)(24)(57)(68), (15)(26)(37)(48) \rangle$.

Let $m = 3$. Then $C(K') \leq \Sigma_6$ and $a \sim (12)(34)(56)$. Then $\langle A^{C_X(a)} \rangle$ contains some $x \sim (12)(34)$, contradicting [MeiStr1, (4.3)].

So let $[A, O^{2'}(C_{C_X(a)}(K'))] = 1$. If $[A, K'] \neq 1$, then $[K', [V, a]] = 1$. If $k \geq 4$, then K' contains some $x \sim (12)(34)$. This again contradicts [MeiStr1, (4.3)]. Let $k \leq 3$. As $[A, O^{2'}(C_{C_X(a)}(K'))] = 1$ and $m > 2$, there is $x \sim (12)$ in A, a contradiction. So we are left with $[A, K'] = 1 = [A, C_{C_X(a)}(K')]$. But this is impossible with $m > 2$.

So we have $m \leq 2$ for all $a \in A^\sharp$. As there is no fours group of transvections we may assume $a = (12)(34) \in A$. Now $A \geq \langle a, b \rangle$, $b = (13)(24)$, $(12)(56)$ or (34). Let $[b, K'] \neq 1$. Then $b = (12)(56)$ and so K' contains no involutions by [MeiStr1, (4.3)]. This shows $k \leq 3$ and so $A \leq \Sigma_7$. But for this group $A = \langle (12)(34), (12)(56) \rangle$ does not act quadratically on the four dimensional module.

Assume now $b = (34)$. Then $[a, E(C_X(b))] \neq 1$. Now $E(C_X(b)) \cong \Sigma_{n-2}$, which is nonsolvable. But then $\langle (34), (12)(56) \rangle$ acts quadratically, a contradiction. □

quadfour

Lemma 3.27 *Let $F^*(R)$ be a quasisimple group such that $R/Z(R)$ is sporadic. Suppose that R acts faithfully on some irreducible $GF(2)$–module V. Let S be a Sylow 2–subgroup with a quadratic normal subgroup W of order at least 4 such that $\langle W^P \rangle$ is abelian and acts quadratically for all $S \leq P < R$, then $R/Z(R) \cong M_{22}$. If W acts quadratically then we have that $|W| = 4$ and $R \cong 3M_{22}$ and V is the 12–dimensional module.*

Proof: By 3.26 we have that $R/Z(R) \cong M_n$, J_2, Co_2, Co_1, Sz. As none of this group possess exactly one maximal P with $S \leq P < R$, we get that R must contain a quadratic group of order at least 8. Hence again by 3.26 we get $R \cong 3M_{22}$. Here we have two maximal parabolics P_1 and P_2 and $W \leq O_2(P_1) \cap O_2(P_2)$, which shows that $|W| = 4$.

Lie2

Lemma 3.28 *Let K be a quasisimple group in* Chev(2) *and L be some automorphism group of K. Let V be a faithfull $GF(2)$–module for L and assume that A is some quadratically acting elementary abelian subgroup of K, which is normal in some Sylow 2-subgroup T of K. Assume further that for any proper parabolic P of K, with $T \leq P$ we have that $\langle A^P \rangle$ is abelian and acts quadratically on V. Then one of the following holds*

a) *K is a rank 1 Lie group*

b) *$K \cong L_n(q)$, $Sp(2n, q)$, $U_n(q)$ and A is in a root group and the natural module is in V.*

c) *$K \cong G_2(q)$ or ${}^3D_4(q)$ and the natural module is in V*

d) *$K \cong Sp(2n,q)$, V contains the spin module and A is in a short root group.*

Proof: Let the rank of K be at least two. We have that A intersects a root subgroup nontrivially. If A is not contained in that root subgroup we have that V is a strong quadratic module. If A is contained in the root subgroup, then as the rank is at least two, there is a minimal parabolic P, such that $\langle A^P \rangle$ is not contained in a root subgroup. Hence again V is strong quadratic. Now application of 3.25 yields K and the module V.

Let first $K \cong L_n(q)$. Then A contains some $V(\lambda_i)$. Let K_i be the parabolic corresponding to λ_i. Let $B = \langle A^{K_i} \rangle$. Then $B = O_2(K_i)$. But this is just quadratic for $i = 1, n-1$. So we have the natural module. Suppose that A is not in a root group. Then there is some parabolic K_1 or K_{n-1}, where $A \not\leq O_2(K_i)$, a contradiction.

Let now $K \cong U_n(q)$ and $V(\lambda_i)$ a submodule. Now as before, we get $i = 1$. As $O_2(K_1)$, K_1 the point stabilizer, is not abelian, we see that A is in the transvection group.

Let $K \cong Sp(2n,q)$. Let K_1 be the point stabilizer in the natural representation. Suppose $O_2(K_1)$ acts quadratically. Then we have the spin module. Now consider K_2, the normalizer of a short root group. But $Z(O_2(K_2))$ does not act quadratically on the spin module, so we have that A is in the short root group. If $O_2(K_1)$ is not quadratic, we have that A is in a long root group. Now $\langle A^{K_2} \rangle$ acts quadratically and so $[N_K(A), [V, A]] = 1$, which implies that we have the natural module.

Let $K \cong F_4(q)$ and $V = V(\lambda_1)$ or $V(\lambda_4)$. Then either $Z(O_2(K_1))$ or $Z(O_2(K_4))$ has to act quadratically, where K_1, K_4 are the maximal parabolics related to the roots. But both is not true.

Let $K \cong \Omega^{\pm}(2n,q)$. Then we have the natural or half spin module. But on the natural module $O_2(K_1)$ does not act quadratically. On the half spin module $O_2(K_n)$ does not act quadratically.

For $K \cong E_6(q)$, $E_7(q)$, or ${}^2E_6(q)$ and $V(\lambda_i)$ we just consider $Z(O_2(K_i))$, which does not act quadratically. □

2Flie

Lemma 3.29 *Let $F^*(X)$ be a group of Lie type in characteristic two and V be an irreducible faithful $2F$–module in characteristic 2, which restricted to $F^*(X)$ remains irreducible. Then V is an F–module, or one of the following holds*

(1) *$F^*(X) = L_m(r^2)$ and $V = V(\lambda_i) \otimes V(\lambda_i)^\sigma$, where $i = 1$ or $n-1$ and σ is a field automorphism of order two.*

(2) *$F^*(X) = Sp(2n, r)$ and $V = V(\lambda_2)$, $n \le 4$.*

(3) *$F^*(X) = L_6(r)$ and $V = V(\lambda_3)$.*

(4) *$F^*(X) = Sp(2n, r)$, $n = 4, 5$, and $V = V(\lambda_n)$.*

(5) *$F^*(X) = Sp(4, r^2)$ and $V = V(\lambda_1) \otimes V(\lambda_1)^\sigma$ or $V(\lambda_2) \otimes V(\lambda_2)^\sigma$, where σ is a field automorphism of order two.*

(6) *$F^*(X) = \Omega^-(8, r)$, $\Omega^-(10, r)$, $\Omega^+(12, r)$ and V is the half spin module.*

(7) *$F^*(X) = U_6(r)$ and $V = V(\lambda_3)$.*

(8) *$F^*(X) = SU_3(r)$ or $Sz(r)$ and $V = V(\lambda_1)$.*

(9) *$F^*(X) = E_6(r)$ and $V = V(\lambda_1)$ or $V(\lambda_6)$.*

(10) *$F^*(X) = F_4(r)$ and $V = V(\lambda_1)$ or $V(\lambda_4)$.*

Proof: [GM1] [GLM] □

$2FAn$

Lemma 3.30 *Let $F^*(X)/Z(F^*(X)) \cong A_n$, $n > 5$, $n \neq 8$, and V be an irreducible faithful $2F$–module in characteristic 2 and $X = \langle A^X \rangle$ for some offender A, then V is the natural permutation module or $n = 7$ and V is a four dimensional one or a direct sum of two of them, $n = 9$ and V is an eight dimensional module or $F^*(X) \cong 3A_6$ and V is a 6-dimensional module or a direct sum of two of them.*

Proof: [GM] □

$2Fodd$

Lemma 3.31 *Let $F^*(X)/Z(F^*(X))$ be a group of Lie type in characteristic r and V be an irreducible faithful $2F$–module in characteristic 2, $2 \neq r$. Suppose there is an offender A such that $X = \langle A^X \rangle$. If $F^*(X)/Z(F^*(X))$ is not a group in characteristic 2, too, then $X \cong 3U_4(3)$ and V is a 12 – dimensional module.*

Proof: [GM] □

$2Fspor$

Lemma 3.32 *Let $F^*(X)/Z(F^*(X))$ be a sporadic simple group and V be an irreducible faithful $2F$–module in characteristic 2. Suppose there is an offender A such that $X = \langle A^X \rangle$. Then one of the following holds*

(i) $F^*(X) \cong M_{12}$, or M_{22} and V is a 10–dimensional module.

(ii) $X \cong M_{23}$, *or* M_{24} *and* V *is an 11 – dimensional module.*

(iii) $X \cong 3M_{22}$, *or* J_2 *and* V *is a 12 – dimensional module.*

Proof: [GM] [GLM] □

$4t$

Lemma 3.33 *Let* $F^*(X) = L$ *be quasisimple and* V *be an irreducible faithful* $GF(2)$*–module for* X*. Let* t *be some involution in* X*. Then*

a) If $|V : C_V(t)| \leq 2$, *then* $L \cong SL_n(2)$, $Sp(2n,2)$ $\Omega^{\pm}(2n,2)$, A_n *and* V *is the natural module.*
b) If $|V : C_V(t)| \leq 4$, *then either* (L,V) *is as in a) or* $L \cong SU_n(2)$, $G_2(2)'$, $SL_n(4)$, $Sp(2n,4)$ *or* $\Omega^{\pm}(2n,4)$ *and* V *is the natural module, or one of the following holds*

(i) $L \cong 3A_6$, $|V| = 2^6$

(ii) $L \cong 3U_4(3)$, $|V| = 2^{12}$

(iii) $L \cong A_7$, $|V| = 2^4$

(iv) $L \cong Sp_6(2)$, $|V| = 2^8$.

Proof: This follows with an easy inspection from 3.16 in case a) and 3.29, 3.30, 3.31 and 3.32 in case b). □

$2Fspor1$

Lemma 3.34 *Let* $F^*(X)$ *be* M_{12}, $3M_{22}$ *or* J_2 *and* V *be the irreducible* 10*–dimensional or* 12*–dimensional module over* $GF(2)$ *as in 3.32. Then there is no offender* A *as a* $2F$*–module such that* $|V : C_V(A)| \leq |A|q < |A|^2$, *for some* 2*-power* q, *with* $|A| = q^s$, *for some* s *and* $|V : [V,A]C_V(A)| \leq q$.

Proof: Let first $F^*(X) = M_{12}$. Let $q > 2$. As $|A| = q^s > q$, we see $q = 4$, $s = 2$ and $A \not\leq F^*(X)$. Now there is some $a \in A^\sharp$ such that $C(a)$ involves A_5 and $|[V,a]| = 2^5$. So $|V : C_V(a)| = 2^5$. As $|V : C_V(A)| \leq 2^6$, we get that A_5 would induce transvections on $C_V(a)$, a contradiction. So we have $q = 2$. We have that $|[V,a]| \geq 2^4$ for all $a \in A^\sharp$. Hence if $A \leq F^*(X)$, we get $|A| = 8$ and $C_V(a) = C_V(A)$ for all $a \in A^\sharp$, which is not possible as centralizers of involutions are maximal subgroups in $F^*(X)$. So there is some $a \in A$ with $|[V,a]| = 2^5$ and then $|A| = 2^4$. This shows $[A,[V,a]] = 1$ and so $\langle A^{C_X(a)} \rangle$ has to act trivially on $[V,a]$, a contradiction.

Suppose next $F^*(X) = J_2$. Suppose that A contains a non 2-central involution a. Then $|[V,a]| = 2^6$. This gives $|A| = 16$. As V is defined over $GF(4)$, we now see that $C_V(A) = C_V(a)$. So $\langle A^{C_X(a)} \rangle$ centralizes $C_V(a)$, which contradicts the $P \times Q$–lemma. Hence A just contains 2–central involutions. In particular $|A| \leq 4$. But for a 2-central involution x we have $|[V,x]| = 16 \geq |A|^2$, a contradiction.

So we are left with $F^*(X) = 3M_{22}$ and $|V| = 2^{12}$. Suppose $q = 2$. Let first $A \leq F^*(X)$. as for involutions $x \in F^*(X)$ we have $|[V,x]| = 16$ and V is not an F–module, we see that $|A| \geq 8$. If $|A| = 16$, then there are just two possibilities. But in one case we would get $|C_V(A)| = 64$ and in the other $|C_V(A)| = 4$. So we have $|A| = 8$. Then $C_V(a) = C_V(A)$ for all $a \in A^\sharp$. Hence $\langle C_{F^*(X)}(a) \mid a \in A^\sharp \rangle \cong 2^4 3A_6$, would act on $C_V(A)$, which is not possible, as this group just induces 6–dimensional modules (recall that elements of order three in the center of $3A_6$ act fixed point freely). Now $q > 2$. As $|A| \leq 2^5$, we get $q = 4$ and $|A| = 16$. Then $|V : C_V(a)| \leq 2^5$ for all $a \in A$. Hence we get $A \leq F^*(X)$ and there are exactly two possiblities. This again shows that $N_X(A)$ involves A_6, and $|C_V(A)| = 64$. Now A acts quadratically and so $|V : C_V(A)| \leq q = 4$, a contradiction.

So assume now $A \not\leq F^*(X)$. If $a \in A \setminus F^*(X)$, then $|[V,a]| = 2^6$, so $|A| = 2^5$ and $C_V(a) = C_V(A)$. As A contains a conjugacy classes of involutions in $X \setminus F^*(X)$, we may assume that $C_X(a) \cong E_8 L_3(2)$. Now A is not normal in $\langle A^{C_X(a)} \rangle$ and so $C_X(a)$ acts trivially on $C_V(a)$, contradicting the $P \times Q$–lemma. $\square$

*split*1

Lemma 3.35 *Let $X \cong A_n, n \geq 5, V$ be a $GF(2)X$-module with $[V,X]$ the natural irreducible permutation module. Assume $C_V(X) = 1$. Then $|V : [V,X]| \leq 2$, and $V = [V,X]$ if n is odd. Furthermore V is a factor of the permutation module*

Proof: This will be proved by induction on n. For $n = 5$ this is well known. So let $n > 5, K \cong A_{n-1}, K \leq X$. If $n-1$ is odd, then $[V,X] = [V,K]$ is the permutation module for K. By induction $V = [V,K] \bigoplus \tilde{T}$. Hence there is $v \in V \setminus [V,X], [v,K] = 1$, i.e. $\langle v^X \rangle = V$ is a factor of the permutation module.

Let $n-1$ be even. Then we have a K-chain. $1 < T < T_1 < [V,X] < V$, with $|T| = 2, T_1/T$ the irreducible permutation module for K and $|[V,X]/T_1| = 2$. Now by induction $C_{V/T}(K) \neq 1$. As $C_{V/T}(K) \not\leq [V,X]/T$, we again get some $v \in V \setminus [V,X], [v,K] = 1$, and so V is a factor of the permutation module. $\square$

split

Lemma 3.36 *(a) Let $X \cong SL_n(q)$, q even, and V be a module over $GF(2)$ with $[V,X]$ the natural module and $C_V(X) = 1$. Then $V = [V,X]$, or one of the following holds:*

(i) *$X \cong L_2(q)$, q even, and $|[V : [V,X]]| \leq q$*

(ii) $X \cong L_3(2)$ *and* $|V| = 16$

(b) Let $X \cong \Omega^{\pm}(2n, q)$*,* q *even,* $n \geq 2$*, and* V *be a module over* $GF(2)$ *with* $[V, X]$ *the natural module and* $C_V(X) = 1$*. Then* $V = [V, X]$ *or* $X \cong \Omega_6^+(2)$ *and* $|V| = 2^7$*.*

(c) Let $X \cong SL_n(q)$*,* $n \geq 5$*,* q *even, and* V *be a module over* $GF(2)$ *with* $[V, X]$ *the exterior square of the natural module and* $C_V(X) = 1$*. Then* $V = [V, X]$*.*

(d) Let $X \cong \Omega^{\pm}(2n, q)$*,* q *even, and* V *be a module over* $GF(2)$ *with* $[V, X]$ *a half spin module and* $C_V(X) = 1$*. Then* $V = [V, X]$*, or* $X \cong \Omega^-(6, 2)$ *and* $|V| \leq 2^{10}$*.*

(e) Let $X \cong Sp(2n, q)$*,* $n \geq 2$*,* q *even, and* V *be a module over* $GF(2)$ *with* $[V, X]$ *the natural module and* $C_V(X) = 1$*. Then* $|V : [V, X]| \leq q$*.*

(f) Let $X \cong Sp_6(q), q$ *even,* V *be a module over* $GF(2)$ *with* $C_V(X) = 0$ *and* $[V, X] \cong V(\lambda_3)$*. Then* $V \cong V(\lambda_3)$*.*

(g) Let $X \cong SU(n, q)$*,* $(n, q) \neq (4, 2)$*, and* V *be a module over* $GF(2)$ *with* $C_V(X) = 1$*. Assume that* $[V, X]$ *is the natural module, then* $V = [V, X]$*.*

Proof: (a) Obviously we may assume $Z(X) = 1$ as otherwise $[V, X] = [V, Z(X)]$ and $V = [V, Z(X)] \oplus C_V(Z(X))$.

If $X \cong L_2(q)$, then for $x \in X, o(x) = 2$, we have $|V : C_V(x)| = q$. We have that X is generated by three conjugates of x. Hence $|V| \leq q^3$, which is (i).

Let now $X \cong SL_3(q)$. If $q \neq 2$, then there are three elements x_1, x_2, x_3 in X acting fixed point freely on $[V, X]$.

$$x_1 = \begin{pmatrix} \omega^{-2} & & \\ & \omega & \\ & & \omega \end{pmatrix}, x_2 = \begin{pmatrix} \omega & & \\ & \omega^{-2} & \\ & & \omega \end{pmatrix}, x_3 = \begin{pmatrix} \omega & & \\ & \omega & \\ & & \omega^{-2} \end{pmatrix},$$

$o(\omega) = q - 1$.

We have $[x_i, V] = [V, X], i = 1, 2, 3$, and so as $[x_i, x_j] = 1$ for all i, j, we get $C_V(x_1) = C_V(x_2) = C_V(x_3)$. But $X = \langle C_X(x_i) | i = 1, 2, 3\rangle$, the assertion.

So let $q = 2$. Let $\nu \in X$, $o(\nu) = 7$. Then $V = C_V(\nu) \oplus [V, \nu]$. Let $x \in C_V(\nu)^\sharp$. As $[V, X]$ is the natural module and so $[V, X] = [V, \nu]$, we see that $|C_X(x)| = 21$. Hence $|x^X| = 8$ and so V is a factor module of the permutation module, which shows $|V| \leq 16$. This is (ii).

Let next $X \cong L_4(2)$. There is $\langle\rho\rangle \times A_5 \leq X, o(\rho) = 3, [V, X] = [V, \rho]$ and $[V, X] = [V, \gamma]$, for $\gamma \in A_5, o(\gamma) = 3$. Hence $C_V(\rho) = C_V(\gamma)$. Now as $\langle C_X(\gamma), C_X(\rho)\rangle = X$, we get $V = [V, X]$.

Let now $n \geq 4$ and $q > 2$ for $n = 4$. Let P be the parabolic in X with $|O_2(P)| = q^{n-1}$ and $P'/O_2(P) \cong SL_{n-1}(q)$ and $|C_{[V,X]}(O_2(P))| = q^{n-1}$. Then $[P', V] = C_{[V,X]}(O_2(P))$, as $P' = P''$. This now shows that $[C_V(O_2(P)), P']$ is the natural module. By induction on n we have that $C_V(O_2(P)) = C_V(P') \bigoplus C_{[V,X]}(O_2(P))$. Now application of [Hu, (I.17.4)] shows $C_V(P') = 1$, as $C_V(X) = 1$. Hence as $|V : C_V(O_2(P))| = q$, we get $V = [V, X]$.

(b) Let first $X \cong \Omega_4^-(q)$. Let $\omega_1 \in X, o(\omega_1) = q^2 - 1$. Then we see that there is some power ω of ω_1 of order $q + 1$ such that $C_{[V,X]}(\omega)$ is of order q^2. We have $V = [V, X]C_V(\omega)$. We have that ω just centralizes a hyperbolic plane in $[V, X]$. Now let $g \in X$ such that $C_{[V,X]}(\omega) \cap C_{[V,X]}(\omega^g) = 1$, this can be achieved by choosing a hyperplane orthogonal to $C_{[V,X]}(\omega)$. Then $X = \langle\omega, \omega^g\rangle$. Now $|V : C_V(X)| = q^4$ and so we get $V = [V, X]C_V(X)$, the assertion.

Let $X \cong \Omega^+(4, q)$. We write X as X_1X_2, $X_i \cong SL_2(q)$. We may assume $q > 2$, as the assertion is obvious for $q = 2$. There is $\omega_i \in X_i$, with $o(\omega_i) = q + 1$ and ω_i acting fixed point freely on $[V, X]$, $i = 1, 2$. Now choose $v \in V \setminus [V, X]$ with $[v_1, \omega_1] = 1$. Then v_1 is uniquely determined in $[V, X]v_1$. So X_2 centralizes v_1 too. Hence $[\omega_2, v_1] = 1$. As there is a unique fixed point of ω_2 in $[V, X]v_1$, we see $[X_1, v_1] = 1$, so $[v_1, X] = 1$, a contradiction.

Let next $X \cong \Omega_6^+(q), q > 2$. Let P be the parabolic with $P'/O_2(P) \cong SL_2(q) * SL_2(q)$. Then let $V_1 = C_{[V,X]}(O_2(P))$. We have $|V_1| = q$. Further as $[O_2(P), P] = O_2(P)$, we get $[V, O_2(P)] = [V, X, O_2(P)]$. Let S be a Sylow 2–subgroup of P. Then $|C_{[V,X,O_2(P)]/V_1}(S)| = q$. As $[P', V] = [V, O_2(P)]$ we now see that $V/V_1 = [V, X]/V_1C_{V/V_1}(O_2(P))$. But then $V = [V, X]C_V(O_2(P))$. As $[P', C_{[V,X]}(O_2(P))] = 1$, we get that $V = [V, X]C_V(P')$, but then by [Hu, (I.17.4)] we get $V = [V, X]$.

Let $X \cong \Omega_6^+(2)$. Now $A_8 \cong \Omega_6^+(2)$ and $[V, X]$ is the permutation module. Hence the assertion follows with 3.35.

Let now finally $X \cong \Omega^-_{2n}(q), n \geq 3$ or $X \cong \Omega^+_{2n}(q), n \geq 4$. Let P be the parabolic with $P'/O_2(P) \cong \Omega^{\pm}_{2n-2}(q)$. Set $V_1 = C_{[V,X]}(O_2(P)), |V_1| = q$. We have $[P', V] = [V, O_2(P)]$, as $P' = P''$. Hence we see $|V/V_1 : C_{V/V_1}(O_2(P))| = q$. Furthermore $C_{V/V_1}(O_2(P)) = [C_{V/V_1}(O_2(P)), P']C_V(O_2(P))/V_1$. Now P acts on $C_V(O_2(P))$ and so we get $[C_V(O_2(P)), P'] = 1$. By [Hu, (I.17.4)] we have $C_V(O_2(P)) \leq [V, X]$ and so $[V, X] = V$.

(c) Let P be the parabolic in X with $O_2(X)$ be the natural module for $X/O_2(X) \cong GL(n-1, q)$. And assume furthermore that we have chosen P such that $C_{[V,X]}(O_2(X))$ is the natural module for X. In particular $C_{[V,X]}(O_2(X))$ is dual to $O_2(X)$ as $X/O_2(X)$–module.

Let $P_1 \leq P$ with $O_2(P) \leq P_1$ and $P_1/O_2(P) \cong A_7$ if $n = 5$ and $q = 2$ and let $P_1 = P'$ else. We have $[V, O_2(P)] = C_{[V,X]}(O_2(P))$ as $n \geq 5$. If $n = 5$, $q = 2$, we see with 3.35 that P_1 acts on some submodule V_1 with $V = V_1[V, X]$ and $V_1 \cap [V, X] = C_{[V,X]}(O_2(P_1))$. If $n = 5$ and $q > 2$, we get the same result with (a). If $n > 5$, we get the result by induction as $[V, X]/C_{[V,X]}(O_2(P))$ is the symmetric square of the natural module for $P/O_2(P)$.

So in any case P_1 acts on V_1, which is a central extension of the natural module. As $O_2(P)$ is not isomorphic to $C_{[V,X]}(O_2(P))$, we see that $[V_1, O_2(P)] = 1$. Hence we have that $V_1 = C_V(O_2(X))$. This shows that P acts on V_1 in any case. But now by (a) we have $V_1 = (V_1 \cap [V, X])C_{V_1}(P')$. Application of [Hu, (I.17.4)] shows $C_V(P') = 1$ and so $V = [V, X]$.

(d) If $X \cong \Omega^+(6, q)$ this is (a). Let now $X \cong \Omega^-(6, q)$. Then $[V, X]$ is the four dimensional unitary module. We will consider it this way. Let first $q > 2$. There are $x_1, x_2, x_3 \in X, (o(\omega) = q + 1)$

$$x_1 = \begin{pmatrix} \omega^{-3} & & & \\ & \omega & & \\ & & \omega & \\ & & & \omega \end{pmatrix}, \; x_2 = \begin{pmatrix} \omega & & & \\ & \omega^{-3} & & \\ & & \omega & \\ & & & \omega \end{pmatrix},$$

$$x_3 = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \omega & \\ & & & \omega^{-3} \end{pmatrix}.$$

As $q > 2$, we have $[V, X] = [V, x_i], i = 1, 2, 3$. Now $C_V(x_1) = C_V(x_2) = C_V(x_3)$. As $C_X(x_i) \cong \langle x_i \rangle SU_3(q)$, we get $X = \langle C_X(x_1), C_X(x_2), C_X(x_3) \rangle$.

Hence we have the assertion.

Let now $q = 2$. Let P be the parabolic with $P/O_2(P) \cong L_2(4)$. Now $|C_{[V,X]}(O_2(P))| = 16$. As $C_{[V,X]}(O_2(P)) \not\cong O_2(P)$ as $P/O_2(P)$-module, we get $V = [V,X] \bigoplus C_V(O_2(P))$. We have $|C_V(O_2(P)) : [C_V(O_2(P)), P]| \leq 4$. Hence $|V : [V,X]| \leq 4$ by [Hu, (I.17.4)].

Let next $X \cong \Omega_8^-(2)$. Then there are $x_1, x_2 \in X, o(x_1) = o(x_2) = 3, x_1 \notin \langle x_2 \rangle, [x_1, x_2] = 1, [V,X] = [V, x_i]$, $C_X(x_i) = \langle x_i \rangle \times \Omega_6^+(2)$, $i = 1, 2$. This shows $C_V(x_1) = C_V(x_2)$ is invariant under $X = \langle C_X(x_1), C_X(x_2) \rangle$. Hence $V = [V,X]$.

Let now $X \cong \Omega^\pm(2n, q)$, $n > 3$ and $q > 2$ for $\Omega^-(8, q)$. Let P be the parabolic with $P'/O_2(P) \cong \Omega^\pm(2n-2, q)$. Then $C_{[V,X]}(O_2(P))$ is the half spin module for $\Omega^\pm(2n-2, q)$ while $O_2(P)$ is the natural $\Omega^\pm(2n-2, q)$-module. This shows $V = [V,X]C_V(O_2(P))$ and $[C_V(O_2(P)), P'] = C_{[V,X]}(O_2(P))$.

Now the restrictions are made such that the half spin module splits for P' by induction. Hence we have $C_V(O_2(P)) = C_{[V,X]}(O_2(P))C_V(P')$. Application of [Hu, (I.17.4)] shows $C_V(P') = 1$ and then $V = [V,X]$.

(e) Let now $X \cong Sp_{2n}(q)$. If $X \cong Sp_4(2)'$, the assertion follows with 3.35. Let now $X \cong Sp_{2n}(q), q > 2$ for $n = 2$. Let P be the parabolic with $P'/O_2(P) \cong Sp_{2n-2}(q), A_6$ for $X \cong Sp_6(2)$. Now $V = [V,X]C_V(Z(O_2(P)))$. Set $V_1 = C_V(Z(O_2(P)))$. Now $[V_1, P'] = [[V,X], P']$. Set $V_2 = [V, Z(O_2(P))]$. Then $|V_2| = q$. We see that $|V_1/V_2 : C_{V_1/V_2}(O_2(P))| = q$. We have $V_1/V_2 = ([[V,X], P']/V_2)C_{V_1}(O_2(P))/V_2$. Hence we have $[C_{V_1}(O_2(P)), P'] = 1$ as $P' = P''$. This shows $C_{V_1}(O_2(P)) \leq [V,X]$ and so $|V : [V,X]| \leq q$ by [Hu, (I.17.4)], or $X \cong Sp_6(2)$ and $[P, C_{V_1}(O_2(P))] \neq 1$. Hence there is some $t \in P \setminus P', o(t) = 2$, with $[C_{V_1}(O_2(P)), t] = V_2$. We may assume that t induces a transvection on the natural module. As $[t, [V,X]/V_2] \neq 1$ we now see $|[V,t]| \geq 4$. But $\langle t \rangle$ is conjugate to $Z(O_2(P))$ and $[V, Z(O_2(P))] = V_2$ is of order 2.

(f) Let now $X \cong PSp_6(q)$ and $[V,X]$ be the spin module. Let $X_1 \leq X, X_1 \cong \Omega_6^+(q) \cong L_4(q)$. Then $[V,X]$ is an extension of the natural $L_4(q)$-module by the natural module. Hence $V = [V,X] \bigoplus C_V(X_1)$. Let $P_1 \leq X_1$ be the parabolic which is the stabilizer of a 2-space in the natural representation of $L_4(q)$. Then $C_V(P_1) = C_V(X_1)$. We have $P_1 \leq P$, P the stabilizer of a 1-space in the natural representation of $Sp_6(q)$. Now $Z(P')$ centralizes P_1 and so $[Z(P'), C_V(P_1)] = 0$. Hence $C_V(P_1)$ is centralized by $\langle X_1, Z(P') \rangle = X$. This shows $C_V(X_1) = 0$ and then $V = [V,X]$.

(g) Let next $X \cong SU_n(q)$. If $X \cong SU_3(q)$. Then there are $x_1, x_2 \in X, x_2 \notin \langle x_1 \rangle, o(x_1) = q+1 = o(x_2), [x_1, x_2] = 1$ and $[V, X] = [V, x_i], i = 1, 2$. Hence $C_V(x_1) = C_V(x_2)$. As $C_V(x_1) \cong \mathbb{Z}_{q+1} \times L_2(q)$, is a maximal subgroup of X, we get $\langle C_X(x_1), C_L(x_2) \rangle = X$, the assertion.

Let now $X = U_4(q), q > 2$. There are $x_1, x_2, x_3 \in X, (o(\omega) = q+1)$

$$x_1 = \begin{pmatrix} \omega^{-3} & & & \\ & \omega & & \\ & & \omega & \\ & & & \omega \end{pmatrix}, \ x_2 = \begin{pmatrix} \omega & & & \\ & \omega^{-3} & & \\ & & \omega & \\ & & & \omega \end{pmatrix},$$
$$x_3 = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \omega & \\ & & & \omega^{-3} \end{pmatrix}.$$

As $q > 2$, we have $[V, X] = [V, x_i], i = 1, 2, 3$. Now $C_V(x_1) = C_V(x_2) = C_V(x_3)$. As $C_X(x_i) = \langle x_i \rangle SU_3(q)$, we get $X = \langle C_X(x_1), C_X(x_2), C_X(x_3) \rangle$. Hence we have the assertion. Let now $q = 2$. Let P be the parabolic with $P/O_2(P) \cong L_2(4)$. Now $|C_{[V,X]}(O_2(P))| = 16$. As $C_{[V,X]}(O_2(P)) \not\cong O_2(P)$ as $P/O_2(P)$-modules, we get $V = [V, X] \bigoplus C_V(O_2(P))$. We have $|C_V(O_2(P)) : [C_V(O_2(P)), P]| \leq 4$. Hence $|V : [V, X]| \leq 4$ by [Hu, (I.17.4)].

Let now $X \cong U_5(2)$. In this case we have x_1, x_2, x_3, x_4,

$$x_1 = \begin{pmatrix} \omega^{-1} & & & & \\ & \omega & & & \\ & & \omega & & \\ & & & \omega & \\ & & & & \omega \end{pmatrix}, x_2 = \begin{pmatrix} \omega & & & & \\ & \omega^{-1} & & & \\ & & \omega & & \\ & & & \omega & \\ & & & & \omega \end{pmatrix}, \ldots$$

and so on, $o(\omega) = 3$.

Further $[x_i, x_j] = 1$. Now $[V, X] = [V, x_j], i = 1, \ldots, 4$, and so $C_V(x_1) = C_V(x_2) = C_V(x_3) = C_V(x_4)$. We have $C_X(x_i) \cong \langle x_i \rangle \times U_4(2)$. Hence $X = \langle C_X(x_i) | i = 1, 2, 3, 4 \rangle$ and then $V = [V, X]$.

If $X \cong SU_6(2)$, then $[Z(X), V] = [V, X]$ and we get $V = [V, X]$, as $|Z(X)| = 3$.

Let now $X \cong SU_n(q), q > 2$ for $n = 5$ or 6. Let P be the normalizer of a root group R in X. We have $|[[V, X], R]| = q^2$. We have $P'/O_2(P) \cong SU_{n-2}(q)$ and $C_{[V,X]}(R)/[[V, X], R] \cong O_2(P)/R$. Now $[V, R] = [[V, X], R]$. Furthermore as $[P', V] \leq C_{[V,X]}(R)$, we see that

$V/[[V,X],R] = [V,X]/[[V,X],R] \cdot C_{V/[[V,X],R]}(O_2(P))$. Let V_1 be the preimage of $C_{V/[[V,X],R]}(O_2(P))$. Then $[V_1/[[V,X],R],P']$ is the natural $SU_{n-2}(q)$-module. By induction $V_1/[[V,X],R] = (V_1/[[V,X],R])C_{V_1/[[V,X],R]}(P')$. Now as $P' = P''$, we get that $[V_2,P'] = 0$ for a preimage V_2 of $C_{V_1/[[V,X],R]}(P')$. Hence $V_2 = 0$ and so $V = [V,X]$, the assertion.

$\square$

split2

Lemma 3.37 *(a) Let $X \cong A_7$ and V be a $GF(2)$–module such that $C_V(X) = 1$ and $[V,X]$ is the four dimensional module, then $V = [V,X]$.*

(b) Let $X \cong A_9$ and V be a $GF(2)$–module such that $C_V(X) = 1$ and $[V,X]$ is the eight dimensional module, which is not the permutation module, then $V = [V,X]$.

(c) Let $X \cong Sz(q)$, $q > 2$, and V be a $GF(2)$–module, such that $C_V(X) = 1$ and $[V,X]$ is the natural module, then $|V : [V,X]| \leq q$.

Proof: (a) Let $X \cong A_7, |[V,X]| = 16$. Let $X_1 \cong L_3(2), X_1 \leq X$, with $|[[V,X],X_1]| = 8$. Then by 3.36 $V = [V,X]C_V(X_1)$. Now by [Hu, (I.17.4)] we see $C_V(X_1) = 1$ and so $V = [V,X]$.

(b) Let $X \cong A_9$ and $X_1 \cong A_8$, $X_1 \leq X$. We have that $[V,X]$ involves exactly two natural $L_4(2)$–modules. So we have that $V = [V,X]C_V(X_1)$ by 3.36. Now by [Hu, (I.17.4)] we get $C_V(X_1) = 1$ and so $V = [V,X]$.

(c) Let $X \cong Sz(q)$. Let ν in X with $o(\nu) = q + \sqrt{2q} + 1$. Then as ν acts fixed point freely on $[V,X]$, we see that $V = [V,X] \times C_V(N_X(\langle\nu\rangle))$. Now let T be a Sylow 2-subgroup of X containing a Sylow 2–subgroup T_1 of $N(\langle\nu\rangle)$. Then $|C_{[V,X]}(T_1)| = q$. Let $x \in \Omega_1(T) \setminus \Phi(T)$, Then we see that $C_{C_V(\nu)}(x) = 1$. As x acts on $C_V(T)$, we see that $|C_V(T)| \leq q^2$, the assertion.

$\square$

Scentral

Lemma 3.38 *Let $L = F^*(X)$ be a quasisimple group and V be an F-module over $GF(2)$ for X. Suppose $C_V(S) \leq C_V(F^*(X))$ for $S \in Syl_2(X)$. Then one of the following holds*

(i) $L \cong L_3(2)$ and $|[V,L]| = 16$.

(ii) $L \cong A_{2^m}$ and $|C_V(L)| = 2, [V,L]/C_{[V,L]}(L)$ is the natural module.

Proof: We may assume $X = LS$ and furthermore $V = [V,L]$. Set $V_1 = C_V(L)$. Assume $C_V(S) \leq V_1$. Let V_2 be an L-submodule of $V, V_1 \leq V_2, V_2/V_1$ irreducible. If $V_1 \cap [V_2,L] = 0$, then $C_{\langle [V_2,L]^S\rangle}(L) = 0$, but $C_{\langle [V_2,L]^S\rangle}(S) \neq 0$.

Hence by 3.16, 3.36 and 3.35 we are left with $L \cong L_2(q)$, $L_3(2)$, $U_4(2)$, $Sp_{2n}(q)$, $G_2(q)$ or A_{2x}, here we consider $Sp(4,2)'$ as A_6. Let C be a Cartan subgroup of L. Then C acts on $C_{V_2}(S \cap L)$, and if $C_{V_2}(S \cap L) \not\leq V_1$, then $C_{V_2}(S \cap L) = V_1[C_{V_2}(S \cap L), C]$. By 3.16 we have that C acts transitively on $[C_{V_2}(S \cap L), C]$ and so $V_1 \cap [C_{V_2}(S \cap L), C] = 1$ As $S = (S \cap L)N_S(C)$. We see that $[C_{V_2}(S \cap L), C] \cap C_V(S) \neq 0$. So we may assume that either $C = 1$ or $C_{V_2}(S \cap L) \leq V_1$.

Suppose the latter. We may assume $V_2 = [V, L]$. If $V_2/C_{V_2}(L)$ is the natural $L_2(q)$-module, then as $[x, V_2]C_{V_2}(L)/C_{V_2}(L) = [L \cap S, V_2]C_{V_2}(L)/C_{V_2}(L)$ for all $x \in S \cap L$, $x \neq 1$, we see that $C_{V_2}(S) \not\leq V_1$.

Let $L \cong L_3(2)$, then $|V_2| = 16$ by 3.36(a). As V is an F-module, we have $V_2 = V$.

Let $V \cong U_4(2)$. By 3.36(d) $|C_{V_2}(L)| \leq 4$. Let P be the parabolic with $P/O_2(P) \cong L_2(4)$. We have $C_{V_2}(O_2(P))/C_{V_2}(L)$ is the natural $L_2(4)$-module. Hence as seen in the $L_2(q)$-case this implies $C_V(S \cap P) \not\leq V_1$.

Let $L \cong Sp_{2n}(q)$. By 3.36(e) $|C_{V_2}(L)| \leq q$. Let R be a transvection on $V_2/C_{V_2}(L)$. Then $[R, V_2]C_{V_2}(L)/C_{V_2}(L)$ is of order q and $[R, [R, V_2]] = 0$. Let $P = N_L(R)$, then $[P', [R, V_2]] = 0$. Hence we have $S \cap L \not\leq P'$. This shows $L \cong Sp_6(2)$. But in this case $|[V_2, R]| = 2$ and so $[S \cap L, [V_2, R]] = 0$, too.

Let next $L \cong G_2(q)$. Let R be a root group, $r \in R^\sharp$. Then $|[V_2, r]| = q^2$ and $C_L(r)$ induces the natural module on $[V_2, r]$. As $[V_2, r] \cap C_{V_2}(L) \neq 0$, we have $C_{V_2}(S) \not\leq V_1$.

Let finally $L \cong A_{2x}$. Then by 3.35 V_2 is a submodule of the permutation module. As $C_{V_2}(S \cap L) \leq V_1$, we have that a Sylow 2-subgroup has to act transitively and so $2x = 2^m$ for some m. So we have

(∗) If $C_{V_2}(S \cap L) \leq V_1$, then $L \cong L_3(2), |V_2| = 16$ or $L \cong A_{2^m}, V_2$ is a submodule of the permutation module.

Let now $C = 1$ and $C_{V_2}(S \cap L) \not\leq V_1$, but $C_{V_2}(L) \neq 0$. Hence we have $L \cong Sp_{2n}(2), G_2(2)'$ or A_{2x}. Now if $L \not\cong A_6$ we have $|X : L| \leq 2$. So assume $|X : L| = 2$ and $V = V_2 + V_2^x$ for some $x \in X \setminus L$. But then x centralizes $u + u^x, u \in C_{V_2}(S \cap L) \setminus V_1$. This leaves us with $X \cong P\Gamma L_2(9)$. Furthermore V has at least four composition factors which are natural $PSp_4(2)$-modules. Now for any $1 \neq A \leq S$, A elementary abelian, $|V : C_V(A)| \geq 16$. As $|A| \leq 8$, this contradicts the fact that V is an F-module. Hence (∗) holds in general. As V is an F-module we now see that $V = V_2$. □

multmod

Lemma 3.39 *Let $E(G)$ be quasisimple and V be some $2F$–module for G, which is faithful for $E(G)$. Assume that $V = V_1 \oplus \cdots \oplus V_c$, where all the V_i are isomorphic irreducible $E(G)$-modules. Then*

(i) If $c > 2$, then $E(G)$ is a classical group, V_1 is the natural module and one of the following holds

  (1) $E(G) \cong L_n(q)$, $c \leq 2(n-1)$

  (2) $E(G) \cong Sp(2n,q)$ or $\Omega^{\pm}(2n,q)$, $c \leq n+1$

  (3) $E(G) \cong U_n(q)$, $c \leq \frac{n}{2}$

(ii) Assume additionally that V is an F-module and $c > 1$, then one of the following holds

  (1) $E(G) \cong L_n(q)$, $c \leq n-1$

  (2) $E(G) \cong Sp(2n,q)$ or $\Omega^{\pm}(2n,q)$, $c \leq \frac{n+1}{2}$

  (3) $E(G) \cong U_n(q)$, $c \leq \frac{n}{4}$

Proof: We will prove (i) and (ii) together. As $c > 1$ we always have that V_1 is an F–module. Now the structure of $E(G)$ and V_1 is given by 3.16. As $c > 2$ in the case of a $2F$–module, we see that $|V_1 : C_{V_1}(A)| < |A|$ for some offender A. Hence we see that either $E(G)$ is classical and V_1 is the natural module or $E(G) \cong Sp(6,q)$ and V_1 is the spin module. Assume the latter. Now we see first that $|A| \neq q^6$, as there is a unique elementary abelian subgroup of this order in $E(G)$ and then $|V_1 : C_{V_1}(A)| = q^7$. Hence A just contains elements of type a_2 and then $|A| \leq q^4$. This shows that $|V_1 : C_{V_1}(A)| = q^2$ and so all elements in A have the same centralizer in V_1. Then $C_V(a), a \in A^\sharp$ is invariant under $\langle C_{E(G)}(b) \mid b \in A^\sharp \rangle = E(G)$, a contradiction.

So we have that $E(G)$ is classical and V_1 is the natural module. Let $E(G) \not\cong L_n(q)$. Set $W = C_{V_1}(A)$. Then we have $W = W_1 \oplus W_2$, where W_1 is some module of dimension m carrying the same form as V_1 and $|W_2| = q^t$, q^{2t} in case of $U_n(q)$. Now we see that $|A|$ is bounded by the size of an elementary abelian group in $Sp(2t,q)$, $\Omega^{\pm}(2t,q)$, $U_t(q)$ respectively, which centralizes a subspace of half of the dimension, i.e. $GF(q)$–dimension t. This gives that $|A| \leq q^{t(t+1)/2}$ or q^{t^2} in case of $U_t(q)$. All these sizes are maximal for $t = n$, $t = \frac{n}{2}$, respectivelly. This shows that we have $cn \leq n(n+1)$, or $\frac{n^2}{2}$. Hence we have (i). For (ii), we just have to multiply these by 2.

Let $E(G) \cong L_n(q)$. Let H be the semidirect product of W by $L_n(q)$, where W is a direct product of copies of the natural module V. We show

($*$) Either $J(H) = W$, or the size of as maximal elementary abelian 2–subgroup E of H is at most $q^{n(n-1)}$.

We prove $(*)$ by induction on n. This is clear for $n = 2$. Let now E be an elementary abelian subgroup of maximal size and $E \not\leq W$. Let $|C_V(E)| = q^m$. Then E is in the centralizer H_m on an m-space in $L_n(q)$. We have $|O_2(H_m)| = q^{m(n-m)}$. Assume that we have x copies of V in W. Suppose $|E/C_V(E)| \leq q^{m(n-m)}$. Then $|E| \leq q^{mx+m(n-m)}$. Further $|E| \geq q^{nx}$. This shows $x(n-m) \leq m(n-m)$, and so $x \leq m$. This shows that $|E| \leq q^{mn} \leq q^{n(n-1)}$, the assertion. So we may assume that $|E/E \cap W| > |O_2(H_m)|$. Hence as $E \leq WO_2(H_m)L_{n-m}(q)$, we get by induction that $|E/E \cap W| \leq q^{(n-m)(n-m-1)}$. This shows that $|E| \leq q^{mx+(n-m)(n-m-1)}$. Now again $(n-m)x \leq (n-m)(n-m-1)$, which shows $x \leq n-m-1$. So $|E| \leq q^{n(n-(m+1))} \leq q^{n(n-1)}$.

Now we come back to our situation of $E(G) = L_n(q)$ and assume first that we have an F–module. Then by $(*)$, we get that we have at most $n-1$ copies, the assertion. If we have a 2F-module and the number of modules is even, then we get that we have twice as many copies, as half of them have to produce an F–module. So assume that we have $2n-1$ natural modules which give a $2F$–module. By $(*)$, we see that equality holds for $n-1$ copies of the natural module, but then the same holds for $2n-2$ as a $2F$-module, hence $2n-1$ copies cannot produce an $2F$–module. □

oddF

Lemma 3.40 *Let G be a subgroup of $Aut(L_2(p^n))$, $n > 1$, p odd, containing a Borel subgroup B of $L_2(p^n)$ as a normal subgroup. Let V be a faithful $2F$–module for G over $GF(2)$, then $p = 3$, or 5, $n = 2$ and $|[V,Y]| = 2^4$, or 2^8, or $p = 3$, $n = 4$ and $|[V,Y]| = 2^8$. In all cases besides $|Y| = 3^2$, the module is exact.*

Proof: Let A be some offender. As a Sylow 2–subgroup of a Borel subgroup of $PGL(2,p^n)$ is cyclic and the group of field automorphisms is also cyclic , we see $|A| \leq 4$.

Let $|A| = 4$. Then there is some $a \in A^\sharp$, which inverts $O_p(B)$. As $|V : C_V(a)| \leq 16$, we get that either $p = 3$ and $|O_p(B)| \leq 3^4$ or $p = 5$ and $|O_p(B)| = 5^2$. If $|A| = 2$, then there is $\omega \in O_p(B)^\sharp$ and $a \in A$ with $\omega^a = \omega^{-1}$. As now $|V : C_V(a)| \leq 4$, we see $o(\omega) = 3$ or 5. Hence in all cases $p = 3$ or 5.

Let first $p = 5$, then there is some $a \in A^\sharp$ with $|[O_p(B), a]| = 5$. This shows $n = 2$, and $|[V, [O_p(B), a]]| = 2^4$. Hence $|[V, O_p(B)]| = 2^8$.

So let $p = 3$. Then $n \leq 4$. Let $n = 4$, then there is some element inverting a subgroup of order 9 and so we get that there are elements ρ of order three with $|[V, \rho]| = 4$, which shows that $|[V, O_p(B)]| = 2^8$. If $n = 3$, then $|A| = 2$ and A inverts $O_p(B)$. But then $|V : C_V(A)| \geq 8$, a contradiction. Let $n = 2$. If $|A| = 2$ we get the assertion as in the case of $p = 5$. So let

$|A| = 4$, then $|V : C_V(A)| \leq 16$ and so $|[V, O_p(B)]| \leq 2^8$. But as a dihedral group of order 8 acts on $O_p(B)$, we get $|[V, O_p(B)]| = 2^4$ or 2^8. $\square$

specialF

Lemma 3.41 *Let $F^*(G) = X \times Y$, where Y is an elementary abelian p–group, p odd, $|Y| = p^n \geq p^2$ and X quasisimple with $m_p(X) = 1$. Assume that G induces on Y a Cartan subgroup of $Aut(L_2(p^n))$ containing the Cartan subgroup of $L_2(p^n)$. Let V be some $2F$–module for G, then one of the following holds*

(a) $[Y, V] = 1$

(b) $C_V(\rho) \neq 1$ *for some p–element $\rho \in X$.*

(c) $X \cong L_2(q)$, $q = 2^{2m}$, *V is a direct sum of two natural modules and* $|Y| = 3^2$.

(d) $X \cong L_3(q)$, $q = 2^{2m+1}$, *V is a direct sum of four natural modules and* $|Y| = 3^2$.

Proof: We may assume that p-elements from X act fixed point freely on V and $[V, Y] = V$. We first show that there is some offender A which acts faithfully on X. Suppose false. Let A be some offender and $B = C_A(X)$. Suppose that also B induces a $2F$–offender on V. Then with 3.40 we see that X acts faithfully on $[V, Y]$, a group of order 2^4 or 2^8. As X is nonsolvable we see that $|[V, \omega]| \geq 2^3$ for $\omega \in Y$. This gives that $n = 2$ and $|[V, Y] = 2^8$. In particular $X \cong A_5$. Hence G contains some involution i centralizing X and acting fixed point freely on Y. In particular there is some $\omega \in Y$ with $|[[V, \omega], i]| = 4$, a contradiction.

So we have that $|V : C_V(B)| > 2|B|$. In particular $C_A(Y) \neq 1$. By assumption $C_A(Y)$ is not a $2F$–offender on V. Then $A/C_A(Y)$ is a $2F$–offender on $C_V(C_A(Y))$ which is a little bit better than $2F$. Hence by 3.40 we get that $|Y| = 3^2$ and so $|[V, Y]| \leq 2^8$. Again we see that $X \cong A_5$, which gives the same contradiction as before.

So we have that V is some $2F$ module with faithful offender A on X. Obviously we have more than one irreducible X-module involved in V. Assume that there are exactly two of them, V_1 and V_2. Then we have ρ_1 and ρ_2 in Y with $V_i = C_V(\rho_i)$, $i = 1, 2$. In particular these are all the conjugates of ρ_1, which shows $|Y| = 3^2$. Now we have that $m_3(X) = 1$, so by 1.1 we get that $X \cong L_2(q)$ or $SL_3(q)$. As V_1 is defined over $GF(2)$, we see with 3.16 that q is a power of 2. As $[V_1, \rho_2] = V_1$ and V_1 is a $GF(q)$–module, we get that 3 divides $q - 1$. Hence we have that $X = L_2(q)$, as $m_3(X) = 1$, and V is a direct sum of two natural modules.

So we may assume that there are more than two modules involved. By 3.39 we get that $X \cong L_n(q)$, $Sp(2n, q)$, $\Omega^{\pm}(2n, q)$, or $U_n(q)$, q a power of 2. Further V is a direct sum of x copies of the natural module. Suppose that all hyperplane orbits of Y under G are of length at least four. Let t be the length of such an orbit. Then we may assume that $V = V_1 \oplus \cdots \oplus V_t$, where each V_i is a direct sum of natural modules. Further there is $\omega_i \in Y$ with $[V_i, \omega_i] = V_i$, $i = 1, \ldots, t$. Suppose that V_i is a direct sum of s natural modules. Then $p \mid q^s - 1$. By 3.39 we get $L_{2s}(q) \leq X$, or $X \cong \Omega^-(4, q)$ and $s = 1$, $t = 4$. In the first case we always have an elementary abelian subgroup of order p^2 in X, a contradiction. So we have $X \cong \Omega^-(4, q)$ and so $p \mid q - 1$. Then V_1 is a sharp F–module, i.e $|V_1 : C_{V_1}(A)| = |A|$ for any offender A and so as $t = 4$, V cannot be a $2F$–module. So we have that $|Y| = 3^2$. Now $m_3(X) = 1$ and so again $X \cong L_2(q)$ or $SL_3(q)$. This shows $X \cong L_3(q)$, $x = 4$ and $q = 2^{2u+1}$, which is (d). □

good

Lemma 3.42 *Let X be a group and V be a nontrivial $GF(2)$–module for X. Assume that there is a component K of $X/C_X(V)$ such that V is an F–module for K. Then one of the following holds*

(1) V is centralized by a good E.

(2) There is a prime p with $m_p(X) \geq 4$ and some nontrivial K–submodule W of V such that any $1 \neq x \in W$ is centralized by a good E in X.

(3) $m_p(X) \leq 3$ for all odd primes p and there is some prime p and nontrivial K–submodule W such that any $1 \neq x \in W$ is centralized by a good E in X.

(4) One of the following holds

  (i) $K \cong L_2(q)$, q even, $[V, K]$ is a nonsplit extension of the trivial module by a natural module. Further $m_p(X) \leq 2$ for all odd p not dividing $q^2 - 1$. For any p with $m_p(X) \geq 3$ there is some p – element ρ centralizing W and $m_p(C_X(\rho)) \geq 3$, or K is normal in X and ρ induces a field automorphism.

  (ii) $K \cong \Omega^-(6, q)$, and $[V, K]$ is the natural module. Further $m_p(X) \leq 3$ for all odd primes p.

  (iii) $K \cong Sp(4, q)$, q even, $[V, K]$ is a nonsplit extension of the trivial module by the natural module and $m_p(X) \leq 3$ for all odd primes p which do not divide $q - 1$. The maximal p-rank is for p which divides $q - 1$.

  (iv) $K \cong G_2(q)$, q even, $[V, K]$ is a nonsplit extension of the trivial module by the natural 6–dimensional module. If p is a prime with $m_p(X)$ maximal, then $m_p(K) = 2$.

- *(v)* *$K \cong U_4(2)$, $[V,K]$ is a nonsplit extension of the trival module by a natural module and $m_p(X) \le 2$ for all primes $p > 3$.*
- *(vi)* *$K \cong L_4(2)$, $[V,K]$ is direct sum of two natural modules. There is some $\rho \in C_X(K)$, $o(\rho) = 3$, acting nontrivially on $[V,K]$ and $m_p(X) \le 2$ for all primes $p > 3$.*
- *(vii)* *$K \cong Sp(6,2)$ $[V,K]$ is the spin module and $m_p(X) \le 2$ for all $p > 3$.*
- *(viii)* *$K \cong U_4(q)$, $[V,K]$ is the natural module $m_p(X) \le 3$ for all odd primes and $m_p(X) \le 2$ for all odd primes not dividing $q+1$.*

(5) $m_p(X) \le 2$ for all odd primes p.

If we have one of (4)(ii)-(viii) then for any $x \in W$ there is some p–element $\rho \in C_X(x)$ such that $m_p(C_G(\rho)) \ge 3$. In (4)(ii) - (v) there is always some $1 \ne x \in [A,W]$, where A is an F–module offender, which is centralized by a good E.

Proof: We go over the possibilities for K and V as given in 3.16. We may assume that we do not have (5). Let first $m_p(C(V)) = 2$ for some prime p with $m_p(X) \ge 3$. Now a Sylow p–subgroup of $C_X(V)$ has a characteristic subgroup which is either elementary abelian of order p^2 or extraspecial of order p^3. By Frattini agrument we get that $C_X(V)$ contains a good E and so we have (1). Hence from now on we may assume that $m_p(C_X(V)) \le 1$ for all odd primes p with $m_p(X) \ge 3$.

Let first $K/Z(K) \cong A_n$ and assume that V involves the permutation module. Then we see that $[V,K] = W$ is an extension of a trivial module by the permutation module. Let $n > 5$. Then $m_3(K) \ge 2$. Suppose that there is $g \in X$ with $[K^g, K] = 1$. Then $[K^g, W] = 1$ and we have (2). So we may assume that K and so W is invariant under X. Suppose first that $n > 11$. Then $m_3(K) \ge 4$ and any element in the permutation module is centralized by a good E, so we have (2) again. So let $n \le 11$. Suppose first that $m_p(X) \ge 4$ for some odd $p > 3$. Then there is some $F \le X/C(V)$, elementary abelian of order p^2, which centralizes K and W. As we may assume not to be in (1), we get that the preimage of F contains a good E. So we may assume that $m_p(X) \le 3$ for all odd primes $p > 3$. Let $n = 9, 10$, or 11, then we see that any $x \in W$ is centralized by a good 3-group E in K, and we are done. So let $n \le 8$. As now $m_3(K) = 2$, we may assume that also $m_3(X) \le 3$. Let $p > 3$, with $m_p(X) = 3$, then again some elementary abelian subgroup F centralizes K and we get a good E as before. So we may assume that $m_p(K) \le 2$ for all primes $p > 3$. Now we have $m_3(X) = 3$. In particular there is some element ρ of order three centralizing K and W. For any x in W there is some element of order three in K centralizing x, so we get a good E.

Assume now that we have some module involved, which is not the permutation module. The case of A_8 on the 4–dimensional module will be handled as $L_4(2)$ later. So we have $n = 7$ and $W = [V, K]$ is the 4-dimensional module or $K \cong 3A_6$ and $W = [V, K]$ is the 6–dimensional module. In both cases W is centralized by any elementary abelian group $F \leq X/C_X(V)$ of order p^2, such that $m_p(KF) = 3$ for $p > 3$, so we get a good E. Hence we have $m_p(X) \leq 2$ for all $p > 3$. Now let $F \leq C(K)$ such that $m_3(KF) = 3$. We may assume that $[F, W] = 1$. But for any $x \in W$ there is some element of order three centralizing x. Together with F this gives a good E centralizing x.

So we are left with $n = 5$ and the permutation module is involved. The case of the $L_2(4)$–module will be handled as $L_2(4)$ later. If K is normal in $X/C_X(V)$, we may argue as above. So assume that we have conjugates of K. If $|K^X| \geq 3$, we have two conjugates centralizing W and so we either are in (2) or (3). Hence we may assume that $\langle K^X \rangle = K \times K^g$. If there is some elementary abelian p-group, centralizing $K \times K^g$, then it also centralizes W, and we are done. So we may assume that $m_p(X) \leq 2$ for all primes $p > 5$ and $m_p(X) = 3$ for $p = 3$ or 5. But then in both cases W is centralized by a good E.

Let t be the maximal p–rank of X and $r = \min(4, t)$. Let p be some odd prime with $m_p(X) = r$.

Let now $K/Z(K) \cong L_n(q)$, q even. Suppose first that there is some K-submodule W in V such that $[K, W] = W$ and W is an extension of a trivial module by the natural module. Then any x in W is centralized by $SL(n - 1, q)$. If $m_p(SL(n - 1, q)) \geq 2$ for some odd p with $m_p(K) \geq 4$, then we have (2). So we may assume that $K \cong SL(2, q)$, $SL(3, q)$, $SL(4, q)$, $SL(5, 2)$, $SL(6, 2)$, or $SL(7, 2)$. Let p be dividing $q^2 - 1$. Then by the same argument we see that $K \cong SL(n, q)$, $n \leq 4$.

Let $K \cong SL(4, q)$, with $q > 2$. Then $SL(3, q)$ does not contain an elementary abelian subgroup of order p^2, or we have (2) or (3), so p divides $q + 1$. As for any prime dividing $q - 1$ the rank of K is three, we now get $r = 4$, i.e. $m_p(X) = 4$. We have that V can involve at most three natural modules, so as the p-rank of $GL(3, q)$ is one, we have that $m_p(C(K)) = 1$. Otherwise as p divides the order of $GL(3, q)$ in K, we get a good E. Hence there is some element of order p inducing a field automorphism on K and normalizing W. But then any element in W is centralized by $SL(3, q)$ extended by the field automorphism and so by a good E.

So let $K \cong SL(4, 2)$. Then $p = 3$. If $m_3(X) > 3$, we may argue as before. So we have $m_3(X) = 3$ and we also have $m_p(X) < 3$ for all $p > 3$.

Further we have more than one module involved in $[V, K]$, which shows that we have (4)(vi). Now any element in $[V, K]$ is centralized by some 3-element from a good E.

Let $K \cong SL(3, q)$. Let p divide $q + 1$, then $m_p(K) = 1$. Hence we have the same situation as before as p divides the order of $SL(2, q)$. So we may assume that p divides $q - 1$. As we now have at most two natural modules in V, we see that $m_p(X) \leq 2$ for all p not dividing $q - 1$. Let F be an elementary abelian p-group such that $m_p(KF) > 2$. If F normalizes W we see that all x in W are centralized by a good E, as all nontrivial elements in W are conjugate and p divides the order of the centralizer in K of such an element. So we may assume that $[K, V]$ is a direct sum of two natural modules on which F acts. But then there are $q + 1$ such modules and so F fixes at least two of them and we are done.

Let finally $K \cong L_2(q)$. Now $W = [V, K]$. Further there is some elementary abelian p–sugroup F of order p^2 which intersects K trivially. Assume that $F \not\leq N_X(K)$. Then we have at least three conjugates of K and two of them centralize W, so we have a good E centralizing W. Hence we may assume that F normalizes K and so $m_p(KF) \geq 3$. If W is just the natural module, we see, as K acts transitively on $W^\sharp$, that any $x \in W$ is centralized by a good E. So W is a nonsplit extension of a trivial module by the natural module. By 3.35, we have that $|W| \leq q^3$. Assume now that no p–element centralizes W. Then K has to be normal, as any conjugate would centralize W and further p divides $q - 1$, $m_p(C(K)) = 1$. Hence some p–element has to induce a field automorphism. This is (4)(i).

So assume now that p does not divide $q^2 - 1$. Assume further that $m_p(K) \geq 2$. Then we see $K \cong L_6(2)$ or $L_7(2)$ and $p = 7$. As $L_6(2)$ contains an elementary abelian subgroup of order 7^2, we see that for $K \cong L_7(2)$ we have a good E. So we have $K \cong L_6(2)$. As $p \neq 3$, we see $r = 4$ and so there is an elementary abelian group F of order 49 centralizing K. As there are at most 5 natural modules in V, we see that there is some element of order 7 centralizing $[V, K]$ and so W. But then there is a good E for W.

So we may assume $m_p(K) \leq 1$. If we have $K \cong L_n(2)$, $n = 6, 7$, then we see that there is an elementary abelian p-group F of order p^3 centralizing K. But then there is some good elementary abelian subgroup E centralizing K and W as well. If $K \cong SL(4, q), q > 2$, then we see that there is some elementary abelian subgroup F of order p^2 centralizing K. Hence F either centralizes W or p divides the order of $SL(3, q)$. In the latter F contains some element of order p which centralizes W, but now in K there is for any $x \in W$ also some p–element which centralizes x, so we have a godd E in any case.

We are left with $K \cong L_4(2)$, $L_5(2)$, $SL(3,q)$ and $SL(2,q)$. In the last two cases, as p does not divide the order of $GL(2,q)$ and we have at most two natural modules in V, we see that any p–element normalizing K normalizes W as well. So we get p divides the order of K, otherwise we have some good E and so $K \cong SL(3,q)$. If W is the natural module, all elements in W are conjugate under K and so there is a good E. By 3.35 we are left with $K \cong L_3(2)$ and $|W| = 16$. Now $p = 7$ and so we have an elementary abelian group of order 7^2 centralizing K and W as well, so we have a good E.

Let finally $K \cong L_4(2)$ or $L_5(2)$. Then there is an elementary abelian subgroup F of order p^2 which centralizes K. So F cannot centralize W. As there are at most 4 natural modules involved, we see that $p = 5$ or 7, where $p = 7$ in case of $K \cong L_4(2)$. Further we see that some p–element in F centralizes W. But the stabilizer of any element of W in K is divisible by p and so we have a good E.

By 3.16 we now may assume that W is an extension of the trivial module by the exterior square. We may assume that $n > 4$. The case of $L_4(q)$ will be handled as $\Omega^+(6,q)$. Then by 3.35 we have W is the exterior square. Now any x in W is centralized by either $Sp(4,q)$ or $L_2(q) \times L_{n-2}(q)$. This shows that we may assume that p does not divide $q-1$. But then as $n \geq 5$, we get $q = 2$. Now we see that $p \neq 3$, and so $n = 5, 6$, or 7. If $m_p(K) = 1$, there is some elementary abelian subgroup F centralizing K. By 3.16 we have that $W = [V,K]$, which shows that F centralizes W and we are in (2). So $m_p(K) = 2$, and $K \not\cong L_5(2)$. But now $m_p(X) \geq 4$ and again there is some elementary abelian subgroup of order p^2 centralizing K and we are done.

Let $K \cong Sp(2n,q)$. Assume first that $W = [W,K]$ is the extension of a trivial module by a natural module. Then any $x \in W$ is centralized by $Sp(2n-2,q)$. If $n > 3$, we may assume that p divides q^2-1, but then $Sp(2n-2,q)$ contains a good E and we are done. So we may assume $n \leq 3$.

Let $K \cong Sp(6,q)$, then we see that p does not divide q^2-1, otherwise we argue as before. Hence $m_p(K) \leq 1$. As $m_3(K) = 3$, we have that $m_p(X) \geq 4$ and so there is some elementary abelian subgroup F of order p^2 centralizing K. Now we see that $[V,K]$ can involve at most two natural modules and as p does not divide the order of $GL(2,q)$ we see that F centralizes W and so we have a good E centralizing W.

Let $K \cong Sp(4,q)$. Now $W = [K,V]$. Suppose $m_p(K) \leq 1$. There is some elementary abelian subgroup F such that $m_p(KF) = 3$. If W is the natural module, then all elements are conjugate under K and so any is centralized by some good E. Hence we must have a nonsplit extension of the trivial module by the natural module. But as p divides q^2+1, we see that there is no field automorphism of order p and so we have a good E centralizing W. So we may assume that p divides q^2-1. Again we must have that W is a nonsplit extension of the trivial module by the natural module, as p divides the order of the point stabilizer. Suppose $m_p(X) \geq 4$. Then there is some p–element centralizing K. If it also centralizes W we have a good E for any $x \in W$. So p divides $q-1$. So we have (4)(iii). Obviously any $x \in W$ is centralized by some p-element from a good E.

Let now $K \cong Sp(6,q)$ on the spinmodule W. Then all elements are centralized by $SL(3,q)$ or $G_2(q)$. So we may assume that p does not divide $q-1$. Let $q \neq 2$. Then we have $m_p(X) \geq 4$. So if $m_p(K) \leq 1$, we get some elementary abelian subgroup F of order p^2 centralizing K. If $q=2$ and $m_p(K) \leq 1$, we also get such a group F. As $W=[V,K]$ and p does not divide $q-1$, we see that F centralizes W and we have (2). So we have $m_p(K) \geq 2$. Then p divides q^2-1. Now p divides $q+1$. Let $q>2$, then $m_p(X) \geq 4$. Now there is some $\rho \in X$, $o(\rho)=p$ and ρ either centralizes K or induces a field automorphism. But then elements in W are either centralized by $G_2(q)$ or some conjugate of $SL(3,q)\langle\rho\rangle$. So we just have to prove that the latter group contains a good E. We have $q=s^p$ and so p divides $s^{p-1}-1$ and s^p+1 as well. Hence p divides $s+1$. This shows that the p-rank of $C_K(\rho)$ is three, hence $SL(3,q)\langle\rho\rangle$ contains a good E. So we have $q=2$ and $p=3$. Further $m_3(X)=3$, which is (4)(vii). Finally any $x \in W$ is centralized by some 3–element from a good E.

Let now $K \cong \Omega^{\pm}(2n,q)$. Assume first that $W=[W,K]$ is an extension of the natural module by a trivial module. Then any element in W is centralized by $\Omega^{\pm}(2n-2,q)$ or $Sp(2n-2,q)$. If $n \geq 4$, $K \not\cong \Omega^-(8,q)$, then for p which divides q^2-1, we have $m_p(K) \geq 4$, so any x in W is centralized by some good E.

Let $K \cong \Omega^-(8,q)$ or $\Omega^+(6,q)$. If p divides q^2-1 we may argue as before. So we have $m_p(K) \leq 1$. If $q \neq 2$ in case of $K \cong \Omega^+(6,q)$, then we see that $m_p(X) \geq 4$. In particular there is some elementary abelian subgroup F of order p^2 centralizing K. As $W=[V,K]$ and p does not divide $q-1$, we see that F centralizes W and we have (2). So let $K \cong \Omega^+(6,2)$. But this case has been handled as A_8 before.

Let $K \cong \Omega^-(6,q)$. Assume $m_p(X) \geq 4$. If $m_p(K) \leq 2$, we may argue as before. Let ρ be some p–element such that $m_p(K\langle\rho\rangle)=4$. Then we see that p divides $q+1$ and so any element in W is centralized by $Sp(4,q)$ or a conjugate of $L_2(q^2)\langle\rho\rangle$. Hence we always have a good E. So we have $m_p(X) \leq 3$ and so we have (4)(ii). Any $x \in W$ is centralized by some p-element from a good E.

Let now W be an extension of the trivial module by the half spin module. The case of $\Omega^+(6,q)$ was treated as $L_4(q)$, the case of $\Omega^-(6,q)$ will be treated as $U_4(q)$. Hence by 3.16, we just have to handle $K \cong \Omega^+(10,q)$. So we have p divides q^2-1. But any $x \in W$ is centralized by $SL(4,q)$ or $Sp(6,q)$ and so we have (2).

Next let $K/Z(K) \cong U_n(q)$, $n \geq 4$. Let $W = [W,K]$ be an extension of a trivial module by the natual module. Now any element in W is centralized by some $SU_{n-2}(q)$. If $n \geq 5$, we may choose p dividing $q+1$. But then the p–rank of $SU_{n-2}(q)$ is at least two, and so we get a good E. So we just have to treat $K \cong U_4(q)$. Further we have that $W = [V,K]$. Now let $m_p(K) = 1$. Then K is normalized by some elementary p–group F such that $m_p(KF) = 3$. Now we see that any $x \in W$ is centralized by some good E. Recall that for $q = 2$, K and so also W is centralized by F, while for $q > 2$ we have that W is the natural module by 3.35. So we have p divides q^2-1. If p divides $q-1$, we again have that W is the natural module and we have some element ρ such that $m_p(K\langle\rho\rangle) = 3$. Now again any $x \in W$ is centralized by some good E. So we are left with p divides $q+1$ and further $m_p(X) = 3$, if W is the natural module, which is (4)(viii). Assume that W is not the natural module, then by 3.35 we have $q = 2$ and we have (4)(v). In both cases any $x \in W$ is centralized by some p–element from a good E.

Let finally $K \cong G_2(q)$. Then $W = [W,K]$ is an extension of the trivial module by the natural module. Suppose that W is the natural module. Then $W = [V,K]$ and all nontrivial elements in W are conjugate under K. In particular we may assume that there is no elementary abelian p group F such that $m_p(KF) = m_p(K)+2$. Hence we have $m_p(K) = 2$ and $m_p(X) = 3$. But now p divides the order of the point stabilizer of K and so again we get a good E centralizing $x \in W$. So we have that the extension is nonsplit and $m_p(K) = 2$, which is (4)(iv). But still any $x \in W$ is centralized by some p–element from a good E.

The last assertion about $[A,W]$ follows as we either have $C_W(K) \leq [A,W]$ or in the case of $\Omega^-(6,q)$ on the orthogonal module we have nonisotropic vectors in $[A,W]$.

□

As we see from 3.41 we have for the situation of 3.42 in the exceptional case, that V is always centralized by a good E.

$good1$

Lemma 3.43 *Let X be a group and V be a nontrivial $GF(2)$–module for X. Assume that there is a component K of $X/C_X(V)$, such that $[V,K]$ is a $2F$–module for K. Let S be a Sylow 2–subgroup of X and let t be the maximal p–rank of X and $r = min(4,t)$. Let p be some odd prime with $m_p(X) = r$.*

Then one of the following holds

(α) $r \geq 3$ *and one of the following holds*

- (1) V *is centralized by a good* E*, or* $[V, K]$ *is centralized by a good* E *and for all primes* p *with* $m_p(X) = r$*, we have that* p *does not divide the order of* K.
- (2) *Let* T *be a Sylow* 2*–subgroup of* K*. Then* $C_V(T)$ *is centralized by a good* E.
- (3) *There is a prime* p *with* $m_p(X) \geq 4$ *and some nontrivial* $N_S(K)K$*–submodule* W *of* $[V, K]$ *such that any* $1 \neq x \in W$ *is centralized by a good* E *in* X*. Further any element in* $C_V(K)$ *is centralized by some* p*–element whose centralizer in* X *contains an elementary abelian group of order* p^3*, or* K *contains a good* E *and in* W *any element is centralized by some* p *– element whose centralizer contains an elementary abelian group of order* p^3*. In particular* $C_V(K)$ *is not contained in* $[V, K]$ *and* $C_V(T)$ *is centralized by a* p*–element whose centralizer contains an elementary abelian group of order* p^3.
- (4)
  - (a) $m_p(X) \leq 3$ *for all odd primes* p *and there is some prime* p *and nontrivial* $N_S(K)K$*–submodule* W *of* $[V, K]$ *such that any* $1 \neq x \in W$ *is centralized by a good* E *in* X*. Further any element in* $C_V(K)$ *is centralized by some* p*–element whose centralizer in* X *contains an elementary abelian group of order* p^3*, or* K *contains a good* E *and in* W *any element is centralized by some* p *– element whose centralizer contains an elementary abelian group of order* p^3*. In particular* $C_V(K)$ *is not contained in* $[V, K]$ *and* $C_V(T)$ *is centralized by a* p*–element whose centralizer contains an elementary abelian group of order* p^3.
  - (b) $m_p(X) \leq 3$ *for all odd primes* p *and there is some prime* p *such that all elements in* $[V, K]$ *are centralized by some good* E *and* $C_V(K) = 1$.
- (5) *One of the following holds*
  - (i) $K \cong L_2(q)$*,* q *even,* $[V, K]$ *is a nonsplit extension of the trivial module by a natural module. Further* $m_p(X) \leq 2$ *for all odd* p *not dividing* $q^2 - 1$.
  - (ii) $K \cong \Omega^-(6, q)$*, and* $[V, K]$ *is the natural module. Further* $m_p(X) \leq 3$ *for all odd primes* p.
  - (iii) $K \cong Sp(4, q)$*,* q *even,* $[V, K]$ *is a nonsplit extension of the trivial module by the natural module and* $m_p(X) \leq 3$ *for all odd primes* p *which do not divide* $q - 1$*. The maximal* p*-rank is for* p *which divides* $q - 1$.

(iv) *$K \cong G_2(q)$, q even, $[V,K]$ is a nonsplit extension of the trivial module by the natural 6–dimensional module. If p is a prime with $m_p(X)$ maximal, then $m_p(K) = 2$.*

(v) *$K \cong U_4(2)$, $[V,K]$ is a nonsplit extension of the trival module by a natural module and $m_p(X) \leq 2$ for all primes $p > 3$.*

(vi) *$K \cong L_4(2)$, $[V,K]$ is direct sum of two natural modules. There is some $\rho \in C_X(K)$, $o(\rho) = 3$, acting nontrivially on $[V,K]$ and $m_p(X) \leq 2$ for all primes $p > 3$.*

(vii) *$K \cong Sp(6,2)$ $[V,K]$ is the spin module and $m_p(X) \leq 2$ for all $p > 3$.*

(viii) *$K \cong U_4(q)$, $[V,K]$ is the natural module $m_p(X) \leq 3$ for all odd primes and $m_p(X) \leq 2$ for all odd primes not dividing $q+1$.*

(ix) *$K \cong A_9$, $[V,K]$ is the spin module and $m_p(X) \leq 2$ for all primes $p > 3$.*

(x) *$K/Z(K) \cong A_n$, $n \leq 7$.*

(xi) *$K \cong 3M_{22}$ or J_2 and $[V,K]/C_{[V,K]}(K)$ is the 12–dimensional module. Further $m_p(K) = 2$.*

(xii) *$K \cong \Omega^-(8,2)$, $[V,K]$ has the half spin module as a submodule and maybe the natural module is also involved, which is not a submodule, $m_p(X) \leq 2$ for all primes $p > 3$, $m_3(X) = 3$.*

(xiii) *$K \cong U_3(q)$, $[V,K]$ is the natural module and $m_p(K) = 2$.*

(xiv) *$K \cong Sp(4,q)$ and $[V,K]/C_{[V,K]}(K)$ is the natural module.*

(xv) *$K \cong L_3(q^2)$, $C_V(K) \leq [V,K]$ and $[V,K]/C_V(K)$ is the tensor product of two algebraically conjugate natural modules. Further $m_p(X) \leq 2$ for all primes p which do not divide q^2-1 and $m_p(X) \leq 3$, if p divides q^2-1.*

(xvi) *$K \cong Sp(4,q^2)$ and $[V,K]/C_{[V,K]}(K)$ is a tensorproduct of two algebraically conjugate natural modules. Further p divides q^2-1.*

(xvii) *$K \cong G_2(q)$ or $U_4(q)$ and $[V,K]$ involves exactly two natural modules.*

(xviii) *$K \cong Sp(6,q)$ and $[V,K]$ involves a spin module and a natural module or a further spin module. Further there is no submodule, which is the natural module. We have that p divides q^2-1. If there are two spin modules, then p does not divide $q-1$.*

(xix) *$K \cong Sp(4,q)$ and $[V,K]$ involves exactly two 4–dimensional modules.*

(xx) *$K \cong \Omega^+(8,2)$ and $[V,K]$ contains two half spin modules, which are interchanged by a Sylow 2–subgroup of X.*

(xxi) *$K \cong L_4(q)$ or $L_3(q)$ and $[V, K]$ contains a direct sum of two natural modules or a natural module and a dual one. In case of $L_4(q)$ also the orthogonal module is involved.*

(xxii) *$K \cong L_2(q)$ and there are exactly two natural modules involved.*

(xxiii) *$K \cong L_5(2)$ or $L_4(2)$, $[V, K]$ is a direct sum of two natural and two dual modules. A Sylow 2–subgroup of X acts transitively on theses modules. Further $m_p(X) \leq 2$ for all primes $p > 3$ and $m_3(X) = 3$.*

(xxiv) *$K \cong L_6(2)$, $[V, K]$ is a direct sum of at least six natural modules, $p = 7$ and $r = 4$.*

(xxv) *$K \cong L_n(2)$, $3 \leq n \leq 5$, $[V, K]$ has a submodule which is a sum of at least three natural modules, $m_p(K) = 1$ and $r = 3$.*

(xxvi) *$L \cong Sz(q)$, $[V, L]$ is a nonsplit extension of a trivial module of order at most q by the natural module.*

(β) *$r \leq 2$.*

Proof: We go over the possibilities for K and V as given in 3.29, 3.30 , 3.31 and 3.32. We may assume that we do not have (5). Let first $m_p(C_X(V)) = 2$ for some prime p with $m_p(X) \geq 3$. Now a Sylow p–subgroup of $C_X(V)$ has a characteristic subgroup which is either elementary abelian of order p^2 or extraspecial of order p^3. By Frattini agrument we get that $C_X(V)$ contains a good E and so we have (1). Hence from now on we may assume that $m_p(C_X(V)) \leq 1$ for all odd primes p with $m_p(X) \geq 3$.

Let first $K \cong A_n$. Let $p = 3$. If $n \geq 9$ then we have (3) or (4), or $n = 9$ and we have some spin module involved. As this module is not an F-module and offenders on the permutation module are not overoffender, we see with 3.37 that $[V, K]$ is the spin module. Now as we not are in (4), we have that $m_p(X) \leq 2$ for all primes $p > 3$, which is (ix). The case $n = 8$ will be handled as $\Omega^+(6, 2)$ and $L_4(2)$.

So we may assume that $p > 3$. Then $n \leq 11$. If $m_p(K) \geq 2$, we have that $p = 5$ and $n = 10$ or 11. Now we have a permutation submodule, such that any element is centralized by some p–element in K and $C_V(K)$ is centralized by some good E, this is (4). So we are left with $m_p(K) \leq 1$. If $m_p(K) = 1$, we have (4). If $m_p(K) = 0$, we have (1).

Let now K be sporadic. Then by 3.32 $K \cong M_{12}$, M_{22}, M_{23}, M_{24}, $3M_{22}$ or J_2. As none of these possesses an F–module, we have that $[V, K]$ involves just one nontrivial irreducible module. Let $m_p(K) \leq 1$, then we have (3),(4) or (2). Hence we may assume that $m_p(K) \geq 2$. If K is one of the Mathieu groups, not $3M_{22}$, then every element in $[V, K]$ is centralized by some 3–element and $C_V(K)$ is centralized by some good E and so we have (3) or

(4). In the remaining case we have (5)(xi).

Let now K be of Lie type in odd characteristic. Then by 3.31 we have $K \cong 3U_4(3)$, and so $C_V(T)$ is centralized by a good E, which is (2).

So we have to treat the case of $K/Z(K)$ is of Lie type in characteristic two. We will first assume that $[V, K]$ involves exactly one nontrivial irreducible module, which is some $V(\lambda)$. Let P be the parabolic corresponding to λ. If P contains a good E, then we have (2). So we may assume that P does not contain a good E. This immediately shows that $K \not\cong E_6(q)$. If we have $F_4(q)$, we get $q = 2$ and so $r = 3$. Now we are either in (3),(4) or (2).

Let $K \cong G_2(q)$. We may assume that we are not in (2) or in (3), (4) with $m_p(K) = 1$. Then we have $m_p(K) \geq 2$ and so p divides $q^2 - 1$. In particular $m_p(K) = 2$. If $C_{[V,K]}(K) = 1$, we would get that any element in $[V, K]$ is centralized by some good E. If $C_V(K) \neq 1$, we have (4)(a) otherwise we have (4)(b). So we may assume that $C_{[V,K]}(K) \neq 1$, then we have (5)(iv).

Let next $K \cong Sz(q)$ and the module involved be the natural module. Then all Sylow p–subgroups of K are cyclic. If $C_{[V,K]}(K) = 1$, so we have that any element in $[V, K]$ is centralized by a good E, so we have either (3) or (4). So we must have a nonsplit extension and so by 3.37 we get (5)(xxvi).

Let now $K \cong \Omega^{\pm}(2n, q)$. Suppose that the nontrivial module in $[V, K]$ is the natural module. If we are not in (2), then $K \cong \Omega^-(4, q)$, $\Omega^{\pm}(6, q)$ or $\Omega^-(8, q)$. Let $K \cong \Omega^-(4, q)$, then all Sylow p–subgroups are cyclic. By 3.37 we have that $C_{[V,K]}(K) = 1$. Hence every element in $[V, K]$ is centralized by some good E, so we have (3) or (4).

Let $K \cong \Omega^+(6, q)$. If p divides $q^2 - 1$, then we have (2). So we may assume that $m_p(K) \leq 1$. Then we see that we have either (3) or (4).

Let $K \cong \Omega^-(6, q)$. By 3.36 we have that $C_{[V,K]}(K) = 1$. So if $m_p(K) \leq 1$, we have (3) or (4). So we may assume that $m_p(K) \geq 2$. In particular p divides $q^2 - 1$. If $m_p(X) \geq 4$, then we get that any element in $[V, K]$ is centralized by some good E, and so we have (3). Hence we have $m_p(X) \leq 3$ and then we have (5)(ii).

Let finally $K \cong \Omega^-(8, q)$. By 3.36 we have $C_{[V,K]}(K) = 1$. If $m_p(K) \leq 1$, we get (3) or (4). So we have that p divides $q^2 - 1$. Then we have (2).

Let next the module in $[V, K]$ be the half spin module. If we are not in (2), we have $K \cong \Omega^{\pm}(6, q)$ or $\Omega^-(8, q)$. Let first $K \cong \Omega^-(8, q)$. If $r \geq 4$ but $m_p(K) \leq 1$, we get (3) or (4). So we have $m_p(K) \geq 3$. Further by 3.36

$C_{[V,K]}(K) = 1$. If $r \geq 4$, then every element in $[V, K]$ is centralized by some good E, so we have (3) or (4). Let $r = 3$. If p divides $q - 1$, then we have (2). Hence $q - 1 = 1$, so $q = 2$ and $p = 3$. This is (5)(xii).

$\Omega^-(6, q)$ on the half spin module will be treated as $U_4(q)$ and $\Omega^+(6, q)$ will be treated as $L_4(q)$ on the natural module.

Let next $K \cong U_n(q)$ on the natural module. Then we have (2) or $K \cong U_3(q)$ or $U_4(q)$. Then by 3.36 we have that $C_{[V,K]}(K) = 1$ or $K \cong U_4(2)$ in which case the centralizer in $[V, K]$ may have order up to four. If $m_p(K) \leq 1$, we get (3) or (4). Let $K \cong U_3(q)$. Then we have (5)(xiiii). If we have $K \cong U_4(q)$ and p divides $q - 1$, then we see that any element is centralized by a good E and so we have (1),(3) or (4). Hence we are left with p divides $q + 1$ and $m_p(K) = m_p(X)$, which is (5)(viii) or (5)(v).

Let next $K \cong Sp(2n, q)$. Let first the natural module be involved. Then we have (2) or $n \leq 3$. Let $K \cong Sp(6, q)$ then we have (2) besides p does not divide $q^2 - 1$, which gives $m_p(K) \leq 1$. But then we have (3) or (4). For $K \cong Sp(4, q)$ we get (5)(xiv).

Let next the spin module be involved. Then we get (2) or $K \cong Sp(6, q)$ and p does not divide $q - 1$. So assume the latter. If $m_p(K) \leq 1$, we have (1), (3) or (4). So we may assume that p divides $q + 1$. If $r > 3$, then we see that all elements in $[V, K]$ are centralized by some good E, as by 3.36 $C_{[V,K]}(K) = 1$. Hence we have (3) or (4). So let $r = 3$, then we get $q - 1 = 1$, and so $q = 2$, which is (5)(vii).

Let finally $V(\lambda_2)$ be involved and so $K \cong Sp(6, q)$ or $Sp(8, q)$. If we are not in (2) we have $K \cong Sp(6, q)$ and p does not divide $q^2 - 1$, hence $m_p(K) \leq 1$. Then we have (1), (3) or (4).

Let next $K/Z(K) \cong L_n(q)$. Suppose that the natural module is involved. Let first $q = 2$. If $p = 3$, we have (2) or $K \cong L_4(2)$ or $L_3(2)$. Now there is a p–element centralizing $[V, K]$ and so we have (3) or (4). So we may assume that $p \neq 3$. If $m_p(K) \leq 1$, we have (1), (3) or (4). So we have $p = 7$ and $K = L_6(2)$ or $L_7(2)$. But then there is a 7–element centralizing $[V, K]$ and so we have (3) or (4).

So we may assume $q > 2$. By 3.36 we have that $C_{[V,K]}(K) = 1$ or $K \cong L_2(q)$. If $m_p(K) \leq 1$, then we have (1), (3) or (4) as all elements in $[V, K]^\sharp$ are conjugate, or $C_{[V,K]}(K) \neq 1$ and we have $K \cong L_2(q)$. But then we have (5)(i). So we may assume $m_p(K) \geq 2$. If p divides $q - 1$, we get (2) or $K \cong L_3(q)$. As $C_{[V,K]}(K) = 1$, all elements in $[V, K]^\sharp$ are conjugate, so we get that any is centralized by a good E, which is (3) or (4). So we now may assume that

p does not divide $q-1$. In particular $n=4$ and p divides $q+1$. As any element is centralized by some p–element in K, we see that any element is centralized by some good E, which is (3) or (4).

Assume next that $V(\lambda_2)$ is involved. Then we have (2) or p does not divide q^2-1. As $L_4(q)$ has been handled as $\Omega^+(6,q)$, we may assume that $n \geq 5$. Hence we must have $q=2$. Now $m_p(K) \leq 1$ and so we have (1), (3) or (4).

Let finally $K \cong L_6(q)$ and $V(\lambda_3)$ be involved, then we have (2) or $q=2$. Further we may assume that $p \neq 3, 7$ as this also would lead to (2). Then $m_p(K) \leq 1$ and so we have (1), (3) or (4).

Let now still just one nontrivial module be involved, but this let be a tensor product of two algebraically conjugate natural modules for either $L_n(q^2)$ or $Sp_4(q^2)$. In the first case we have (2) or $n \leq 4$. Let $n=4$, we get $m_p(K) \leq 1$, otherwise we have (2). But then we get (1), (3) or (4). Let $n=2$, this is just the orthogonal module, a case handled before. Let $n=3$. If $m_p(K) \leq 1$, we get (1),(3) or (4). So let $m_p(K)=2$, in particular p divides q^2-1. If $C_V(K) \neq 1$, then as any element in $[V,K]$ is centralized by a p–element, we have (3) or (4). So we may assume that $C_V(K)=1$ and then $r=3$. This is (5)(xv).

So we have $Sp(4,q^2)$. If $m_p(K) \leq 1$, we get that p does not divide q^2-1. Then we get (1), (3) or (4). Hence we have that p divides q^2-1 and so we have (5)(xvi).

Now we may assume that $[V,K]$ involves at least two nontrivial modules, where one of these now has to be an F-module. This gives $K \cong G_2(q)$, $L_n(q)$, $U_n(q)$, $\Omega^{\pm}(2n,q)$ or $Sp(2n,q)$.

If we have $K \cong G_2(q)$ then there are exactly two natural modules involved, as there are just exact F–module offender on the natural module, which is (5)(xvii).

Let $K \cong U_n(q)$, then just natural modules are involved. So we have (2) or $n=4$ and as in the $G_2(q)$–case we get (5)(xvii).

Let $K \cong Sp(2n,q)$. Suppose $n \geq 4$. Then we may assume that p divides q^2-1. If we just have natural modules, then we have (2). As the spin module is not an F–module, we get that one of the modules involved has to be a natural module. So we have the natural module and the spin module involved. Hence we are in (2) or $q=2$. But in the spin module and also in the natural module every element is centralized by a good E, so we have (3) or (4).

So we have $n \leq 3$. Let $n = 3$. Let first p divide $q^2 - 1$. Then as above we have (2) or there is a spin module or $V(\lambda_2)$ involved. As there are at least two modules involved, we see that $V(\lambda_2)$ is not possible. Hence we have twice the spin module or the spin module and the natural module. In any case there is no module in $[V,K]/C_{[V,K]}(K)$ which is the natural module. This is (5)(xviii). Suppose now that $m_p(K) \leq 1$. If $m_p(K) = 0$, there is a good E centralizing K and so it centralizes $[V,K]$, which is (1). So we have $m_p(K) = 1$. Now also $r > 3$ and so there is a good E centralizing K and then also $[V,K]$, which gives (3).

Let finally $K \cong Sp(4,q)$, then there are exactly two 4–dimensional modules involved, which is (5)(xix).

Let next $K \cong \Omega^{\pm}(2n,q)$. Then the modules involved are natural ones or half spin modules. Let first $n \geq 5$, then we have (2) or $n = 5$, $q = 2$ and both natural and half spin modules are involved. If we have $\Omega^-(10,2)$, then we have a natural or half spin submodule W, which is invariant under $N_S(K)$. But then we have (3) or (4) or (1). Suppose we have $\Omega^+(10,2)$, then we have the same conclusion, or there are two half spin modules interchanged by a Sylow 2–subgroup of X. Further there is a natural module involved. But this cannot be a $2F$– module as on the half spin modules we have exact offenders as F–modules.

Let now $K \cong \Omega^+(8,q)$. If there are three different modules involved then we do not have a $2F$–module. Hence we have that $C_{[V,K]}(K)$ is centralized by $SL_3(q)$, so we have (2) or $q = 2$. If one half spin module is invariant under a Sylow 2-subgroup of X we have (3). So we just have that there are two of them interchanged by a Sylow 2–subgroup of X, which is (5)(xx).

Let next $K \cong \Omega^-(8,q)$. As the half spin modules are not F-modules, we have at most one involved. So we have a natural module in $[V,K]$. If $m_p(K) \leq 1$, we see that we get (1), (3) or (4). So we have that p divides $q^2 - 1$. If we have a natural submodule, we get (3) or (4). So we just have the half spin module as submodule. Then we have $m_p(X) = 3$ and $q = 2$. This is (5)(xii).

$K \cong \Omega^+(6,q)$ will be handled as $L_4(q)$, $K \cong \Omega^-(6,q)$ has been handled as $U_4(q)$ and $K \cong \Omega^-(4,q)$ will be handled as $L_2(q^2)$.

Let now $K/Z(K) \cong L_n(q)$. We may assume that we are not in (2). Then $n \leq 7$. Suppose first that no natural modules are involved. Then $V(\lambda_2)$ is involved. Hence $L_2(q) \times L_2(q)$ centralizes $C_V(T)$, which gives (2) or $q = 2$, or $n = 4$.

Suppose first $n = 4$. Then p does not divide $q^2 - 1$. In particular $m_p(K) \leq 1$. As there are at most two nontrivial modules involved, we see that there is a good E, centralizing $[V, K]$ and so we have (1), (3) or (4).

Let next $q = 2$, $n \geq 5$. We now have $p \neq 3$. Again there are at most two nontrivial modules involved. If we do not have (1), (3) or (4), we see that there is no good E centralizing K. In particular $r = 3$. Then there is no elementary abelian 3-subgroup of order 27 in K, so $K = L_5(2)$, but then $m_p(K) = 1$, a contradiction.

So we have shown that there are natural modules involved. Let first $q > 2$. As not $V(\lambda_2)$ and $V(\lambda_{n-2})$ can both be involved, because of exact offenders, we have that $n \leq 5$, otherwise we have (2).

Let $n = 5$. Then we must have $V(\lambda_1)$, $V(\lambda_2)$ and $V(\lambda_4)$ be involved in $[V, K]$. But then there are no more modules. Now we either have $V(\lambda_1) \oplus V(\lambda_4)$ or $V(\lambda_2)$ as a submodule. But in both modules any element is centralized by a good E, so we have (3).

Let $n = 4$. Then as we have a $2F$–module not all three types of modules can be involved. So assume that $V(\lambda_2)$ and $V(\lambda_1)$ are involved. If $V(\lambda_1)$ occurs just once, then we see that we have (1), (3) or (4). Hence we have (5)(xxi).

Let now $n = 3$. Then just natural and dual modules are involved, altogether at most 4 such modules. This is (5)(xxi).

Let next $n = 2$. As the tensor product is not an F–module and offenders for the natural module are exact, we have that at least two natural modules are involved, which is (5)(xxii).

So let now $q = 2$. Suppose first $p = 3$. Then as above, we get $n \leq 5$. In case of $L_3(2)$, we have (5)(xxi). Let $n = 4$ or 5. Let ρ be a 3–element in $C_X(K)$. Suppose $[[V, K], \rho] = [V, K]$, then $r = 3$ and any module involved occurs twice. Let now first $n = 5$, then $V(\lambda_2)$ is not involved. Hence we have two natural and two dual modules involved. Further S has to act transitively on these modules. This is (5)(xxiii). So let $C_{[V,K]}(\rho) = W \neq 1$. Again any submodule in here has to occur at least twice, otherwise we have (3) or (4). This shows that we must have two natural modules and two dual ones, which then shows that $[[V, K], \rho] = 1$, again (5)(xxiii).

Let next $K = L_4(2)$. Suppose first $[[V, K], \rho] = [V, K]$. Again $V(\lambda_2)$ cannot occur. If we just have two natural modules involved we have (5)(vi). So assume that we have two natural and two dual modules involved, we have (5)(xxiii). So assume that $W = C_{[V,K]}(\rho) \neq 1$. Then again submodules of W have to show up at least twice. We cannot have just two natural modules in W, this would lead to (3) or (4). So we must have two natural and two dual ones, which is (5)(xxiii) again.

So we are left with $p \neq 3$. Let first $m_p(K) = 2$, then $n = 6$ or 7 and $p = 7$. Further $r > 3$. In particular there is a good E centralizing K. Let $E = \langle \nu, \tau \rangle$ and assume $W = C_{[V,K]}(\nu) \neq 1$. Then $[W, \tau] = W$, as $[E, V] \neq 1$. This now shows that we have $C_V(K) \leq [V, K]$, otherwise we would have (3) or (4). Assume next $[\nu, V] = 1$. Then $W = [V, K]$ and we have (2) as just natural or dual modules are involved. So we have $W_1 = [C_V(\tau), \nu] \neq 1$. Now we see that there are just natural and dual modules in $[V, K]$. If $n = 7$, we have to have both types. This shows that we have exactly three natural modules and three dual ones. As in the direct sum of three natural modules still any element is centralized by some $L_4(2)$, we get (4). So we have $K \cong L_6(2)$. If we have both types of modules, we get as before (4). So we just have natural modules involved. This in (5)(xxiv).

So we now have $m_p(K) \leq 1$. If $r - m_p(K) > 2$, we have (1), (3) or (4). So we may assume $r = 3$ and $m_p(K) = 1$. Hence we have that $n \leq 5$. Let again $E = \langle \nu, \tau \rangle$ in $C_X(K)$. We may assume that $W = [C_{[V,K]}(\nu), \tau] \neq 1$. Hence any module shows up at least three times. This gives that just natural or dual modules can be involved. An easy inspection shows that we can just have a $2F$–module if all these modules are equal. This is (5)(xxv). □

Sp6over

Lemma 3.44 *Let $G = Sp(6, q)$, q even. Let V be some $GF(2)G$–module, which involves exactly two nontrivial irreducible modules, where one is the spin module and the other is either the natural module or the spin module again. Let A be a quadratic offender on V as a $2F$–module. If $|A| \geq q^4$, or $|A| = q^3$, then for both modules W we have that $|W : C_W(A)| = q^4$, q^3, respectively.*

Proof: First of all we see that A has to induce an F–module offender on one of the two modules.

Let first $|A| \geq q^4$. If $A \leq O_2(P)$, where P is the point stabilizer in the natural module, then we have that for the spin module W we have $|W : C_W(A)| = q^4$ and $|A| \leq q^5$. But then on the natural module A cannot act quadratically, so we have that the second module again is a spin module and we have the assertion.

So we may assume that $A \not\leq O_2(P)$. Then as $|A| \geq q^4$, we have at least $A \cap O_2(P) \neq 1$. As $[C_W(O_2(P)), A] \neq 1$, W the spin module, we see that $A \cap O_2(P)$ just can contain elements of type a_2. In particular $|A \cap O_2(P)| \leq q^2$. If A intersects more than one root group in $O_2(P)$, then we get that $|[W, A \cap O_2(P)]| = q^3$ and so A has to induce a transvection group on $C_W(O_2(P))$, which shows that $|A/A \cap O_2(P)| \leq q$, a contradiction. So we have that $|A \cap O_2(P)| = q$ and $|A/A \cap O_2(P)| = q^4$. Now A corresponds to the centralizer of a 2–space in the natural module, as $|[W, A \cap O_2(P)]| = q^2$. This gives that $|W : C_W(A)| = q^4$. If both modules in V are spin modules, we are done. So assume that W_1, the natural module is involved. Then $|C_{W_1}(O_2(P))| = q$ and on $[W_1, O_2(P)]/C_{W_1}(O_2(P))$, we have that P induces a four dimensional module which is not isomorphic to the ones induced in W. Hence we see that $|W_1 : C_{W_1}(A)| = q^4$, the assertion.

Let now $|A| = q^3$. It is enough to show that A is not an over offender on the spin module W and the natural module W_1. Suppose first that A does not contain transvections. Then with 3.17 we get that A has to induce a strong F–module offender on W_1, i.e. $|W_1 : C_{W_1}(A)| = q^2$. Let P be as before, then we get that $|A \cap O_2(P)| \leq q$. But as all elements in A have the same centralizer, we get that $A/A \cap O_2(P)$ is a group of order q^2 in $Sp(4, q)$, which induces transvections to a hyperplane in the natural module, or $A \cap O_2(P) = 1$. As $Sp(4, q)$ does not have a transvection group of order q^2, we get that $A \cap O_2(P) = 1$. But now we have that A corresponds to the 2-space stabilizer in the $Sp(4, q)$–modules in the spin module and so it corresponds to the point stabilizer in the module involved in the natural $Sp(6, q)$–module and so $|W_1 : C_{W_1}(A)| \geq q^3$, a contradiction.

So we have that A contains transvections x. In particular $[W, x] = C_W(A)$ and so $A \leq O_2(P)$. But there are no over offender on the natural module in $O_2(P)$. □

minmod

Lemma 3.45 *Let X be a group, denote by min(X) the minimal dimension of a nontrivial X-module over $GF(2)$. Ten we have for $r = 2^n$ that min$(G_2(r)) = 6n$, min$(Sp(2m, r)) = 2mn$, min$(L_m(r) = mn$, min$(U_5(r)) = 10n$, min$(\Omega^-(2m, r)) = 2mn$, min$({}^3D_4(r)) \geq 12n$ and min$({}^2F_4(r)) \geq 12n$.*

Proof: For all the values in the assertion there is one module, the natural one. Hence we just have to show that this module in fact is the minimal one. Let X be one of $G_2(r)$, $Sp(2m, r)$, $L_m(r)$, $U_5(r)$, $\Omega^-(2m, r)$, and p be a Zsygmondy prime dividing $r^6 - 1$, $r^{2m} - 1$, $r^m - 1$, $r^5 + 1$, $r^m + 1$ respectively. Then the smallest $GF(2)$–module on which an element of order p can act nontrivially is of dimension $6n$, $2mn$, nm, $2mn$, $8n$, respectively. Hence in these cases the bounds are sharp. Assume that we do not have a Zsigmondy prime, then we have $G_2(2)$, $Sp(6, 2)$, $L_6(2)$ or $\Omega^-(6, 2)$. In all cases we have an nonabelian Sylow 3–subgroup and so the smallest dimension for this group will be 6.

Let now $X \cong^2 F_4(r)$. Then r^6+1 divides $|X|$. If $X \cong^3 D_4(r)$, then $r^8+r^4+^1$ divides $|X|$. In both cases we may choose p as a Zsigmondy prime dividing $r^{12}-1$, which yields the assertion. □

lower

Lemma 3.46 *Let $F^*(X) = L$ be a quasisimple Lie group in odd characteristic $p, Z(L)$ a p'-group, $L \not\cong L_2(q)$, ${}^2G_2(q)$, $G_2(q)$, ${}^3D_4(q)$ or $PSp_4(3)$. Let V be a faithful $GF(2)$-module for X and $t \in X$ be some involution. Then $m([V,t]) \geq (q-1)q^w d(p)/2\varepsilon p$, where $d(p)$ is the degree of the smallest nontrivial representation of Z_p, q^{2w+1} is the order of $O_p(C_L(R)), R$ a long root group (see table below) and $\varepsilon = 1$ or $L \cong PSp_{2n}(q), \varepsilon = 2, q > p$.*

L	$L_n(q)$	$U_n(q)$	$\Omega_n^\pm(q)$	$PSp_{2n}(q)$	$F_4(q)$	$E_6(q)$	${}^2E_6(q)$	$E_7(q)$	$E_8(q)$
w	$n-2$	$n-2$	$n-4$	$n-1$	7	10	10	16	28

Proof: This is [Asch, (10.4)]. We just sketch his proof. Let R be a long root group in X and $Q = O_p(C_L(R))$. Then Q is a special group of order q^{1+2w}, where w is as described above.

By [Asch, (10.1)] we may assume that t inverts some $U, |U| = p, U \leq Q, U \not\leq R$, or $X = PSp_{4k}(q)$ and t induces a field automorphism. In the former we get the assertion with [Asch, (7.2)].

So assume the latter. Now $E(C_L(t)) \cong PSp_{2k}(q^2)$ and $t \sim tz, tz$ induces a nontrivial inner automorphism on $C_L(t)$.

We can proceed by induction. First of all by 3.26 $[[V,t], E(C_L(t))] \neq 0$. Set $W = [V,t]$. Let first $k = 1$. Then $E(C_L(t)) \cong L_2(q^2)$. Let R_1 be a subgroup of order q^2 in $E(C_L(t))$. Then $N_{E(C_L(t))}(R_1)$ has at most two orbits on the hyperplanes of R_1. Now there is some orbit $\triangle$ with

$$[W, R_1] = \bigoplus_{H \in \triangle} C_{[W,R_1]}(H)\,.$$

This shows $m([W, R_1]) \geq (q^2-1)d(p)/2(p-1) > (q-1)\,q^w d(p)/2p$, since $w = 1$.

Let now $k > 1$. Then by induction

$$m([V, tz]) \geq (q^2-1)q^{2u} d(p)/4p\,,$$

where $u = k-1$ and $w = 2k-1$. Hence $w = 2u+1$ and so

$$m([V,t]) \geq (q^2-1)q^{w-1}d(p)/4p > (q-1)\,q^w d(p)/4p\,.$$

□

Lemma 3.47 *Let $F^*(X) = L$ be a quasisimple group such that $L/Z(L)$ is a Lie group over a field of odd characteristic which is not a Lie group over a field of characteristic 2 too. Let $t \in X$ be an involution and V be an irreducible faithful $GF(2)$-module for X. Then one of the following holds:*

(1) $|V : C_V(t)| \geq 2^8$

(2) $L \cong L_3(3)$, $|V : C_V(t)| \geq 2^4$

(3) $L \cong U_4(3), L_4(3), G_2(3), 3 \cdot G_2(3), PSp_6(3)$ *and* $|V : C_V(t)| \geq 2^6$

(4) $L \cong 3 \cdot U_4(3)$

(5) $L \cong L_2(25)$ *or* $L_2(p), p$ *prime,* $|V : C_V(t)| \geq 2^4$

Proof: ([Asch, (10.5)]). Assume $|V : C_V(t)| \leq 2^7$. Let first $L \cong L_2(q), q$ odd. Suppose furthermore $q = p^f > p$. Let $t \in PGL_2(q)$. Then $L\langle t\rangle$ is generated by three conjugates of t and so $|V| \leq 2^{21}$. Let $P \in \mathrm{Syl}_p(L)$ and $\triangle$ be one orbit of hyperplanes under $N_G(P)$. Then $V = \bigoplus_{U \in \triangle} C_V(U)$. We have $|\triangle| = (q-1)/(p-1)$ or G does not induce $PGL_2(q)$ on L and $|\triangle| = (q-1)/2(p-1)$. Let $d\,(p) = |C_V(U)|$. We get $d\,(p)(q-1) \leq 42(p-1)$. As $d\,(p) \geq 2$, we get a rough bound by $p+1 \leq 21$, and so $p \leq 19$. If $p > 7$, then $d\,(p) \geq 8$ and so $p+1 \leq 5$, a contradiction. Hence $p = 3, 5$ or 7. Let $p = 7$, then $d\,(p) = 3$ and so $q - 1 \leq 14(p-1)$. This shows $L \cong L_2(7^2)$ and we have exactly two orbits on the hyperplanes. But now t inverts P, a contradiction.

Let $p = 3$. Then $f \geq 3$ as $L_2(9) \cong Sp_4(2)'$. Now $L \cong L_2(27)$. But then there is just one orbit on the hyperplanes and we have $2 \cdot 26 \leq 21 \cdot 2$, a contradiction. Let finally $p = 5$. Then we get $L \cong L_2(25)$. As $|V| \geq 2^9$, we get $|V : C_V(t)| \geq 2^4$.

Suppose next that t induces a field automorphism on L. Let $L_1 = E(C_L(t))$ (recall $q > 9$). Suppose $[[V,t], L_1] = 0$. Then we have a quadratic fours group and so we get a contradiction with 3.26. Suppose now $[[V,t], L_1] \neq 0$. Then L_1 acts faithfully on $[V,t]$ and $|[V,t]| \leq 2^7$. In particular $L_1 \cong L_2(p), p \Big| |L_7(2)|$, or $L_1 = L_2(9)$. But in the latter $PGL_2(9)$ acts on $[V,t]$ and so $|[V,t]| \geq 2^8$. We are left with $L_2(5), L_2(7), L_2(31)$ and $L_2(127)$. But in all cases $PGL_2(p)$ is involved and so there is some involution s which inverts a Sylow p-subgroup. This shows $|[V,t]| \geq 2^{14}(L_2(127)), |[V,t]| \geq 2^{10}(L_2(31))$. Let $C_L(t) \cong PGL_2(7)$. Then $|[V,t]| \geq 2^6$. There is $z \in L_1, o(z) = 2$, such that $t \sim tz$. Hence we see that $|[V,P]| \leq 2^{18}$ for $P \in \mathrm{Syl}_7(L_1)$. As $L_2(7^2) \not\leq L_{18}(2)$, we see that $C_V(P) \neq 1$. Let $\triangle$ be the orbit of P in

$N_G(E), E \in \mathrm{Syl}_p(L), P \leq E$. Then $|\triangle| = 4$ and $V = \bigoplus_{U \in \triangle} C_V(U)$. Now there is some $U \in \triangle$ with $[U,t] = U$. This shows $|C_V(P) \cap [V,t]| \geq 8$. But then $|[V,t]| \geq 2^9$, a contradiction.

Let now $L \cong L_2(25)$. We may assume $|[V,t]| \leq 8$. Hence $[[V,t], L_1] = 1$. As t inverts some elements of order 13, we see that $L\langle t\rangle$ is generated by three conjugates of t. Now $|V| \leq 2^9$ but $13 \not| |L_9(2)|$.

So we are left with $L \cong L_2(p), p$ prime, $p \geq 11$. Suppose $|[V,t]| \leq 8$, then $|V| \leq 2^9$ and so $L \lesssim L_9(2)$. This shows $p = 31, 127, 73, 17$. As $37 \Big| |L_2(73)|$, but $37 \not| |L_9(2)|, L \ncong L_2(73)$. In $L_9(2)$ the normalizer of a Sylow 127-subgroup is of order $2 \cdot 3 \cdot 7 \cdot 127$, hence $L_2(127) \not\leq L_9(2)$. On a Sylow 31-subgroup just a group of order 5 is induced, as this is true in $L_5(2)$, hence $L_2(31) \not\leq L_9(2)$. As t cannot invert an element of order 17, we get $L\langle t\rangle \cong PGL_2(17)$ if $L \cong L_2(17)$. Now t centralizes a group of order 9. Hence we get some element of order 3 which centralizes $[V,t]$ and $V/C_V(t)$ as well. But $|C_V(t) : [V,t]| \leq 8$ and so this element centralizes V, a contradiction.

Let now $[[V,t], E(C_L(t))] = 0$, then we get the assertion with 3.26. So assume $[[V,t], E(C_L(t))] \neq 0$.

If $L \cong {}^2G_2(q)$, then $E(C_L(t)) \cong L_2(q), q \geq 27$. But $L_2(q) \not\leq L_7(2)$.

Let $L/Z(L) \cong G_2(q)$. If $t \in L$, then $C_L(t) \cong (SL_2(q) * SL_2(q)) \cdot 2$. Hence the structure of $L_7(2)$ shows $q = 3$ or 7. If $q = 3$, then t inverts some element of order 13 and so $|V : C_V(t)| \geq 2^6$. If $q = 7$, t inverts some element of order 817.

Let $t \notin L$. Suppose t induces a field automorphism. Then $E(C_L(t)) \cong G_2(\sqrt{q}) \not\leq L_7(2)$. So assume that $q = 3^f$ and $E(C_L(t)) \cong {}^2G_2(q)$. This shows again $q = 3$ and $L_2(8)$ acts on $[V,t]$, i.e. $|[V,t]| \geq 2^6$.

Let now $L/Z(L) \cong {}^3D_4(q)$. Then t acts on some $G_2(q)$ in L. Hence we may assume $q = 3$. Now t induces $PGL_2(27)$ on a subgroup $SL_2(27)$ of L. Hence we see $|[V,t]| \geq 2^8$ as before.

Suppose now that L is none of these groups but $p \not| |Z(L)|$. Then the conclusion follows from 3.46. If $L \cong L_3(3)$ then either t inverts an element of order 13 or an elementary abelian group $E \cong E_9$, with $N_L(E)$ transitive on $E^\sharp$. In both cases $|[V,t]| \geq 2^6$.

So we are left with $p \Big| |Z(L)|$. This now leaves us with $3 \cdot \Omega_7(3)$. As $3 \cdot \Omega_7(3) \not\leq L_{14}(2)$, we get $t \in C(Z(L))$. Furthermore we may assume $[Z(L), V] = V$. Now $E(C_L(t)) \leq GL_3(4)$. As $t \in L$, we see $E(C_L(t)) \cong U_4(3)$

or $Sp_4(3)$, or we have $C_L(t)/Z(L) \cong \Sigma_4 \times (SL_2(3) * SL_2(3)) \cdot 4$ [CCNPW]. But the first two are not in $GL_3(4)$. For the latter the embedding in $GL_3(4)$ gives some kernel which contains a fours group. Now we have a quadratic fours group and can apply 3.26 for a contradiction. □

cododd

Lemma 3.48 *Let $F^*(X) \cong G(r)$ be a Lie group over a field of odd characteristic which is not a Lie group in characteristic two too. Let V be a faithful irreducible $GF(2)$-module for X and t be an involution in X with $|[V,t]| \leq 2^{m_2(X)+1}$. Then $F^*(X) \cong 3 \cdot U_4(3)$, $L_3(3)$, $U_4(3)$, $L_4(3)$, $L_2(25)$, where we have equality in the last four cases and $m_2(F^*(X)) < m_2(X)$.*

Proof: (The proof follows [Asch, (10.9)]). Let first $F^*(X) \cong L_2(r)$. Then $m_2(X) \leq 3$ or $m_2(X) = 2$ for r prime . Now the assertion follows with 3.47. If $F^*(X) \cong {}^2G_2(r)$, $(S)L_3(r)$, $(S)U_3(r)$, $(S)L_4(r)$, $(S)U_4(r)$, $3 \cdot U_4(3)$, $PSp_4(r)$, $G_2(r)$, $3 \cdot G_2(3)$ or ${}^3D_4(r)$, then $m_2(X) \leq 6$ and the assertion follows with 3.47, unless $F^*(X) \cong L_3(3), U_4(3), L_4(3)$ or $3 \cdot U_4(3)$, where we have equality in the first three cases.

Let now $T \in \mathrm{Syl}_2(X), \triangle = \mathrm{Fun}(T), k = |\triangle|, Y = \langle \triangle \rangle$, and $YT^* = YT/C_{YT}(Y)$. Then Y^* is a direct product of k copies of $L_2(r)$ permuted by T. Let B^* be an elementary abelian subgroup of T^* of maximal order, $K \in \triangle, D^*$ a complement to $E^* = N_B(K)^*$ in $B^*, t = m_2(D^*)$, and $S = \langle K^D \rangle \cap T$. Then S^* is the direct product of 2^t copies of $L_2(r)$ and $\varepsilon = m_2(C_{K^*}(E^*)) = 1$ or 2. Hence $m_2(C_{S^*}(E^*)) = \varepsilon 2^t$ and $m_2(S^* \cap B^*) = m_2(C_{S^*}(B^*)) = \varepsilon$, so $m_2(E^* C_{S^*}(B)) \geq m_2(E^*) + \varepsilon 2^t - \varepsilon \geq m_2(E^*) + t = m_2(B^*)$, so that $E^* C_{S^*}(E^*)$ is also elementary abelian of maximal order. Thus we may choose B to fix each member of $\triangle$. Let C^* be the subgroup of B^* inducing inner automorphisms in $PGL_2(r)$ on each member of $\triangle^*$. Then $m_2(B/C) \leq 1$ and $m_2(C^*) \leq 2k$, so $m(B^*) \leq 2k+1$. Set $Z = T \cap Z(Y)$. Then $m_2(Z) \leq k$ and $m_2(C_T(Y)) \leq m_2(Z) + i(F^*(X))$, where $i(F^*(X))$ can be found in [Asch, (10.8)]. So we have

$$m_2(X) \leq 3k + 1 + i(F^*(X))\,, i(F^*(X)) \leq 3\,.$$

Let m be the lower bound for $[V,t]$ supplied by 3.46. Then

$$m \geq (p-1)p^{w-1}d(p)/2\varepsilon \geq 2(3^{w-1})\varepsilon,$$

where $r = p^s, w$ is given by $|O_p(C_{F^*(X)}(R))| = r^{1+2w}, R$ some root group in $F^*(X), \varepsilon = 1$ or $F^*(X) = PSp_{2n}(r), r > p$, where $\varepsilon = 2$. Furthermore k is listed in [Asch1, Theorem 2]. In particular $k \leq w$ unless $F^*(X) = PSp_{2n}(r)$, where $k = n$ and $w = n-1$.

So assume $F^*(X) \neq PSp_{2n}(r), n \geq 3$. Then

$$2(3^{w-1}) > 3w + 5 \geq 3k + 2 + i(F^*(X))\,.$$

Hence the lemma holds.

So let $F^*(X) \cong PSp_{2n}(r)$. Then $i(F^*(X)) = 0, k = n, w = n-1$. Hence

$$(p-1)r^{n-1}d(p)/2\varepsilon p > 3n+2 \geq m_2(X)+1\,,$$

unless $F^*(X) \cong PSp_6(3)$. But then $|[V,t]| \geq 2^6$. As $m_2(\mathrm{Aut}\ (PSp_6(3))) = 4$, the lemma holds. □

codspor

Lemma 3.49 *Let $F^*(X)$ be a perfect central extension of a sporadic simple group, V be a faithful $GF(2)X$-module. Then $m_2(G)$ and the minimal codimension of $C_V(t)$ in V, for t an involution in X are listed in table 1*

Proof: This is [Asch, (11.1)]. □

L2q

Lemma 3.50 *Let $X \cong Sz(q)$, $U_3(q)$ or $L_2(q), q$ even, and V be some faithful $GF(2)$-module for X.*

(i) We have $|[V,t]| \geq q^2, q^2, q$, respectively, where t is some involution in X.

(ii) Let $X \cong Sz(q)$, V be irreducible and $|V : C_V(t)| \leq q^2$ for some involution t. Then V is the natural module.

(iii) Let $q > 2$, $T \in Syl_2(X)$ and $S = \Omega_1(T)$. If $V = C_V(S) \bigoplus C_V(S^g)$ for some $g \in X$, then V is a direct sum of natural modules.

(iv) Let $X \cong L_2(q)$ or $Sz(q)$, $T = \Omega_1(S)$, $S \in Syl_2(X)$. If V is irreducible with $[V,T,T] = 0$, then V is the natural module.

Proof: (i) Let $X \not\cong U_3(q)$. We have that t inverts an element of order $q+\sqrt{2q}+1, q+1$, respectively, and so $|[V,t]| \geq q^2, q$.

So let $X \cong U_3(q)$. Let $U \leq X$ with $U \cong \mathbb{Z}_{q+1} \times L_2(q)$ and $t \in U$. Assume further that $|[V,t]| < q^2$. Then we get that there is just one nontrivial irreducible $L_2(q)$–module in V, which then is centralized by $\mathbb{Z}_{q+1}$. Hence we have that $\mathbb{Z}_{q+1}$ acts trivially on $[V,t]$ and so, as this group acts irreducibly on $S/Z(S)$ for a Sylow 2-subgroup S containing t, we have that $Z(S)$ acts quadratically and so $Z(S) \leq U$ yields $|[V,Z(S)]| = q$. But we can generate X by three conjugates of $Z(S)$, which now gives $|V| \leq q^3$ contradicting the fact that the order of X is divisible by q^3+1.

(ii) By 1.14 X is generated by three conjugates of t. Hence $|V| \leq q^6$.

$F^*(X)/Z(F^*(X))$	$m_2(X)$	*codimension of* $C_V(t)$
M_{11}	2	4
M_{12}	4	4
M_{22}	5	3
M_{23}	4	4
M_{24}	6	4
J_1	3	8
J_2	4	4
J_3	4	6
Mc	4	8
Ly	4	33
HS	5	6
He	6	10
Sz	6	8
Ru	6	12
$O'N$	4	21
Co_3	4	8
Co_2	10	6
Co_1	11	8
$M(22)$	10	18
$M(23)$	11	18
$M(24)'$	12	18
F_5	6	40
F_3	5	9
F_2	22	54
F_1	24	54
J_4	11	50

Table 1

Set $W = V \bigotimes_{GF(2)} GF(q)$. By the tensor product lemma $W = \bigoplus M\sigma$ where M is a tensor product of algebraic conjugates of the natural module N. Let $q = 2^n$, then $\dim W \le 6n$. But this shows $W = N$.

(iii) We have that for $t \in S, t$ inverts some $\omega \in X, o(\omega) = 3$ for $X \cong L_2(q)$ and $o(\omega) = 5$ for $X \cong Sz(q)$. As $\langle S, S^g\rangle = X$ by 1.14, we may assume $\omega = g$. Now $C_V(\omega) = 0$. The assertion follows with [Hi, (8.2)] and [Mar].

(iv) By 1.14 there is $g \in X$ with $\langle T, T^g\rangle = X$. Hence $V = [V, T] + [V, T^g]$. Furthermore $C_V(T) \cap C_V(T^g) = 0$. As $[V, T, T] = 0$, we get $V = C_V(T) \bigoplus C_V(T^g)$. The assertion now follows with (iii). $\square$

normalpar

Lemma 3.51 *Let $X = G(r)$ be a Lie group, $r = 2^n$, $X \not\cong Sz(r)$, $L_2(r)$, $U_3(r)$. Let $S \in Syl_2(X), A \lhd S, A$ elementary abelian. Then there is some parabolic P of $X, O^{2'}(P/O_2(P)) \cong L_2(r), L_2(r^2)$ or $U_3(r)$, such that $A \le O_2(P)$. If $X \cong {}^2F_4(r), A \le O_2(P)$ for both minimal parabolics. If $X \cong {}^3D_4(r)$ then $O^{2'}(P/O_2(P)) \cong L_2(r^3)$.*

Proof: By way of induction it is enough to prove the assertion for Lie groups of rank two, i.e. $X \cong L_3(r)$, $Sp_4(r)$, $U_4(r)$, $U_5(r)$, $G_2(r)$, ${}^3D_4(r)$, ${}^2F_4(r)$.

Let P_1, P_2 be the two minimal parabolics. In case of $X \cong U_4(r)$, $U_5(r)$, ${}^3D_4(r)$, ${}^2F_4(r)$ choose notation such that $O^{2'}(P_1/O_2(P_1)) \cong L_2(r)$, $SU_3(r)$, $L_2(r^3)$, $Sz(r)$, respectively.

If $X \cong L_3(r)$ or $Sp_4(r), O_2(P_1)$ and $O_2(P_2)$ are the only maximal elementary abelian subgroups of X. Hence $A \le O_2(P_1)$ or $O_2(P_2)$.

Let $X \cong U_4(r)$. Let $A \not\le O_2(P_1)$. We have that $P_1 \le N_X(R)$, R a root group. Further $O_2(P_1)/Z(O_2(P_1))$ is elementary abelian and for $a \in A \setminus O_2(P_1)$ we have that $|[O_2(P_1)/Z(O_2(P_1)), a]| = r^2$. This implies $|\langle Z(O_2(P_1)), A\rangle| > r^3$.

Let now $A \not\le O_2(P_2)$. We have $\Omega_1(O_2(P_2))$ is elementary abelian of order r^4 and $O^{2'}(P_2/O_2(P_2)) \cong L_2(r^2)$. Furthermore $\Omega_1(O_2(P_2))$ is the $\Omega^-(4, r)$-module for $L_2(r^2)$. As A acts quadratically on $\Omega_1(O_2(P_2))$, we see $|A : A \cap O_2(P_2)| \le r$. Now $|\langle Z(O_2(P_1)), A\rangle \cap \Omega_1(O_2(P_2))| > r^2$. Hence $|[\Omega - 1(O_2(P_2)), a]| < r^2$ for $a \in A \setminus A \cap O_2(P_2)$, contradicting 3.50(1).

Let $X \cong G_2(r)$. Let Let P_1 be the normalizer of a root group normal in a Sylow 2–subgroup. Then $O_2(P_1)/Z(O_2(P_1))$ is elementary abelian of order r^4. If $a \in A \setminus O_2(P_1)$, then $|[a, O_2(P_1)/Z(O_2(P_1))]| = r^2$. Hence $|\langle A, Z(O_2(P_1))\rangle| > r^3$, contradicting the fact that $G_2(r)$ contains no elementary abelian subgroup of order greater than r^3 by 1.5.

Let $X \cong {}^3D_4(r)$. Then $V = O_2(P_1)/Z(O_2(P_1))$ is the 8-dimensional $GF(r)$-module for $L_2(r^3)$. Suppose that $A \not\leq O_2(P_1)$. As $O_2(P)/Z(O_2(P)$ is a tensor product of three algebraically conjugates of the natural module for $L_2(r^3)$, we see that $|[a, O_2(P_1)/Z(O_2(P_1))]| = r^4$. Then $[a, O_2(P_1)] = C_{O_2(P_1)}(a)$. Let $\gamma \in N_X(S), o(\gamma) = r^2 + r + 1$. Then $[\gamma, C_V(S)] = 0$, as $|C_V(S)| = r$. Hence $C_{O_2(P_1)}(\gamma)$ is a Sylow 2-subgroup of $L_3(r)$. But we may assume there is $a^g = b \in L_2(r^3)$ with $\gamma^b = \gamma^{-1}$. But there is no automorphism b of a Sylow 2-subgroup T of $L_3(r)$ such that $[T, b]$ is elementary abelian of order r^2, since a Sylow 2–subgroup of $L_3(r)$ contains exactly two elementary abelian subgroups of order r^2 which bot are either normalized by b or interchanged.

Let $X \cong {}^2F_4(r)$. We have $Z_2(O_2(P_1)) = \Omega_1(O_2(P_1))$. Now $[a, O_2(P_1)] \leq Z_2(O_2(P_1))$ for $a \in A$, and so $A \leq O_2(P_1)$. Hence $A \leq Z_2(O_2(P_1)) \leq C(Z(O_2(P_1))) \leq O_2(P_2)$. □

$fourL2$

Lemma 3.52 *Let V be a nonsplit extension of a trivial module by the natural module for $X = L_2(q)$, q even. Let S be a Sylow 2-subgroup of X and A be a fours group in S. Then $[V, A] = [V, S]$.*

Proof: Let $\nu \in X$, $o(\nu) = q + 1$ and $\nu^a = \nu^{-1}$ for some $a \in A$. We have that $|[V, \nu]| = q^2$ and so $[V, a] \leq [V, \nu]$. Let $A = \langle a, b \rangle$. We have that $\langle [V, \nu], [V, b] \rangle$ is invariant under $\langle A, \nu \rangle = X$. Hence we have that $\langle [V, \nu], [V, b] \rangle = V$ and so $[V, A] = [V, S]$. □

$Sp4$

Lemma 3.53 *Let $K = Sp(4, q)$, $q = 2^n > 2$. Let V be an indecomposable $GF(2)$–module for K such that $V/C_V(K)$ is the natural module and $C_V(K) \neq 1$. Then the following holds*

(i) Let $U \leq R$, R the transvection group in K and $|U| = 4$, then $C_V(K) \leq [U, V]$.

(ii) Let $A \leq K$ be an elementary abelian 2–subgroup, which is a quadratic offender on V as an F–module, then $C_V(K) \leq [V, A]$.

(iii) Let $A \leq K$ be an elementary abelian 2-subgroup which is a quadratic offender on V as an F–module. If $|V : C_V(A)| < |A|$, then $[V, A] = [V, B]$ for any quadratic group B in K with $A \leq B$.

Proof: (i) Let P be a parabolic of K with $U \cap O_2(P) = 1$. Then there are two conjugates of U which generate $P/O_2(P)$. hence we can generate P with three conjugates of U. As P is a maximal subgroup in K we can generate K with four conjugates. Let $W = [U, V] \cap C_V(K)$. Then we have that $|[U, V]/W| = q$. Hence $|[V, K]/W| = q^4$ and then $W = C_V(K)$.

(ii) We have $|A| \geq q$. If A induces a transvection group then the assertion follows from (i). So we have $|V : C_V(A)| \geq q^2$, and then $|A| \geq q^2$. Let now P be the parabolic which stabilizes $[V,A]C_V(K)/C_V(K)$. Then $A \leq O_2(P)$. If $A \cap Z(O^{2'}(P)) = 1$, we get that the other parabolic P_1 is generated by two conjugates on A. Set again $W = C_V(K) \cap [V,A]$ then we get that $V = U \times [V,P_1]$, where U is a complement of W in $C_V(K)$. By [Hu, (I.17.4)] we get that $W = C_V(K)$. So we have that $L = A \cap Z(O^{2'}(P) \neq 1$. Let W be as before that we see that $[V,A] = W[L,V]$. Hence we have $[V,A^g] = W[L,V] = [V,A]$ for all $g \in O^{2'}(P)$. This shows that $[V,O_2(P)] = [A,V]$. But $C_V(K) \leq [V,O_2(P)]$ by (i).

(iii) We have $|A| > q$ and so $|V : C_V(A)| \geq q^2$. Hence $|A| > q^2$. By (ii) we have that $C_V(K) \leq [V,A]$. So we have that $[V,A] = [O_2(P),V]$, where P is the stabilizer of $[V,A]C_V(K)/C_V(K)$. But $B \leq O_2(P)$. □

$G2$

Lemma 3.54 *Let $G = G_2(q)$, q even, and V be a nonsplit extension of a trivial module by the natural module. Let A be an offender in G on the natural module as an F–module. Then $[V,A] \cap C_V(G) \neq 1$.*

Proof: By 3.18 we have that $|A| = q^3$. So suppose that $|[V,A]| = q^3$. Let R be a root group $R \leq A$. Then $[V,R] \leq [V,A]$ and so $A \leq O_2(N_G(R))$. Let P be the other parabolic of G containing S, S a Sylow 2–subgroup of $N_G(R)$. Then there is a conjugate A^g, $g \in N_G(R)$ such that $A^g \not\leq O_2(P)$. Hence we can generate P by two conjugates of A. Hence we have that $|[V,P]| = q^6$ and so complements $C_V(G)$. But this contradicts [Hu, (I.17.4)]. □

Szq

Lemma 3.55 *Let $X \cong Sz(q)$ or $L_2(q)$, $q > 2$ and U be a 2-group on which X acts. Let V be a normal subgroup of U of order 2 and U/V be the natural module for X. In case of $X \cong Sz(q)$ assume additionally that U contains an elementary abelian subgroup U_1 with $|U_1|^2 = 2|U|$. Then U is abelian.*

Proof: If $X \cong L_2(q)$, then X acts transitively on $(U/V)^\sharp$. As $q > 2$ there are involutions in $U \setminus V$, so all elements in U are involutions, the assertion. So let $X \cong Sz(q)$. Then elements of order 5 act fixed point freely on U/V. We may assume that U is extraspecial. By assumption it is of $+$–type. But as $q = 2^{2n+1}$, we get $|U/V| = 2^{8n+4}$ and so U is a central product of $4n+2$ dihedral groups. On such a group U an element of order 5 cannot act fixed point freely on U/V. □

$noquadoff$

Lemma 3.56 *Let $K \cong L_3(q^2)$ or $Sp(4,q^2)$ and V be the tensor product of two algebraically conjugate modules. Then V does not admit a quadratic $2F$–module offender.*

Proof: Let A be a quadratic subgroup in K. By 3.25 A is contained in a root group or in case of $Sp(4,q^2)$ we might have $|A| \leq q^4$. Let first $K \cong L_3(q)$. Then A is contained in some $L_2(q^2)$, which induces on V a natural and an orthogonal module. As A has to act quadratically on the orthogonal module, we see $|A| \leq q$. But then on the natural module W we have $|W : C_W(A)| = |A|^2$, a contradiction.

So we have $K \cong Sp(4,q)$. Let S be a Sylow 2–subgroup containing A and assume that $|A : A \cap Z(S)| \geq 4$. Let E be an elementary abelian subgroup of S with $A \not\leq E$, then $|A : A \cap E| \geq 4$. As $N_K(E)$ acts indecomposably on E, we get with 3.52 that $\langle A^S \rangle$ contains a root group. But then there is some $a \in A \cap Z(S)$ and some root element r such that $\langle a, r \rangle$ acts quadratically , contradicting 3.25. So we have that $|A : A \cap Z(S)| \leq 2$. On the other hand there is some $g \in K$ with $(A \cap Z(S))^g \cap E = 1$. So $(A \cap Z(S))^g$ acts as a subgroup of $L_2(q^2)$ on the chieffactors of $N_K(E)$ in V. But there are tensor products for this group, so $|A \cap Z(S)| \leq q$. This shows that $|A| \leq 2q$ and then $|V : C_V(A)| \leq 4q^2$. But as elements in A are conjugate into $N_K(E) \setminus E$ and $N_K(E)/E$ has at least three chieffactors, we get with 3.50, that $|V : C_V(A)| \geq q^6$, a contradiction. □

4 Small modules for small groups

module

Lemma 4.1 *Let G be a group with $F^*(G) = O_2(G) \neq 1$ and $A \leq G$ be elementary abelian with $A \not\leq O_2(G)$ but $A \trianglelefteq S$ for some Sylow 2–subgroup S of G. Then either there is some $g \in G$ such that for $X = \langle A, A^g \rangle$ the following hold*

(1) *$X/O_2(X) \cong L_2(q)$, $Sz(q)$ or $X/O_2(X)$ is a dihedral group of order $2u$, u odd.*

(2) *$S \cap X$ is a Sylow 2–subgroup of X*

(3) *$Y = (A \cap O_2(X))(A^g \cap O_2(X)) \trianglelefteq X$*

(4) *$Y \neq A \cap O_2(X)$*

(5) *$|A : C_A(Y)| \leq |Y : C_Y(A)|q \leq |Y : C_Y(A)|^2$, where $q = 2$ if $X/O_2(X)$ is dihedral.*

(6) *If $X/O_2(X)$ is not dihedral, then $Y/(A \cap A^g)$ is a direct sum of natural modules for $X/O_2(X)$.*

or there is some $g \in G$ with $g^2 \in N_G(A)$ such that $A^g \leq S$, $1 \neq [A^g, A] \leq A \cap A^g$ and $|A : C_A(A^g)| = |A^g : C_{A^g}(A)|$.

Proof: We start the proof with some general remarks. Let X be as in (1) and (2). Then obviously (3) follows. If (4) would be false, then as $[O_2(G), A] \leq O_2(G) \cap A \leq O_2(X) \cap A$, we get that $[O_2(G), X, X] = 1$, which contradicts $C_G(O_2(G)) \leq O_2(G)$. Hence also (4) holds. Next we see that $C_Y(A) = A \cap Y$ and so we see that $C_{Y/(A\cap A^g)}(A) = (A \cap Y)/(A \cap A^g)$ and $Y/(A \cap A^g) = (Y \cap A)/(A \cap A^g) \oplus (Y \cap A^g)/(A \cap A^g)$. So (5) follows. Further we see that elements of odd order in X act fixed point freely on $Y/(A \cap A^g)$. Hence [Hi] and [Mar] yield (6). So we see that in case (1) we just have to prove (2) which will become clear by the particular construction.

Set $\bar{G} = G/O_2(G)$. Let r be some odd prime and R be a r–subgroup of $\bar{G}$ with $1 \neq [R, \bar{A}] \leq R$. Then $R = \langle C_R(\bar{B}) \mid |\bar{A} : \bar{B}| = 2 \rangle$. Hence there is some $\bar{B}$ with $C_R(\bar{B}) \neq 1$ and $[C_R(\bar{B}), \bar{A}] \neq 1$. So there is some element $\omega \in C_R(\bar{B})$, with $o(\omega) = r$ and $\langle A, \omega \rangle / O_2(\langle A, \omega \rangle) \cong D_{2r}$. Suppose there is some component L with $1 \neq [L, \bar{A}]$ and $|\bar{A} : C_{\bar{A}}(L)| = 2$. Then as $\bar{A} \not\leq O_2(\langle L, \bar{A} \rangle)$ there is some $\omega \in \langle L, \bar{A} \rangle$, $o(\omega)$ odd, which is inverted by some $\bar{a} \in \bar{A} \setminus C_{\bar{A}}(L)$. Then $\langle A, \omega \rangle / O_2(\langle A, \omega \rangle) \cong D_{2u}$, u odd. In both cases of course $S \cap X$ is a Sylow 2-subgroup of X.

So we may assume that $F^*(\bar{G}) = E(\bar{G})$. We have that A acts quadratically on $O_2(G)$. Further for any component L we may assume that $|\bar{A} : C_{\bar{A}}(L)| \geq 4$.

Hence by 3.24 we have $[L, \bar{A}] \leq L$.

Assume first that L is of Lie type in odd characteristic, which is not also of Lie type in even characteristic. Then by 3.26 we have that $L/Z(L) \cong U_4(3)$. Let B be the projection of $\bar{A}$ onto $\mathrm{Aut}_{\bar{G}}(L)$. As $A \trianglelefteq S$, there is some 2–central involution s in B. If $B \not\leq O_2(C_L(s))$, then there is a conjugate B^g such that $W = \langle B, B^g \rangle \cong D_6$ and $\bar{S} \cap W$ is a Sylow 2-subgroup of W. Hence we may set $X = \langle A, A^g \rangle$. So we may assume that $B \leq O_2(C_L(s))$. As we may generate $C_L(s)$ by elements g with $g^2 \in S$, the action of $C_L(s)$ on $O_2(C_L(s))$ gives us some $B^g \leq O_2(C_L(s))$ with $1 \neq [B, B^g] \leq B \cap B^g$. Then also $1 \neq [A, A^g] \leq A \cap A^g$ and $A^{g^2} = A$. Obviously $|A : C_A(A^g)| = |A^g : C_{A^g}(A^{g^2})| = |A^g : C_{A^g}(A)|$.

Let next $L \cong G(r)$ be a group of Lie type in even characteristic. Let R be a root subgroup in $Z(\bar{S} \cap L)$. Let B be again the projection of A onto $\mathrm{Aut}_{\bar{G}}(L)$. Suppose $B \not\leq O_2(N_{\mathrm{Aut}_{\bar{G}(L)}}(R))$. Then we have induction and the lemma holds. So we may assume that $B \leq O_2(N_{\mathrm{Aut}_{\bar{G}(L)}}(R))$. If we may generate $C_L(R)$ by elements g with $g^2 \in O_2(N_L(R))$, then we get the second alternative , or $\langle B^{N_L(R)} \rangle$ is abelian. If $B \leq R$, then $B \leq \tilde{L} \leq L$, with $\tilde{L} \cong L_2(r)$ or $Sz(r)$ and $S \cap \tilde{L}$ is a Sylow 2–subgroup of $\tilde{L}$.

Hence we just have to handle rank 1 groups or $L \cong L_n(r)$, $Sp(2n, r)$, $3 \cdot A_6$ $F_4(r)$, ${}^2F_4(r)$.

If we have a rank 1 group then as $|B| \geq 4$, we either get X such that $X/O_2(X)$ is dihedral or we get X with $X/O_2(X) \cong L_2(q)$ or $Sz(q)$ and a Sylow 2-subgroup of X is contained in $\bar{S}$. So we may assume that $B \not\leq R$.

Assume next $L \cong L_n(r)$, $n \geq 3$. Assume that B acts trivially on the Dynkin diagram. Let P_1, P_{n-1} be the two parabolics containing $\bar{S} \cap L$ which involve $L_{n-1}(r)$. If $B \not\leq O_2(P_i)$ for one i, then we have induction. So we have $B \leq O_2(P_1) \cap O_2(P_{n-1}) = R$, a contradiction. Now let $b \in B$ acting nontrivially on the Dynkin diagram. Then we get a parabolic P_3 with $P_3/O_2(P_3) \cong L_2(r) \times L_2(r)$ such that B acts nontrivially on $P_3/O_2(P_3)$. If $r > 2$, we have induction. If $r = 2$ this is solvable and we get a dihedral group $X/O_2(X)$.

Let next $L \cong Sp(2n, r)'$, $n \geq 2$. We may assume that $B \leq Z(O_2(N_{\mathrm{Aut}_{\bar{G}}}(R)))$. So we may embed B into some $\tilde{L} \cong Sp(4, r)'$ with $S \cap \tilde{L}$ a Sylow 2–subgroup of $\tilde{L}$. Hence we may assume $L \cong Sp(4, r)'$. Now we have two parabolics P_1, P_2, containing $\bar{S} \cap L$. By induction we may assume that $B \leq O_2(P_1) \cap O_2(P_2)$. As B is not contained in a root subgroup we see that $\langle B^{P_i} \rangle = O_2(P_i)$ for $i = 1, 2$. Let H_i be the preimage of P_i, i.e. $H_i/O_2(G) = P_i$. Now suppose that $\langle A^{H_i} \rangle$ is abelian. Then we see that $O_2(H_i) \leq C_{S\cap L}(A)O_2(G)$. If this is true for both i, we get $S \cap L = C_{S\cap L}(A)O_2(G)$. As A acts quadratically on $O_2(G)$ by 3.25 there is a chief factor V in $O_2(G)$ which is the natural module. We have $|[V, B]| = r^2$, while $|C_V(S \cap L)| = r$. As $[V, B]$ is covered by A this is a contradiction. So we have the latter. Now by quadratic action $B \leq A_6$ and then $|B| = 2$, a contradiction.

Let next $L \cong 3 \cdot A_6$ and the 6–dimensional module be involved in $O_2(G)$. Then by quadratic action we get $B \leq L$ and further $B \leq O_2(P_1) \cap O_2(P_2)$. But then $|B| = 2$, a contradiction.

Let next $L \cong F_4(r)$. By induction we may assume that B acts trivially on the Dynkin diagram. We have two root groups R_1 and R_2 and we may assume that $B \leq Z(O_2(N_L(R_1))) \cap Z(O_2(N_L(R_2)))$. But this group is contained in some $Sp(4, r)$ and we get the assertion by induction.

Let next $L \cong {}^2F_4(r)$. As B acts quadratically we get with 3.25 $B \leq R$, a contradiction.

Let now $L \cong A_n$, $n \geq 5$. So we may assume $n = 7$ or $n \geq 9$. If n is odd, then there is $\tilde{L} \leq L$, $\tilde{L} \cong A_{n-1}$, which is normalized by $\bar{S}$. Hence we may assume n to be even right from the beginning. So $n \geq 10$. Let $n = 2^m$. Then there is a subgroup $\tilde{L} \leq L$ with $S \cap L \leq \tilde{L} \leq \Sigma_{\frac{n}{2}} \wr \mathbb{Z}_2$. As $n \geq 16$ we may apply induction. Let $m_1, \ldots, m_r$ be the dyadic decomposition of n. Let $\tilde{L}$ be the subgroup of L with $S \cap L \leq \tilde{L} \leq \Sigma_{m_1} \times \cdots \times \Sigma_{m_r}$. Let X_1 be one of the components of $\tilde{L}$ on which B acts nontrivially. Then we may apply induction. So as $|B| > 2$ and B acts nontrivially on $\tilde{L}$, we see that $B \leq \Sigma_4 \times \mathbb{Z}_2$. If we can embed B into some $X_2 \cong \Sigma_6$ such that $\bar{S} \cap X_2$ is a Sylow 2–subgroup of X_2. Now again we may apply induction. Hence we may assume that $4 \mid n$ and B is a Sylow 2–subgroup of $X_2 \cong A_4$. Then $B \sim \langle (12)(34), (13)(24) \rangle$ and so there is some conjugate B^g with $\langle B, B^g \rangle \cong A_5 \cong L_2(4)$, the assertion.

Let finally L be sporadic. By 3.26 we get that $L/Z(L) \cong M_{12}$, M_{22}, M_{24}, J_2, Co_1, Co_2, or Suz. Now we choose $s \in Z(\bar{S} \cap L \cap B)$. If $B \not\leq O_2(C_{\mathrm{Aut}_{\bar{G}}}(s))$, then by induction we get the assertion again. If there is some involution g in $C_L(s)$ with $[B, B^g] \neq 1$, we have the second alternative. So we may assume that $\langle B^{C_L(s)} \rangle$ is abelian. This gives $L/Z(L) \cong M_i$. If $L \cong M_{24}$ there is a subgroup $\tilde{L} \leq L$ with $S \cap L \leq \tilde{L}$ and $L \cong E_{16}A_8$. Now by induction we may assume $B \leq O_2(\tilde{L})$. But there is no quadratic foursgroup in $O_2(\tilde{L})$ according to [MeiStr2]

Let next $L/Z(L) \cong M_{22}$. Then we may embed B into a subgroup $SL(3, 4)$ and again we get the assertion by induction.

So we are left with $L \cong M_{12}$. If $B \not\leq L$, then with [MeiStr2] we see that $B \not\trianglelefteq S \cap L$, so we have $B \leq L$. Now in L there are two parabolics P_1, P_2 such that $P_i/O_2(P_i) \cong \Sigma_3$. So if $B \not\leq O_2(P_i)$ for some i we have induction again. Hence we may assume that B is contained in $O_2(P_1) \cap O_2(P_2)$ and $\langle B^{C_L(s)} \rangle$ is elementary abelian of order 8. Then this group contains an involution i which acts fixed point freely on the 12 points moved by L. So $C_L(i) \cong \mathbb{Z}_2 \times \Sigma_5$. Further S contains a Sylow 2–subgroup of $C_L(i)$. As $B \leq C_L(i)$, we get the assertion by induction. □

$2Fmodule$

Lemma 4.2 *Suppose Y and H are subgroups of a group G with a common Sylow 2–subgroup S and $F^*(Y) = O_2(Y)$, $F^*(H) = O_2(H)$. Assume further $Y_Y \not\leq O_2(H)$. Then one of the following holds.*

(1) *There is some $g \in H$, $g^2 \in N_H(Y_Y)$ with $Y_Y^g \leq S \leq Y$, $Y_Y \leq Y^g$. Further $1 \neq |Y_Y : C_{Y_Y}(Y_Y^g)| = |Y_Y^g : C_{Y_Y^g}(Y_Y)|$. In particular Y_Y is an F–module.*

(2) *There is some $g \in H$ such that for $L = \langle Y_Y, Y_Y^g \rangle$ we have $L/O_2(L) \cong L_2(q)$, $Sz(q)$, q even, or D_{2u}, a dihedral group of order $2u$, u odd. Set $q = 2$ in the latter. Further we have that $A = Y_Y^g \cap O_2(L) \leq S \leq Y$. For the action of A on Y_Y we have $[Y_Y, A, A, A] = 1$, If $x \in Y_Y \setminus O_2(L)$, then $C_A(x) = A \cap Y_Y$, and $|Y_Y : C_{Y_Y}(A)| \leq q|A/(A \cap Y_Y)|$. In particular Y_Y is a $2F$–module with offender $A/(A \cap Y_Y)$ and an $F+1$–module in case of $q = 2$.*

(3) *If we are in (2) then $|Y_Y : C_{Y_Y}(A)| < |A/(A \cap Y_Y)|^2$.*

(4) *If we have that A acts quadratically then Y_Y is an F–module.*

Proof: We find everything for (1) and (2) in 4.1 where $G = H$ and $A = Y_Y$.

For (3) assume that we have $|Y_Y : C_{Y_Y}(A)| = |A/(A \cap Y_Y)|^2$. Then $|(Y \cap O_2(L))(Y^g \cap O_2(L))/Y \cap Y^g| = q^2$. Hence we have that $L/O_2(L) \cong L_2(q)$ or L induces Σ_3 on $(Y \cap O_2(L))(Y^g \cap O_2(L))/Y \cap Y^g$. In both cases L acts transitively on $((Y \cap O_2(L))(Y^g \cap O_2(L))/Y \cap Y^g)^\sharp$ and so $(Y \cap O_2(L))(Y^g \cap O_2(L))$ is abelian. But then $|Y_Y : C_{Y_Y}(A)| = |A/(A \cap Y_Y)|$, a contradiction.

In (4) we have that $[A, Y_Y \cap O_2(H)] = 1$. So the assertion follows with 4.1(6). □

$Knormal1$

Lemma 4.3 *Let the notation be as in 4.2. Let K be a component of $Y/C_Y(Y_Y)$ with $[K, A] \neq 1$. Suppose 4.2(2) with $[K, A] \not\leq K$ then $|A| > 4$, $K \cong L_n(2)$ for some n. Let $a \in A$ with $K^a \neq K$. Then $|[Y_Y, a]| = 2^n$ and A induces the full transvection group on $[Y_Y, a]$. In particular $|Y_Y^g : A| = 2$.*

Proof: Suppose first $q > 2$. By 4.1 we know that $Y := (Y_Y \cap O_2(L))(Y_Y^g \cap O_2(L))/(Y_Y \cap Y_Y^g)$ is a direct sum of natural modules. So let A_1 be contained in the intersection of A with one of this modules V_1, with $|A_1 : A_1 \cap Y_Y| = q$. We have $[Y_Y, A_1, A_1] \leq Y_Y \cap Y_Y^g$. Suppose $[Y_Y, A_1, A_1] \neq 1$. Let R be a hyperplane in $Y_Y \cap Y_Y^g$ with $[Y_Y, A_1, A_1] \not\leq R$. As $|Y_Y \cap V_1|^2 = 2|V_1|$, we have the assumptions of 3.55, a contradiction. So A_1 acts quadratically on Y_Y. Hence by 3.24 we have three posibilities

(1) $[K, A_1] \leq K$

(2) $|A_1 : C_{A_1}(K)| > 2$, $[K, A_1] \not\leq K$ and $K \cong L_2(2^n)$

(3) $|A_1 : C_{A_1}(K)| = 2$ and $[K, A_1] \not\leq K$.

Let $[K, A_1] \not\leq K$. Assume (3). Let $a \in C_{A_1}(K)$. Then K^{A_1} acts on $[Y_Y, a]$. By quadratic action we have $[Y_Y, a, K] = 1$. In particular we get that $[Y_Y, K]$ is centralized by a and so by 4.2 $[Y_Y, K] \leq O_2(L)$. Hence A acts quadratically on $[Y_Y, K^A]$. Assume now $|A : C_A(K)| = 2$. Then $[Y_Y, A](Y_Y \cap Y_{M^g}) = [Y_Y, C_A(K)](Y_Y \cap Y_{M^g})$. In particular $[Y_Y, K] \leq Y_Y \cap Y_{M^g}$, a contradiction.

So we may assume that $K^A = \Omega^+(4, 2^n)$. In particular we may assume that we are in (2). Suppose first $[Y_Y, K] \leq O_2(L)$. By 3.36 there is some $y \in C_{Y_Y}(K) \setminus O_2(L)$. Hence we see $[y, A](Y_Y \cap A) = Y_Y \cap O_2(L)$. Now $[Y_Y, K, A] \leq [Y_Y \cap O_2(L), A] = [y, A, A]$. But $[y, A] \leq C_{Y_Y}(K)$, a contradiction. So we have $[Y_Y, K] \not\leq O_2(L)$. In particular there is some minimal module V such that $V \not\leq O_2(L)$ and $V/C_V(K)$ is the natural $\Omega^+(4, 2^n)$–module. As above we see that $C_{A_1}(K) = 1$. Choose $y \in V \setminus O_2(L)$, then we get $[y, A_1](Y_Y \cap A) = [Y_Y, A_1]$. As $|[V/C_V(K), A_1]| > |A_1|$, we see that $V \cap A \not\leq C_V(K)$. Let $a \in C_A(K)$. Then $[V, a] < V$, and so $[V, a] = 1$, which gives $a = 1$, as $V \not\leq O_2(L)$. So we have $C_A(K) = 1$. But then A acts quadratically on V, which then gives that V is an F–module, a contradiction.

So we have shown that $K^{A_1} = K$. As A is generated by such subgroups A_1, we have the contradiction $[K, A] \leq K$.

So we have $q = 2$. If $[Y_Y, K] \leq O_2(L)$, then again A acts quadratically on $[Y_Y, K]$. If we have $|A : C_A(K)| > 2$, we may argue as before. So let $|A : C_A(K)| = 2$. Then for a K^A–module W we have that A induces transvections, as W must be in $[Y_Y, a]$ for some $a \in C_A(K)$, which is not possible.

So we have $[Y_Y, K] \not\leq O_2(L)$. Let $a \in A$ with $K^a \neq K$. Assume first $[Y_Y, a, A] = 1$, then by 3.24 either $|A : C_A(K)| = 2$, or $K \cong L_2(r)$ and orthogonal $\Omega^+(4, r)$–modules are involved. Suppose the latter. Then as before, we get some $y \notin C_{Y_Y}(K)$ with $[A, y](A \cap Y_Y) = [Y_Y, A]$, a contradiction. So we have $|A : C_A(K)| = 2$. Now we have that $[K, Y_Y, C_A(K)] = 1$, but as $[Y_Y, K] \not\leq O_2(L)$ this shows $|A : A \cap Y_Y| = 2$. Now A induces transvections on Y_Y and so A has to normalize K, a contradiction.

So we have that $[Y_Y, a, A] \neq 1$. Now A induces transvections on $[V, a]$ to a hyperplane. Let $b \in C_A(K)$ and assume first that $[Y_Y, b, K] = 1$. Then also $[Y_Y, K, b] = 1$ and so $[Y_Y, K] \leq O_2(L)$, a contradiction. Hence K acts nontrivially on $[Y_Y, b]$. But b induces a transvection on $[Y_Y, a]$, a contradiction. So we have that $C_A(K) = 1$. Further $K^A = K \times K^a$. As $[Y_Y, a, A] \neq 1$, we see that $K_1 = C_{K \times K^a}(a) \cong K$ acts faithfully on $[Y_Y, a]$, and so, as A induces transvections to a hyperplane, we get with 3.16 $K \cong L_n(2)$, $Sp(2n, 2)$, $\Omega^{\pm}(2n, 2)$ or A_n. We have $|A| > 2$. So assume first $|A| = 4$. Then $|[Y_Y, a]| \leq 4$, but K_1 has to act nontrivially on $[V, a]$, a contradiction. So $|A| > 4$ and then $K \cong L_n(2)$. □

$2Fsmall$

Lemma 4.4 *Let Y, H be as in 4.2. Assume further that Y is a minimal parabolic with respect to S. Set $X = Y/O_2(Y)$. Assume $m_3(X) \leq 3$. If X is nonsolvable and P a normal p–subgroup, then assume that $m_p(P) \leq 3$. Set $V = Y_Y$. Assume that $|V : C_V(A)| < |A|^2$. Let finally $C_X(V)$ be nilpotent. Then one of the following holds:*

(1) $E(X) \cong SL_3(r)$ and V is a direct sum of the natural and the dual module.

(2) $E(X/C_X(V)) \cong L_2(r), r = t^2, V$ is the orthogonal module.

(3) $E(X) \cong X_1X_2, X_1 \cong X_2 \cong L_2(r)$, $V = V_1 \bigoplus V_2$, $[V_i, X_{3-i}] = 1$, $[V_i, X_i] = V_i, i = 1, 2$, and V_i is orthogonal, a direct sum of two natural modules, or $r = 4$ and V_i is a direct sum of two orthogonal modules.

(4) $E(X) = X_1X_2, X_1 \cong X_2 \cong L_2(r)$ and V is the natural $O_4^+(r)$-module.

(5) $E(X) = X_1X_2, X_1 \cong X_2 \cong L_3(r)$, $V = V_1 \bigoplus V_2$, $[V_i, X_{3-i}] = 1$, $[V_i, X_i] = V_i, i = 1, 2$, and V_i is a direct sum of the natural and the dual module.

(6) $E(X) \cong Sp_4(r), V$ is a direct sum of the natural and the dual module.

(7) $E(X) \cong 3 \cdot A_6, [V, Z(E(X))] = 1$ and V is a direct sum of the natural 4–dimensional module and its dual.

(8) $E(X) = X_1 * X_2$, $X_1 \cong X_2 \cong 3 \cdot A_6$, $[V, Z(E(X))] = 1$, $V = V_1 \bigoplus V_2$, $[V_i, X_{3-i}] = 1$, $[V_i, X_i] = V_i, i = 1, 2$ and V_i is a direct sum of the 4–dimensional module and its dual.

(9) $E(X) \cong A_9$

(10) X is a solvable

Proof: We may assume that X is nonsolvable. Suppose $E(X) = 1$. Let P be some normal p–subgroup of X on which X induces a nonsolvable group. Now by assumption $m_p(P) \leq 3$ and so on some critical subgroup C of P the group X induces a subgroup of $L_3(p)$ or $Sp_4(p)$. Let R be the preimage of $O_p(X/C_X(C))$ and R_1 the preimage of $O_2(X/R)$. Then $N_X(S \cap R_1)R_1 = X$. As $R_1S \neq X$, we get $S \cap R_1 = 1$. This shows that X induces a subgroup of $L_3(p)$. Now we have $|A| = 4$ as $4 \leq |[V : C_V(A)| < |A|^2$. Now there is a subroup Y which is generated by three conjugates of A which covers $X/C_X(C)$. Hence this group is in $GL(9, 2)$. As $X/C_X(C)$ is a minimal parabolic, we see with 2.6 that $p = 5$ or 7, C is elementary abelian of order p^3 and $E(X/C_X(C)) \cong L_2(p)$. This with 2.1 shows that we must have $C \leq C_X(V)$. Now 3.29 gives (i) or (ii).

So we may assume that $E(X) \neq 1$. Then we have that $X = E(X)S$. Let us go over the list of possibilities for X. Let $E(X)$ be quasisimple. As S is in a unique maximal subgroup, we see with 1.1 that $E(X) \cong L_2(r)$, $Sz(r)$, $U_3(r)$, $Sp_4(r)$, $L_3(r)$ or A_9. Set $W = V/C_V(E(X))$.

Let $E(X) \cong Sz(r)$. Then with 3.50 we get that A has to act quadratically, a contradiction.

Let $E(X) \cong L_2(r)$. Let first $A \leq E(X)$. We have $|A| > 2$ then by 1.14 $E(X) = \langle A, A^g \rangle$ for suitable $g \in E(X)$. As $|A| \leq r$, we get $|W| \leq r^4$. If $|W| = r^4$, then $A \in \mathrm{Syl}_2(E(X))$ and by 3.50 $W = V_1 \bigoplus V_2, V_i$ the natural module for $E(X)$. But then A acts quadratically, a contradiction.

So assume $|W| < r^4$. Then W involves exactly one nontrivial irreducible module. So we have (2) by 3.29.

Assume next $A \not\leq E(X)$. Then $r = t^2$ and $|A| \leq 2t$. As $|A| > 2$ we have $|A \cap E(X)| \neq 1$, then as before $|W| \leq 16t^4 = 16r^2$. If $|W| < r^4$ we may argue as before. So assume $r = 4$ and $|W| = r^4$. Then $|A| = 4$ and so W is a direct sum of two orthogonal modules, which is (3).

Let $E(X) \cong (S)L_3(r)$. Suppose first $A \leq E(X)$. If W is not irreducible as $(S)L_3(r)$–module then we get some submodule W_1 of W which is an F–module. By 3.16 W_1 is a natural module. But now as some element in S has to induce a diagram automorphism also W_1^* is involved. This is (1).

So assume that W is an irreducible $(S)L_3(r)$-module. Then by 3.29 W is the natural module or a tensor product of the natural module with an algebraically conjugate module. As X contains some x inducing a Dynkin automorphism on $E(X)$ this is impossible.

So we have $A \not\leq E(X)$. Suppose first $|A : A \cap E(X)| = 4$. Then $r = t^2$ and $|A \cap E(X)| \leq t$. So $|W : C_W(A)| \leq 16t^2 = 16r$. As $|W : C_W(A)| > r^2$, we get $r = 4, |A| = 8$. There is some $x \in A$ such that $C_X(x)$ contains $U \cong L_3(2) \cdot 2$. There is $t \in U'$ such that $x \sim xt$. As $|W : C_W(t)| \geq 32$, we get $|W : C_W(x)| \geq 8$. But there is some $y \in U \setminus U', y \in A$. Hence $|C_W(x) : C_{C_W(x)}(\langle y, t\rangle)| \geq 2^4$. This implies $|W : C_W(A)| \geq 2^7$, a contradiction.

So we have $|A : A \cap E(X)| = 2$. Suppose A contains a field × diagram automorphism. Then $r = t^2$ and $|A| \leq 2t$. So $|W : C_W(A)| \leq 4t^2 = 4r$. As $|W : C_W(A)| > r^2$, we get a contradiction. Let p be a primitive prime divisor of $r^3 - 1, p = 9$ if $r = 4$. Then there is some $a \in A \setminus E(X)$ inverting some $\omega, o(\omega) = p$. Now we get $|W : C_W(a)| \geq t^3$, where $r = t^2$ or $r = t$, if r is not a square.

Let a be the diagram automorphism. Then $C_{E(X)}(a) \cong L_2(r)$. Hence $|W : C_W(A)| \geq rt^3$. So $4 \geq t$. This shows $E(X) \cong L_3(2), (S)L_3(4)$ or $(S)L_3(16)$. In case $(S)L_3(16)$ we have $|W : C_W(a)| = 4^3$ and so $[[W, \omega], a] = 1$. Hence $|C_W(\omega) \cap C_W(A)| = 1$. But this implies $|C_W(\omega)| \leq 16$, i.e. $|W| \leq r^4$, a contradiction.

Let $E(X) \cong (S)L_3(4)$. Now $|C_W(\omega) : C_{C_W(\omega)}(a)| \leq 2$. Hence $|C_W(\omega)| \leq 2 \cdot 4 = 8$. This shows $|W| \leq 2^9$, again $|W : C_W(t)| \leq 2^4$ for $t \in E(X), t^2 = 1$, a contradiction.

Let $E(X) \cong L_3(2)$. Then we get the natural and the dual module and a interchanges both modules. This is (1).

So we have $r = t^2$ and A contains a, a field automorphism. Hence $C_X(a)$ contains $U \cong (S)L_3(t) \cdot 2$. Again $|W : C_W(a)| \geq t^3$. As $|W : C_W(A)| > r^2 = t^4$, we get $|A \cap U| > t$. Now $|C_W(a) : C_W(A)| > t^2$. Hence $|W : C_W(A)| > t^5 = r^2 t$, so $4 > t$, i.e. $r = 4$. Now $|W : C_W(a)| = 8$. So $|[W, \omega]| = 2^6$ and $C_{C_W(\omega)}(A) = 1$. This shows $|C_W(\omega)| \leq 2^3$. So $|W| \leq 2^9$, a contradiction.

So we are left with $A \cap E(X) = 1$. Then we have $|A| = 4$ and $r = t^2$. There is some $a \in A^\sharp$ with $|W : C_W(a)| \geq t^3$. This shows $t = 2$. But then for this a we may assume $C_X(a) \geq U \cong L_3(2) \cdot 2$ and $|C_W(a) : C_W(A)| = 2$. But there is now $b \in (U \cap A)^\sharp$ inducing a transvection on $C_W(a)$, a contradiction

Let $E(X) \cong (S)U_3(r)$. Then $|A| \leq r$. So $|W : C_W(a)| \leq r^2$ for any $a \in A^\sharp$. Now 3.50 gives a contradiction with the quadratic action..

Let $E(X) \cong Sp_4(r)$, including $E(X) \cong A_6$ or $3A_6$. Suppose first $A \leq E(X)$. If W is a reducible $E(X)$–module, then we have some submodule W_1 which is an F–module. By 3.16 we have that W_1 is the natural module or $E(X) \cong 3A_6$ and W_1 is the 6–dimensional module. As X contains some diagram automorphism also the dual module is involved. The two 6–dimensional modules for $3A_6$ do not have the same offender. So we have A_6 and then (6) or (7).

Suppose that W is irreducible as $Sp_4(r)$–module. Then by 3.29 it is the natural module or a tensor product module, but as a diagram automorphism is involved, this is not possible.

So we have $A \not\leq E(X)$. Assume first $A \cap E(X) \neq 1$. Recall $|W : C_W(t)| > r^3$ for any $a \in A \cap E(X), a \neq 1$. Suppose $|A \cap E(X)| \leq r$. Then $|W : C_W(A)| \leq 16r^2, 4r^2$, if r is not a square. Hence $r \leq 4$.

Let $r = 4$, then $|A| \leq 8$, so $|W : C_W(A)| \leq 64 = r^3$, a contradiction. So let $r = 2, |A| = 4$. But now inspection of the A_6-modules just implies (7).

So assume $|A \cap E(X)| > r$. Then $r = t^2$ and A contains a field automorphism. So $|A| \leq 2t^3$. Hence $|W : C_W(A)| \leq 4t^6 = 4r^3$. Now as there is some $x \in E(X)^\sharp$ such that $a \sim ax, a$ the field automorphism, we get $|W : C_W(a)| \geq 2t^3$. So $|C_W(a) : C_W(A)| \leq 2t^3$ and by induction just the natural and its dual is involved. As $|A \cap E(X)| = t^3$, we then get $|C_W(a) : C_W(A)| \geq t^5$, a contradiction.

So we are left with $A \cap E(X) = 1$. Hence $|A| = 4$. For $u \in E(X)^\sharp, u^2 = 1$, we have $|W : C_W(u)| \leq 2^8$. This implies $r = 4$. Now $|W : C_W(a)| = 16$ for any $a \in A^\sharp$. This yields that $E(X)A = \langle C_{E(X)}(a) | a \in A^\sharp \rangle$ acts on $C_W(A)$, a contradiction.

Let $E(X) \cong 3 \cdot A_6$ and $[W, Z(E(X))] \neq 1$. Hence by 3.30 we get a contradiction as there is a diagram automorphism x in X.

From now on let $E(X)$ not be quasisimple but nontrivial. Let first $X_1^a = X_2 \neq X_1$ for some component X_1. Then by 4.3 we have $X_1 \cong L_3(2)$. But as with the natural X_1–module also the dual one is involved this is not possible. So we have that any component is normalized by A. If A acts faithfully on some component, we get the assertion. So we may assume that $C_A(X_1) \neq 1$. As all components are conjugate, they have to induce $2F$–modules. Hence we have the same possibilities for X_1 as before. In particular we see that $Sz(r)$ and $U_3(r)$ again are not possible. In particular we have exactly two components.

Let $[V, X_1, X_2] = 1$. Assume further that $[V, X_1]$ involves just one nontrivial irreducible module, then we get (3). So assume that $[V, X_1]$ contains more than one irreducible module. Then these modules have to be F-modules and with 3.16 we see that there are exactly two such modules involved, which gives (3), (5) and (8).

So we may assume that $[V, X_1, X_2] \neq 1$. In particular $[V, X_1]$ involves at least two nontrivial irreducible isomorphic F–modules for X_1. This now shows that we must have $X_1 = L_2(r)$ and there are exactly two natural modules, which is (4). □

sol

Lemma 4.5 *Let R be a p-group, p odd, and E be an elementary abelian 2-group acting faithfully on R. Let V be a $GF(2)$–module for RE on which E acts qudratically. Then we have $V = \langle v \mid |E : C_E(v)| \leq 2\rangle$.*

Proof: By 2.1 we may assume that RE is a direct product of dihedral groups $D_1 \times D_2 \times \cdots \times D_r$ of order $2p$. Further we may assume that RE acts faithfully and $C_V(R) = 1$. As E acts quadratically we see that $C_E(O_p(D_1))$ acts trivially on $[O_p(D_1), V]$. This implies that $V = [O_p(D_1), V] \oplus \cdots \oplus [O_p(D_r), V]$ and if $v \in [O_p(D_i), V]$ then $|E : C_E(v)| \leq 2$. □

*FG*2

Lemma 4.6 *Let V be a nontrivial F-module for X, where X is a minimal parabolic with respect to the Sylow 2–subgroup S, $O_2(X) = 1$. Assume further that $m_3(X) \leq 3$, $C_X(V)$ is nilpotent, $O_2(X/C_X(V)) = 1$ and $m_p(P) \leq 3$ if P is a normal p–subgroup of X and X is nonsolvable. Then one of the following holds*

(i) $E(X/C_X(V)) \cong L_2(r), r$ even, Z_2 involves exactly one nontrivial irreducible module which is the natural module, or $r = 4$ and this module is the orthogonal module. An offending subgroup A is a Sylow 2-subgroup of $E(X)$ or $r = 4$ and $|A| \leq 4, |A \cap E(X)| \leq 2$.

(ii) $E(X) = X_1 \times X_2 \cong L_2(r) \times L_2(r), r$ even. Denote by W the group $[E(X), V]C_V(E(X))/C_V(E(X))$. Then $W = W_1 \oplus W_2, [W_i, X_{3-i}] = 1, i = 1, 2$. Furthermore W_i is the natural X_i-module, or $r = 4$ and W_i is the orthogonal X_i-module. In the first case an offending subgroup is a Sylow 2-subgroup of X_1, X_2 or $E(X)$, while in the second case the offending subgroup A normalizes X_1 and X_2 and $|A \cap X_1| \leq 2 \geq |A \cap X_2|$, $|A| \leq 16$.

(iii) $E(X) \cong A_9$ and there is exactly one nontrivial irreducible module involved which is the natural module.

(iv) X is solvable and X is a $\{2, 3\}$-group.

Proof: Let $E(X) \neq 1$. Then as $X = E(X)N_X(S \cap E(X))$. Hence $X = E(X)S$. By 3.24 any quadratic offender normalizes any component. Hence components induce F–modules. By 3.16 we have that components of X are $L_2(r)$, $SL_3(r)$, $Sp_4(r)$ or A_9. But in the case of $SL_3(r)$ and $Sp_4(r)$ the Sylow 2–subgroup S also induces a diagram automorphism and so with the natural module also the dual is involved, which shows that V cannot be an F–module. If we have a component $L_2(r)$, then as S acts transitively on the components, we see that we have at most two of them, otherwise $m_3(X) \geq 4$. If we have a component A_9 we just have one component. Now the assertion about the modules follows easily with 3.16.

We now determine the offending subgroups. Let $E(X) = X_1 \cong L_2(q), q$ even. By 3.16 V involves exactly one irreducible module. This is the natural one, or $q = 4$ and it is the orthogonal module. Furthermore an offending subgroup is a Sylow 2-subgroup or $X \cong \Sigma_5$ and $|A| = 2, 4$ with $|A : A \cap X_1| = 2$.

Let now $E(X) = X_1X_2, X_1 \cong L_2(q) \cong X_2, q$ even. Let $[A, X_1] \neq 1$. Then $C_V(C_A(X_1))$ is an F-module for X_1. So V involves exactly one irreducible nontrivial X_1-module, as $|V : C_V(C_A(X_1))| \leq |C_A(X_1)|$. Now $[[V, X_1], X_2] = 0$. As there is some $x \in X$ with $X_1^x = X_2$, we get the assertion.

So assume now $E(X) = 1$. Let next X be nonsolvable. Let P be some normal p–subgroup of X on which X induces a nonsolvable group. Now by assumption $m_p(P) \leq 3$ and so on some critical subgroup C of P the group X induces a subgroup of $L_3(p)$ or $Sp_4(p)$. Let R be the preimage of $O_p(X/C_X(C))$ and R_1 the preimage of $O_2(X/R)$. Then $N_X(S \cap R_1)R_1 = X$. As $R_1S \neq X$, we get $S \cap R_1 = 1$. This shows that X induces a subgroup of $L_3(p)$. Now three conjugates of some offender, if the offender has order 4 and five conjugates in case of an offender of order two, generate a subgroup Y which covers $X/C_X(C)$. Hence this group is in $GL(6, 2)$. Now 2.6 gives, as $X/C_X(C)$ is a minimal parabolic, that $p = 5$, C is elementary abelian of order 5^3 and $E(X/C_X(C)) \cong L_2(5)$. In particular an offender has order at most 4. This with 2.1 shows that we must have $C \leq C_X(V)$. This gives $|[V, X]| = 16$, and so $X/C_X(V)$ satisfies (i).

Let now X be solvable, then by minimality it is a $\{2, r\}$–group for some prime r. By 2.1 and the fact that we have some quadratic offender, we get $r = 3$. □

We will now treat the solvable case separately.

Fmin

Lemma 4.7 *Let X be a minimal parabolic,i.e. a Sylow 2–subgroup is in exactly one maximal subgroup, and V be a faithful F-module for X over $GF(2)$. Assume further that X is a $\{2, 3\}$–group with $m_3(X) \leq 3$. Then*

$X \cong \Sigma_3$ or $\Sigma_3 \wr Z_2$ and $|[V, F(X)]| \leq 16$.

Proof: Set $F(X) = P$. Then P is a 3-group. Let T be a Sylow 2–subgroup of $O_{3,2}(X)$, then $N_X(T)$ contains a Sylow 2–subgroup S, the same applies for $F(X)S$. As S is in exactly one maximal subgroup we get $X = PS$. Let C_1 be a critical subgroup of P and $C = \Omega_1(C_1)$. Let first C be elementary abelian. Then $|C| \leq 27$ and so by 2.1 we have that $|A| \leq 8$. Let $|A| = 8$. Then $|[V, C]| = 2^6$ and so $X \leq GL(6, 2)$. As S cannot act irreducibly on C, we see that $P = Z_3 \wr Z_3$ and so P cannot be normal in X. Hence we have that $|A| \leq 4$. If C is extraspecial then $|C| \leq 3^5$. If now $|A| = 8$, there is some $a \in A$ which inverts $Z(C)$. Hence we have that $|[V, Z(C)]| \leq 2^6$. But C cannot be a subgroup of $GL(6, 2)$. So in any case we have that $|A| \leq 4$. If $|A| = 4$, then by 4.5 there we must have elements in $a \in A^\sharp$ with $C_V(a) \neq C_V(A)$, hence we have elements a with $|V : C_V(a)| = 2$. This of course is true for $|A| = 2$.

If first P be cyclic. Then we get $|[V, P]| = 4$. So $|P| = 3$ and $X \cong \Sigma_3$. So let P be noncyclic. Let $a \in A^\sharp$ such that $|V : C_V(a)| = 2$, then there is some $\omega \in P \setminus \Phi(P)$ with $|[V, \omega]| = 4$. By irreducible action there is a minimal generating system for P with elements ω such that $|[V, \omega]| = 4$. Hence we see that $[\Phi(P), V] = 1$. So P is elementary abelian and as S acts irreducibly we have that $|P| = 9$ and $X \cong \Sigma_3 \wr Z_2$ and $[V, O_3(X)]$ is the orthogonal module. □

min2Fquad

Lemma 4.8 *Let $F^*(L) \cong (S)L_3(q)$, $(S)U_3(q)$, $L_2(q)$, or $Sz(q)$, q even and further let S be a Sylow 2-subgroup of L which is contained in a unique maximal subgroup of L. Let V be a faithful irreducible F– or $2F$–module for L with a non quadratic offender in case of a $2F$–module. Then $L \cong L_2(q)$.*

Proof: Let $L \not\cong L_2(q)$. Let V be an F–module. Then by 4.6 we get the assertion. So we may assume that V is a $2F$–module. Let A be the offender which does not act quadratically. Then $|A| \geq 4$. We have that V restricted to $F^*(L)$ remains irreducible. As in case of $L \cong (S)L_3(q)$ we have some diagram automorphism induced on $F^*(L)$, we see with 3.29 that $F^*(L) \cong Sz(q)$ or $SU_3(q)$ and V is the natural module. Now we have that $|V : C_V(A \cap F^*(L))| = q^2$. As A does not act quadratically, we see that $A \not\leq F^*(L)$ and so $F^*(L) = SU_3(q)$ and $|A| = 2q$. But for involutions i not in $F^*(L)$ we have that $|V : C_V(i)| = q^3$. But then we get $|V : C : V(A)| \geq q^4$, a contradiction.

5 Uniqueness groups

In this section M is always some uniqueness group satisfying the assumptions of this paper.

ex2

Lemma 5.1 *Let M be exceptional with respect to p and $E \leq P$ with $m_p(E) \geq 2$, then $N_G(E) \leq M$, in particular $N_G(P) \leq M$.*

Proof: If $E \leq Q$, the assertion is clear. So it is enough to consider $|E| = p^2$, $|E \cap Q| = p$. Let $g \in N_G(E)$. Then $(E \cap Q)^g \leq M^g$ and so $P \leq C_G((E \cap Q)^g) \leq M^g$. But then $M = M^g$ and so $g \in M$.

goodEex

Lemma 5.2 *Let M be exceptional with respect to p. Set $C_M = C_M(Y_M)$ and $M_0 = N_M(S \cap C_M)$, for S a Sylow 2–subgroup of M. Let R be the preimage of $O_p(M/O_2(M))$. Then either $R \leq C_M$ or $R \leq M_0$.*

Proof: Let $R \not\leq C_M$. As M acts irreducibly on $R/O_2(M)$, we get $R \cap C_M = O_2(M)$. Now $[S \cap C_M, R] \leq R \cap C_M = O_2(M) \leq S \cap C_M$, the assertion. □

stark

Lemma 5.3 *Let M be a uniqueness group. Let $p \in \sigma(M)$ and E be a p-subgroup of M, E elementary abelian of order at least p^2, with $\Gamma_{E,1}(G) \leq M$. Let further $R \leq G$, with $E \leq R \cap M$, Then $R \leq M$ or one of the following holds*

(a) *$E(R/O_{p'}(R)) = L$ is simple and we have one of the following, where $P \in Syl_p(R \cap M)$:*

  (i) $L \cong L_2(p^n), n > 1, U_3(p^n)$ *or* ${}^2G_2(3^n), n > 1; M \cap L = N_L(P \cap L)$

  (ii) $p = 3\,, L \cong L_2(8)\,, L_3(4), M_{11}, A_6; M \cap L = N_L(P \cap L)$

  (iii) $p = 5\,, L \cong Sz(32), McL, {}^2F_4(2)'; M \cap L = N_L(P \cap L)$

  (iv) $p = 5$ *or* 7 *and* $L \cong HS$*,* Ru*,* He*,* $O'N$*, or* $M(24)'$ *;* $m_p(L) = 2$ *and* $\Gamma_{P \cap L,1}(G) \not\leq M$

  (v) $p = 5\,, L \cong A_{10}$

  (vi) $p = 11\,, L \cong J_4; M \cap L = N_L(P \cap L)$.

  (vii) $p = 5$, $L \cong M(22)$ *and* $E(M) = F^*(M)$*. Further* $E(M)$ *involves* $D_4(2)$ *and* $e(G) \geq 4$.

(b) *M is exceptional with respect to p and $E(R/O_{p'}(R)) = X_1L$, where $X_1 \leq X$ and L is as in (a)(i) or (ii).*

(c) *M is exceptional with respect to p and $F^*(R/O_{p'}(R)) \cong \mathbb{Z}_p \times L$, with L as in (a)(i) or (ii).*

(d) $p = 3$, *$R/O_{p'}(R) \cong 3^2SL_2(3)$ or $3^2GL_2(3)$, further a Sylow 3–subgroup T of M is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$ and $\Gamma_{T,2}(G) \not\leq M$.*

In any case $O_{p'}(R) = O_{2'}(R)$ and $O_2(R \cap M) = 1$ or $L \cong L_3(4)$ and some graph $\times$ field automorphism is involved.

Proof: Let $R \not\leq M$. Set $K = O_{p'}(R)$. As $K = \langle C_K(x) | x \in E^\sharp \rangle$ we get $K \leq M$. Set $U = O_{p',p}(R)$. Let $P \in \mathrm{Syl}_p(R)$ with $E \leq P$ and $1 \neq x \in Z(P)$. Then $[x, E] = 1$ and so $x \in M$. If $x \in E$, then $P \leq M$. Assume $x \notin E$. Now $m_p(\langle x, E \rangle) \geq 3$ and so $P \leq N_G(\langle x \rangle) \leq M$ for M being not exceptional. In the other case we have that P is abelian by 5.1 and so $[P, E] = 1$, again $P \leq M$.

So $U \leq M$ in any case. We show next that $U = K$. Suppose $P \cap U \neq 1$. Let first M be exceptional with respect to p. Then by 5.1 and the Frattini argument we have $|U/K| = p$. Further $Q \cap U = 1$. Then we get with Gaschütz lemma a subgroup R_1 of R containing E such that $R_1 \cap U = K$ and $R_1 \not\leq M$, so we are in (c). Hence we may assume $K = U$. So let now M be not exceptional. If $m_p(P) \geq 3$, then $N_R(Z(U \cap P)) \leq M$. So we have $m_p(P) = 2$. Let first $P \cap U$ be cyclic. But as $m_p(P) = 2$, we get that $E \cap U \neq 1$, a contradiction. So we have that $m_p(Z(U \cap P)) = 2$ and $p = 3$. Further $\mathbb{Z}_3 \wr \mathbb{Z}_3$ is a Sylow 3–subgroup of G. As $N_G(E) \leq M$, we have that $E \not\leq U$. Hence $P \cong 3^{1+2}$ and R induces $SL_2(3)$ or $GL_2(3)$ on $U \cap P$, which is (d).

Hence from now on we may assume that $U = K$. Let now W be the preimage of $E(R/U)$. Then $W > K$ and $p \,\Big|\, |W|$. We first show that W/K is simple. Let $E \leq P \in \mathrm{Syl}_p(R)$, then as before $P \leq M$. Let $W_1/K \cdot W_2/K \cdots W_r/K = W/K$, where W_i/K are the components. Suppose there is $\omega \in E$ with $(W_1/K)^\omega \neq W_1/K$. Then we see that $r \geq p$. As $p \geq 3$, we get that $W_2/K \cdots W_r/K \leq N_{W/K}((P \cap W_1)K/K) \leq M$ and $W_1/K \cdots W_{r-1}/K \leq N_{W/K}((P \cap W_r)K/K) \leq M$. Hence $W \leq M$ and as $R = WN_R(W \cap P)$ we see $R \leq M$.

So we have $E \leq N(W_i/K)$, for all i. Let first M be exceptional with respect to p. Then by 5.1 we see that $N_W(P \cap W) \leq M$, as $r \geq 2$. As $N_{W_1}(P \cap W)$ normalizes $Q \cap W$, we see that either $Q \cap W_1 \neq 1$ or $Q \cap W_2 \cdots W_r \neq 1$. Hence we may assume that $W_2 \cdots W_r \leq M$. So $Q \cap W \leq W_1$. Hence we have that $r = 2$ and $W_2 \leq X$. So we are in (b). All what is left to show is that W_1 is as in (a)(i) or (ii). This will be done later.

Let now M be not exceptional with respect to p. There is $x \in Z(P \cap W_1)$ with $N_G(\langle x \rangle) \leq M$, so $W_2W_3 \cdots W_r \leq M$. But the same is true for W_r, i.e. $W_1W_2 \cdots W_{r-1} \leq M$. Hence we have $W \leq M$. By Frattini argument we get

$R \leq M$. So we have that W/K is simple.

Now we may apply [GoLy, (24-9)] to W or W_1. Recall that if M is exceptional for the prime p we have $\Gamma_{P,2}(G) \leq M$ by 5.1. We get a list of possibilities for L.

Assume that L is not one of (i) - (vii). Then we have that $L \cong PSp_4(p)$, $L_3(p)$, A_{2p}, A_{3p}, or $p = 3$ and $L \cong G_2(8)$, $Sp_4(8)$, $Sp_6(2)$, J_3, M_{12}, ${}^2F_4(2)'$ or $p = 5$ and $L \cong {}^2F_4(32)$. We first show that $L \not\cong PSp_4(p), L_3(p)$ or A_{3p}. If $\Gamma_{P,2}(G) \leq M$, then we see that we cannot have one of these groups. So we have that $\Gamma_{P,2}(G) \not\leq M$. Then we have that $p = 3$ and that a Sylow 3–subgroup of G is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. Let $L \cong A_9$ or $PSp_4(3)$. Then there is an elementary abelian subgroup F of P of order 27. We have that $\Gamma_{F,1}(G) \leq M$. But as $\Gamma_{F,1}(G)$ covers L we cannot have one of these. We are left with $L \cong L_3(3)$. We have $N_L(P \cap L) \leq M$ and E is contained in the elementary abelian subgroup F of order 27 in a Sylow 3-subgroup of M. Now $N_L(E) \cong EGL_2(3)$. As F is a Sylow 3–subgroup of $C_G(E)$, we see that $N_M(F)$ involves $GL_2(3)$. But $FGL_2(3)$ does not have $\mathbb{Z}_3 \wr \mathbb{Z}_3$ as a Sylow 3–subgroup.

Suppose next $p = 3$ and $L \cong G_2(8)$, $Sp_4(8)$, $Sp_6(2)$, J_3, M_{12}, ${}^2F_4(2)'$. Then $\Gamma_{P \cap L,1}(G) \not\leq M$. This shows $m_3(M) = 3$. Let first $L \cong G_2(8)$ or $Sp_4(8)$, then we have that $E \not\leq L$. This gives that E induces a field automorphism. Then some element from E centralizes $G_2(2)$ or $Sp_4(2)$ in L. But by [GoLy, 24-10] we have that $M \cap L \cong SU_3(8)$ or $L_2(8) \wr \mathbb{Z}_2$, a contradiction. Let next $L \cong Sp_6(2)$. Then there is some elementary abelian subgroup F in L of order 27 with $\Gamma_{F,1}(L) = L$, a contradiction. Let next $L \cong J_3$, then $m_3(C_L(x)) = 3$ for any element of order three in $P \cap L$ and then again $\Gamma_{P \cap L,1}(L) = L$, a contradiction. Let $L \cong M_{12}$. Then $E \leq L$ and we have that $\Gamma_{P \cap L,1}(L) \neq L$. Hence again $\mathbb{Z}_3 \wr \mathbb{Z}_3$ is a Sylow 3–subgroup of G. Now we get the same contradiction as in the $L_3(3)$–case above. Let finally $L \cong {}^2F_4(2)'$. By [GoLy, 24-10(2)] we have that L contains some $L_3(3)$ in $M \cap L$. But then we have the same contradiction as before.

So let $p \geq 5$. Let $L \cong {}^2F_4(32)$. Then $m_p(P) \geq 3$ and so also W satisfies the assumption for some $\widetilde{E} \leq P \cap W$. But this contradicts [GoLy, (24-9)].

Let $L \cong A_{2p}$, then $A_p \times A_p$ is in $M \cap R$. If $p > 5$, then A_p contains a 2–group which is normalized by an elementary abelian group of order 9. As now $3 \in \sigma(M)$ and $m_3(L \cap M) \geq 4$, we get $R \leq M$, a contradiction.

In all cases the assertions about $M \cap L$ follow from [GoLy, 24-10(1)].

If $L \cong M(22)$, then $M \cap R$ involves $D_4(2)$. If $1 \neq O_2(M)$ or $e(G) = 3$, then $3 \in \sigma(M)$. As $m_3(M \cap R) \geq 4$, we have that R also satisfies the assumptions with respect to the prime 3. But then we would get $R \leq M$.

It remains to show that in (b) and (c) we have L of type (a)(i) or (ii). As P is abelian the other possibilities are (v) or (iii). In (v) we get that L contains a subgroup $A_5 \times A_5$, which is in M. But this contradicts the structure of M being exceptional. So we have (iii). Then $L \cong Sz(32)$ or ${}^2F_4(2)'$. In ${}^2F_4(2)'$, some $SL_2(3)$ acts on a Sylow 5–subgroup, which also contradicts the structure of M. In $Sz(32)$ there is a cyclic subgroup of order 25. Hence we must have an automorphism of order 5 in E. This shows that a Sylow 5–subgroup of G is nonabelian. But a Sylow 5–subgroup of M is abelian in the exceptional case.

We just have to prove the additional assertion about $M \cap R$. Let $2 \,\big|\, |K|$. Then let $T \in \mathrm{Syl}_2(K)$. So $R = K\, N_R(T)$. As we may assume $E \leq N_R(T)$, we get $N_R(T) \leq M$, a contradiction. So $2 \nmid |K|$.

Let $O_2(R \cap M) \neq 1$. Then obviously we are not in (d). Assume that we have $M \cap L = N_L(P \cap L)$. We see that $O_2(N_L(P \cap L)) = 1$. So assume there is some $\omega \in \mathrm{Aut}\ (L)$ with $[\omega, P \cap L] = 1, o(\omega) = 2$. Then we just have $L \cong L_3(4)$. So we are left with (a)(iv), (v) and (vii). As there is no involution in the automorphismen group of L centralizing $D_4(2)$ or $A_5 \times A_5$, we cannot have (v) or (vii). In (iv) it is easy to see that there are no 2–locals of L containing a Sylow p–subgroup. Hence again we just have to investigate outer automorphisms. But there is no such, which centralizes a Sylow p–subgroup of L. (see [CCNPW]). □

$rank2$

Lemma 5.4 *Let M be a uniqueness group, S a Sylow 2–subgroup and assume that $N_G(S) \leq M$. Let $p \in \sigma(M)$ and H be some 2–local containing S. Suppose H contains some elementary abelian p–subgroup E such that $|E| = p^2$ and $\Gamma_{E,1}(G) \leq L$ for some uniqueness group L, then $H \leq M$.*

Proof: We have that $H \leq L$. Now as both L and M contain a Sylow p–subgroup of G, we have some $x \in G$ such that $M \cap L^x$ contains a common Sylow p–subgroup of M and L^x. This now shows that $L^x \leq M$. Now there is some $y \in M$ such that $S \leq L^{xy}$. So we may assume hat $xy \in N_G(S)$. By assumption we have $xy \in M$, so $L \leq M$ and then $L = M$, the assertion. □

Proof: Let P be a Sylow p–subgroup of M with $x \in P$. By assumption $\Omega_1(Z(P))$ is not cyclic. Hence $m_p(C_P(x)) \geq 3$ and so $C_G(x) \leq M$. □

$uniqueMg$

Lemma 5.5 *Let M be a uniqueness group, $g \in G$ and $\omega \in M \cap M^g$ be a p–element, $p \in \sigma(M)$. Suppose that $N_G(\langle\omega\rangle) \leq M$. Then $M = M^g$, or $p = 3$,* *a Sylow 3–subgroup of M is isomorphic to $Z_3 \wr Z_3$ and not for all 3–elements $\rho \in M$ we have that $C_G(\rho) \le M$. If we have that $N_G(\langle\omega\rangle) \le M \cap M^g$, we get $M = M^g$ without restrictions.*

Proof: Let R be a Sylow p–subgroup of $M \cap M^g$. If $N_G(R) \le M$, then R is a Sylow p–subgroup of M and so $M = M^g$. If $m_p(R) = 1$, then $N_G(R) \le N_G(\langle\omega\rangle)$, hence we may assume that $m_p(R) \ge 2$. Suppose first that M is exceptional. Then with 5.1 we get the assertion. So M is not exceptional. Then we have that $p = 3$, R is elementary abelian of order 9 and a Sylow 3–subgroup of G is isomorphic to $Z_3 \wr Z_3$. Further not all elements in R have centralizers in M. This settles the first assertion.

So assume now additionally that $N_G(\langle\omega\rangle) \le M^g$. Let R_1 be a Sylow 3–subgroup of $N_{M^g}(R)$. Now in R there is exactly one subgroup of order three, whose normalizer is in M, this is $\langle\omega\rangle$. Also there is exactly one subgroup of order three, whose normalizer is in M^g, which by assumption again is $\langle\omega\rangle$. But then we have that $R_1 \le N_G(\langle\omega\rangle) \le M$, a contradiction, as $R_1 \ne R$. □

We now collect some properties of the exceptional uniqueness groups.

$ex3$

Lemma 5.6 *Let M be exceptional with respect to p. Let $\tau \in P$ be an element of order p. If $C_{O_2(M)}(\tau) \ne 1$, then $C_G(\tau) \le M$.*

Proof: We have $P \le C_G(\tau)$. Now we may apply 5.3. This gives $p = 3$ and $E(C_G(\tau)/O_{3'}(C_G(\tau)) \cong L_3(4)$. Hence $|P| = 27$. Further we see that $[P, C_{O_2(M)}(\tau)] = 1$, which contradicts the fact hat P contains elements acting fixed point freely on $O_2(M)$.

exp

Lemma 5.7 *Let M be exceptional with respect to p. Let $S \le M \cap H$ and $Y_M \le Y_H$ and $H \not\le M$. Then p does not divide $|C_H(Y_H)|$.*

Proof: Let P be a Sylow p–subgroup of $C_H(Y_H)$ and assume $P \ne 1$. As $Y_M \le Y_H$, we get that that $C_H(Y_H)) \le M$. As $[Y_M, P] = 1$, we get with 5.6 that $N_G(P) \le M$. As $H = C_H(Y_H)N_H(P)$, we get the contradiction $H \le M$. □

$ex4$

Lemma 5.8 *Let B_M be normal in M and $|B_M : Y| = 2$, and $|B_M : Y| = 4$ for $3 \notin \sigma(M)$. Then there is some $1 \ne y \in Y$ which is centralized by some $E \le M$, $E \cong E_{p^2}$, $p \in \sigma(M)$ with $\Gamma_{E,1}(G) \le M$.*

Proof: Let F be elementary abelian of order p^3, $p \in \sigma(M)$, where we choose F in the exceptional case such that $F \cap X \ne 1$. Then there is some $E \le F$ with $|C_{B_M}(E)| \ge 4$, ≥ 8 for $p \ne 3$. Now by 5.6 we get the assertion. □

exF

Lemma 5.9 *Let M be exceptional with respect to p and $V \leq Y_M$ be some F–module for M, then we have $[V, Q] = 1$.*

Proof: This is 3.41. Recall that (c) and (d) of 3.41 are not F–modules. $\square$

ex2F

Lemma 5.10 *Let M be exceptional with respect to p and $V \leq Y_M$ be some $2F$–module for M. Suppose $[V, Q] \neq 1$. Then an offender acts quadratically.*

Proof: We have 3.41(c) or (d). Now we have a direct sum of F–modules on which an offender acts quadratically. $\square$

pnormal

Lemma 5.11 *Let M be a uniqueness group for a prime p and K be a normal subgroup of M with $m_p(K) \geq 2$. Then K contains some elementary abelian subgroup E of order p^2 with $\Gamma_{E,1}(G) \leq M$.*

Proof: This is evident if M is exceptional. So let M not be exceptional. Then all we have to show is that there is some $E \leq K$ such that $C_M(E)$ contains an elementary abelian subgroup of order p^3. In particular we may assume that $m_p(K) = 2$. Let P be a Sylow p–subgroup of K and R a Sylow p–subgroup of M with $P \leq R$. Let C be some characteristic elemententary abelian subgroup of P. Suppose $|C| = p^2$. Then we have that $C = \Omega_1(C_R(C))$. This gives $m_p(R) = 2$, a contradiction. So we have that any characteristic abelian subgroup of P is cyclic and then $\Omega_1(P)$ is extraspecial. As $m_p(P) = 2$, we get that $|\Omega_1(P)| = p^3$ and so M induces a subgroup of $GL_2(p)$ on $\Omega_1(P)$. In particular $|R : C_R(\Omega_1(P)P| \leq p$. As $\Omega_1(P) = \Omega_1(C_R(P))P$, we get that there is some elementary abelian subgroup of order p^3 which intersects $\Omega_1(P)$ in a group of order p^2. $\square$

gengood

Lemma 5.12 *Let M be a uniqueness group and K be a normal component in $M/O_2(M)$. Let further p be a prime with $p \in \sigma(M)$. Assume that M is not exceptional with respect to p. Suppose that p divides $|K|$ and also $|C_{M/O_2(M)}(K)|$. If p does not divide $|Z(K)|$ then for all p–elements $x \in M$ we have that $C_G(x) \leq M$.*

Proof: Let P be a Sylow p–subgroup of M with $x \in P$. By assumption $\Omega_1(Z(P))$ is not cyclic. Hence $m_p(C_P(x)) \geq 3$ and so $C_G(x) \leq M$. $\square$

For a uniqueness group M with $F^*(M) = O_2(M)$ we set $C_M = C_M(Y_M)$. Let S be a Sylow 2–subgroup of M then set $M_0 = N_M(S \cap C_M)$.

CM

Lemma 5.13 *Suppose $\mathcal{M}(M_0) \neq \{M\}$, then for any $p \in \sigma(M)$ there is an elementary abelian subgroup E of C_M such that $\Gamma_{E,1}(G) \leq M$.*

Proof: Let $H \neq M$, $H \in \mathcal{M}(M_0)$. We have $M = M_0C_M$. Suppose that for some p we have that $m_p(C_M) \leq 1$. Then by 2.5, 5.2 we have that M_0 contains a good E As $M_0 \leq H$ we get a contradiction. So we have that $m_p(C_M) \geq 2$. Hence we may assume that M is not exceptional by 5.2. If $m_p(C_M) > 2$, we are done. So assume $m_p(C_M) = 2$. Then there is a Sylow p–subgroup P of M and a normal subgroup Q, which is elementary abelian of order p^2 or extraspecial of order p^3 and $Q \leq C_M$. Hence Q contains a good E as $m_p(P) \geq 3$. □

CentY

Lemma 5.14 *Suppose $\mathcal{M}(M_0) \neq \{M\}$. Let $x \in Y_M^\sharp$ then $C_G(x) \leq M$.*

Proof: This follows from 5.13. □

PinM

Lemma 5.15 *Let M be some uniqueness group with $F^*(M) = O_2(M)$. Let H be a group with $S \leq M \cap H$, S a Sylow 2–subgroup of M, and $F^*(H) = O_2(H)$, but $H \not\leq M$. Let further P be a Sylow p–subgroup of $F(H/C_H)$ and $x \in S$ with $[P, x] \neq 1$ and $|Y_H : C_{Y_H}(x)| = 2$. Assume that for any $1 \neq V \leq Y_M$ we have that $N_G(V) \leq M$. If $Y_M \leq Y_H$ then the preimage of $[P, x]$ in H is not containied in M.*

Proof: Suppose that $[P, x] \leq M \cap H/C_H$.First of all we have that $p = 3$. Set $U = \langle [x, P]^S \rangle$ and $W = C_U(Y_M)$. But as S acts on $[W, Y_H]$ and $C_{Y_H}(S) \leq Y_M$, we see that $[W, Y_H] = 1$, i.e. $W = 1$. We have that $[x, P]$ is generated by elements u with $|[Y_H, u]| = 4$. Now as $[Y_M, u] \neq 1$, we see that $[Y_H, u] \leq Y_M$. So we have that $1 \neq [Y_H, [x, P]] \leq Y_M$. As $[x, P]$ is normal in P we now have that $P \leq M \cap H/C_H$. In particular $U_1 = \langle [x, P]^H \rangle \leq M \cap H/C_H$. Set $U_2 = C_{U_1}(Y_M)$. Then U_2 is S–invariant and so as before we see that $[U_2, Y_H] = 1$, i.e. $U_2 = 1$. Again we see that $[Y_H, U_1] \leq Y_M$ and then we have that $H \leq M$, a contradiction. □

l323

Lemma 5.16 *Let N be a subgroup of the uniqueness group M, with $S \leq N$. Let $3, 7 \notin \sigma(M)$. Assume further that one of the following holds*

(i) N has a factor group N/R isomorphic to $L_3(2) \times L_3(2) \wr Z_2$.

(ii) $N/O_2(N) \cong L_3(2) \wr Z_2$

Then there is a 3-element in at least two of the components $L_3(2)$ whose centralizer is in M.

Proof: In case of (i) we first show that a Sylow 3–subgroup of N is elementary abelian of order 27. Let T be a Sylow 3–subgroup of R, then $N_N(R)$ involves N/R. As $m_3(T) \leq 3$, we get that N/R has to act on a group of order at most 3^5 which is the $\Omega_1(C)$ for some critical subgroup C. This shows that just trivial action is possible, as the smallest faithfull representation of $L_3(2)$ over $GF(3)$ is of dimension 6. Hence N/R is covered by $C(T)$. As $L_3(2)$ has no 3–elements in the Schur multiplier we get that $T = 1$, otherwise $m_3(N) > 3$. Hence in both cases a Sylow 3–subgroup of N is elementary abelian.

We will study the action of N on $F^*(M/O_2(M))$. We set $R = O_2(N)$ in case (ii) and $N = N_l N_t$ where $N_l \cap N_t = R$ and $N_l N_t = N$, $N_t/R \cong L_3(2) \wr Z_2$,$N_l/R \cong L_3(2)$ in case (i) and $N_l = R$ in case (ii). Let first K be some component of $M/O_2(M)$. Let $N_1 = N_N(K)$. If 3 divides the order of K, we see that N_1 covers $E(N/R)$. So suppose that K is a $Sz(q)$ and N_1 does not cover $E(N_t)$. Then we have at least 7 components under the action of N_t . So the Sylow 3-subgroup of a component of N_t centralizes in K^{N_t} an elementary abelian p–subgroup of order p^3. As this component contains an frobenius subgroup of order 21, the element also centralizes nontrivial elements in $O_2(M)$. Hence application of 5.3 shows that its centralizer is in M.

So we may assume that $E(N_t)$ normalizes any component. Assume that $C_{N_t}(K) \leq R$. Then we see that $E(N_t)$ induces inner automorphisms on K, in particular as a parabolic, since $S \leq N$. We have $m_3(K) \leq 3$. So with 1.1 we get that $K \cong L_6(2)$ or $L_7(2)$, or S does not normalize K and then $K \cong L_3(2)$. Suppose we have the latter, then $E(N_t)$ is centralized by some E, $|E| = p^2$ with $\Gamma_{E,1}(G) \leq M$. Then as above, we get the assertion with 5.3. Hence we have the former. We have $m_p(K) \leq 1$. Hence again K is centralized by some E, $|E| = p^2$ and $\Gamma_{E,1}(G) \leq M$. As before we get the assertion.

We have shown that $E(N_t/R) \leq C_{N_t}(E(M/O_2(M)))R/R$. Let now P be a Sylow p–subgroup of $F(M/O_2(M))$ with $C_{N_t}(P) \leq R$. Let C be a maximal elementary abelian characteristic subgroup of P. Suppose $C_{N_t}(C) \leq R$. Then $|C| \geq p^4$ and so $p \neq 3, 7$. There is a subgroup of order 21 in N_t projecting in one of the components which acts faithfully on C. Hence the element of order 3 has fixed points. The other component acts on the fixed points and on the commutator as well. Hence by the same argument an element of order three now has fixed points on both modules, recall that C is completely reducible. But then this 3-element centralizes a good E and so also the other does. As before application of 5.3 gives the assertion. So we may assume that $C_{N_t}(C)R/R$ contains $E(N_t/R)$. Hence $E(N_t/R)$ centralizes any characteristic abelian subgroup of P. So there is a special subgroup U on which N_t acts nontrivially. Further $U = \Omega_1(U)$.

Let first $p \in \sigma(M)$. Then we see that $|\Phi(U)| = p$, otherwise we may apply 5.3. Hence U is extraspecial. Now let x be of order three in one of the components of N_t. Then $C_U(x) \not\leq Z(U)$. Hence x centralizes an elementary abelian subgroup of order p^2 in U. If this group is good, we get the assertion as before. So we have that $|U| \leq p^3$, a contradiction to $C_{N_t}(U) \leq R$. So we have that $p \notin \sigma(M)$. We have $|U/\Phi(U)| \geq p^4$. If $|\Phi(U)| = p^2$, then for $y \in U \setminus \Phi(U)$, we get that $C_U(y) = \langle y, Z(U)\rangle$. This gives the contradiction $|U : Z(U)| \leq p^3$. So again we have that U is extraspecial and so $|U| = p^5$. So N_t is isomorphic to a subgroup of $Sp_4(p)$. By [Mi2] we see $p = 7$ and $SL_2(7) \wr Z_2$ is induced. Now there is some element of order 7 in N_t which centralizes $7^{1+2}SL_2(7)$. But the 7-rank of that group is three and so we get $m_7(M) \geq 4$, a contradiction. □

*ln*2

Lemma 5.17 *Let M be a uniqueness group and $M_0 = N_M(S \cap C_M(Y_M))$, S a Sylow 2–subgroup of M. Let $K \leq M_0$, containing S such that $E(KC_{M_0}(Y_M)/C_{M_0}(Y_M))$ is a component of $M_0/C_{M_0}(Y_M)$ which is isomorphic to $L_7(2)$, $L_6(2)$, $L_5(2)$ or $L_4(q)$, q even. Then K contains a 3–element ρ with $N_G(\langle\rho\rangle) \leq M$, or we have $E(KC_{M_0}(Y_M)/C_{M_0}(Y_M)) \cong L_4(q)$ and there is some ρ with $o(\rho)$ divides $q-1$, such that $N_G(\langle\rho\rangle) \leq M$.*

Proof: Assume otherwise. We have that all groups contain some $L_4(2)$. Hence it is enough to show that this group contains such a 3–element. Set $L = K^\infty$. Then $L/O_{2,2'}(L) \cong L_7(2)$, $L_6(2)$, $L_5(2)$ or $L_4(q)$.

We first prove that L acts trivially on $F(M/O_2(M))$. Let T be a Sylow t–subgroup of $F(M/O_2(M))$. Assume that there is some elementary abelian characteristic subgroup $C \leq T$, with $C_L(C) \leq O_{2,2'}(L)$. Then in particular $m_t(C) \geq 4$. As centralizers in C of 3–elements of L are of order at most t by 5.3, we see that $|C| = t^4$. We claim that $GL(4,t)$ does not involve A_8. We see that $L/C_L(C)$ either has a subgroup A_8 or $2A_8$. In both cases there is some elementary abelian subgroup of order 16 in $L/C_L(C)$, which by 2.1 implies that we have transvections on C. But those are not in $GL(4,t)'$. Hence we have that L centralizes any characteristic abelian subgroup of T. So assume now that it acts on a special subgroup C with $C = \Omega_1(C)$. Again $|C/\Phi(C)| \geq t^5$. But then $m_t(C) \geq 4$ and 3-elements in L which are in a Frobenius group of order 21 centralize $Z(C)$ and some t-element in $C \setminus Z(C)$ and so some good E. As they also centralize nontrivial elements in $O_2(M)$, we get the assertion with 5.3.

So we may assume that L centralizes $F(M/O_2(M))$. Let now U be some component of $M/O_2(M)$. If L does not normalizes this component, we are immediately done. So we may assume that $[U, L] \leq U$. If $U \neq L/O_{2,2'}(L)$, then

L centralizes U. Hence there is some U which is not centralized by L, which shows that $L/O_2(L)$ is some component. Suppose now first $L/O_2(L) \not\cong L_4(q)$, $q > 2$. If $3 \in \sigma(M)$ then L contains elements ρ of order three, which are centralized by some elementary abelian subgroup of order 27 in M. Hence $N_G(\langle\rho\rangle) \leq M$. So we may assume that $3 \notin \sigma(M)$, we now see that $L/O_2(L)$ is centralized by some good E, a contradiction. So we have $L/O_2(L) = L_4(q)$, $q > 2$. By the same argument as before we may assume that there is no uniqueness prime dividing $q-1$. But then there is some good E normalizing L and so centralizing a Sylow 3–subgroup of L. □

alperin

Lemma 5.18 *Let M be a uniqueness group, $p \in \sigma(M)$ and P a Sylow p–subgroup of M. Suppose N is a normal subgroup of M such that $P = (N \cap P)Z$, with a cyclic group $Z \notin N$. If $N_M(ZN/N) \neq C_M(ZN/N)$, then $p = 3$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M.*

Proof: Assume that $P \not\cong Z_3 \wr Z_3$. Then for any subgroup X of P with $m_p(X) \geq 2$, we have that $N_G(X) \leq M$. Let H be any subgroup of P. Then by assumption we have that $[H, N_M(H)] \leq N$. If $m_p(H) > 1$, then $N_M(H) = N_G(H)$, so $[H, N_G(H)] \leq N$. Let $m_p(H) = 1$. If $C_P(H)$ is cyclic, then $H \cap Z(P) \neq 1$ and so $N_G(H) = N_M(H)$. So assume that $C_P(H)$ is not cyclic. As normalizers of p-goups of rank at least two are in M, we get that a Sylow p–subgroup R of $C_G(H)$ is contained in M. Now $N_G(H) = C_G(H)N_G(R)$, where again $N_G(R) \leq M$. As $[H, N_G(H)] = [H, N_G(R)]$, we see that $[H, N_G(H)] = [H, N_M(H)] \leq N$. Hence we have that $\langle [H, N_G(H)] \mid 1 \neq H \leq P\rangle \leq N$. Application of [Go, (7.4.1.9)] gives the contradiction that G has a subgroup of index three.

6 The nonconstrained case

The purpose of this chapter is to prove that a uniqueness group M has to satisfy $F^*(M) = O_2(M)$. For this we assume that M is a uniqueness group with $F^*(M) = KO_2(M)$, where $O_2(M)$ might be trivial and K is some component which is a group of Lie type in characteristic 2, not $L_2(q)$, $U_3(q)$, $Sz(q)$, $L_3(q)$, $Sp_4(q)$ or ${}^2F_4(q)$, $Z(K) = O_2(K)$, and for every $p \in \sigma(M)$ we have $m_p(K) \geq 2$ and $m_p(C_M(K)) \leq 1$. In particular we have $m_3(K) \geq 2$. For the remainder of this chapter we assume that M does not satisfy the conclusion of the main theorem.

$3sigma$

Lemma 6.1 *Let* $m_{2,3}(G) \geq 4$*, then* $3 \in \sigma(M)$*.*

Proof: Let $3 \notin \sigma(M)$. Let further H be a uniqueness group for the prime 3. Let P be a Sylow 3-subgroup of M, $P \leq H$. As $m_3(H) \geq 4$, we have that $\Gamma_{P,1}(G) \leq H$, if H is not exceptional for the prime 3. So assume first $\Gamma_{P,1}(G) \leq H$. We have that $m_3(K) \geq 2$, so by 5.3 we get that $M \leq H$. If $O_2(H) \neq 1$, then we now have $H \leq M$, the assertion. So assume that $O_2(H) = 1$. Let $p \in \sigma(M)$, then $p > 3$. With 5.3 we see that $H \leq M$, the assertion.

Hence we are left with H exceptional for the prime 3. Suppose that K has an elementary abelian Sylow 3-subgroup E of order 9, with $\Gamma_{E,1}(G) \not\leq H$. Application of 1.1 shows $K \cong L_4(q)$, $L_5(q)$, $U_4(q)$ or $U_5(q)$. Now by 5.1 we have that $N_K(E) \leq H$. But $N_K(E)$ acts irreducibly on E, which shows that $E \leq O_3(H/O_2(H))$, a contradiction, as $C_G(x) \leq H$ for any x, $o(x) = 3$ with $xO_2(H) \in O_3(H/O_2(H))$. So we have $m_3(K) \geq 3$. Now P contains some E, $|E| = 9$ and $EO_2(H)/O_2(H) \leq O_3(H/O_2(H))$. Application of 5.3 gives $M \leq H$, a contradiction as $m_3(K) \geq 2$. □

$3sigma1$

Lemma 6.2 *Let K be defined over $GF(2)$, then one of the following holds*

(i) $3 \in \sigma(M)$.

(ii) $K \cong {}^3D_4(2)$, $O_2(M) \neq 1$, $e(G) = 3$, $\sigma(M) = \{7\}$ *and* $7 \| C_M(K)|$.

Proof: Let $3 \notin \sigma(M)$. Let $p \in \sigma(M)$, $p > 3$, with $m_p(K) \geq 2$. If $O_2(M) \neq 1$ we have that $m_3(K) \leq 3$. Then by 1.1 we have that $K \cong L_6(2)$, $L_7(2)$ or ${}^3D_4(2)$ and $p = 7$. In the first two cases we have $m_3(K) = 3$ and so we must have that $m_7(M) \geq 4$, which contradicts $m_7(C_M(K)) \leq 1$. So we have $K \cong {}^3D_4(2)$. We have $m_{2,7}(K) = 1$. Hence $7 \| C_M(K)|$. As $m_7(K) = 2$, we also see that $m_7(M) = 3$ and so $e(G) = 3$. This is (ii).

So we have that $K = F^*(M)$. As we do not have an outer automorphism of order p, we get that $m_{2,p}(K) \geq 3$. Now K possesses maximal parabolics P which involve $L_n(2)$, $L_n(4)$, $U_n(2)$, or $\Omega^{\pm}(2n,2)$. As $m_3(P) \leq 3$ we get with 1.1 that we have $L_n(2)$, $2 \leq n \leq 7$, $L_n(4)$, $2 \leq n \leq 4$, $U_4(2)$ or $\Omega^{-}(6,2)$. But none of them contains an elementary abelian subgroup of order p^3 for some $p > 3$. □

Suppose in case of $K \cong F_4(q)$ that S does not induces a diagram automorphism on K. Then for the remainder of the proof we fix a long root group R in $K/Z(K)$, with $R \leq Z(S \cap K)$. Let $\tilde{R}$ be a Sylow 2–subgroup of the preimage in K and $G_1 = N_M(\tilde{R})$. Then $G_1 \cap K$ is 2-constrained. In case of $K \cong F_4(q)$ and S induces a diagram automorphism we choose G_1 such that $G_1 \cap K$ is the parabolic with $Sp_4(q)$ on top. Hence in all cases $S \leq G_1$.

*goodE*1

Lemma 6.3 *There is a prime $p \in \sigma(M)$ and $E \leq G_1, E \cong E_{p^2}$ with $\Gamma_{E,1}(G) \leq M$. Further $C_{G_1}(Z(S))$ involves some $L_2(q)$, $U_3(q)$ or $L_3(2)$, which contains a good p–element.*

Proof: Let first $K \cong G(q), q > 2$. Then application of 1.3 shows that we have the assertion or $K/Z(K) \cong L_4(q), Sp_6(q), U_n(q), n \leq 7, \Omega_8^-(q)$, ${}^3D_4(q)$ or $G_2(q)$.

Let $K/Z(K) \cong L_4(q), Sp_6(q), U_4(q)$, or $\Omega_8^-(q)$. If we have $m_p(K) \geq 3$, then $p \mid q-1, q^2-1, q+1, q^2-1$ respectively, and the assertion holds. So let $m_p(K) = 2$. As there is always some prime r with $m_r(K) \geq 3$, we get $e(G) > 3$ and so $m_p(C_M(K)) = 1$ and $m_p(\mathrm{Aut}_M(K)) = 3$, which gives the assertion again.

Let $K/Z(K) \cong U_n(q), 5 \leq n \leq 7$. We have that $m_p(K) \geq 4$ for $p | q+1$. Now G_1 involves $SU_{n-2}(q)$ and so $m_p(G_1) \geq 3$ and we are done, or $n = 5$, $p = 3$ and $m_3(G_1) = 2$. But as $m_3(K) = 4$, we have that all elementary abelian subgroups of order 9 are good, the assertion.

Let $K/Z(K) \cong {}^3D_4(q)$ or $G_2(q)$. We have $m_p(K) = 2$, so either $m_p(\mathrm{Aut}_M(K)) = 3$, or $m_p(C_M(K)) = 1$. Then in the case of $G_2(q)$ we are done, as p divides q^2-1, and in case of ${}^3D_4(q)$ as p divides q^6-1, so there is always some $E \cong E_{p^2}$, $E \leq G_1$, which is centralized by some elementary abelian group of order 27.

Let now $K = G(2)$ be defined over $GF(2)$. By 6.2 we have $3 \in \sigma(M)$ or $K \cong {}^3D_4(2)$. Let first $3 \in \sigma(M)$. Let E be an elementary abelian group of order 9 in $G_1 \cap K$. If $m_3(M) \geq 4$, we are done. So assume $m_3(M) = 3$. If $m_3(C_M(K)) = 1$, also any element of order 3 is centralized by some elementary abelian group of order 27, the assertion. So we have $m_3(\mathrm{Aut}_M(K)) = 3$.

With 1.1 we get $K \cong U_4(2)$, $Sp_6(2)$, $\Omega^-(8,2)$, $L_6(2)$, or $L_7(2)$. But in all these groups an element of order 3 is centralized by an elementary abelian group of order 27.

So we may assume that $G_1 \cap K$ does not contain an elementary abelian subgroup of order 9. Then $K \cong G_2(2)'$, ${}^3D_4(2)$, $L_4(2)$, or $L_5(2)$. If 3 divides $|C_M(K)|$, then all elements of order 3 are good and G_1 contains an elementary abelian subgroup of order 9, so we are done. So we may assume that $C_M(K)$ is a $3'$–group. So $m_3(\mathrm{Aut}_M(K)) = 3$. This shows $K \cong {}^3D_4(2)$. Now we have an elementary abelian subgroup E of order 9 in G_1, where $E \leq K_1 \cong \mathbb{Z}_3 \times G_2(2)$, and so E is good.

Let finally $K \cong {}^3D_4(2)$ and $p = 7$. By 6.2 we have that $7 || C_M(K)|$. Hence G_1 contains an elementary abelian subgroup E of order 49. As $\Omega_1(Z(T))$ contains an elementary abelian subgroup of order 49, for T a Sylow 7–subgroup of M, we see that all 7–elements are good. □

Let G_2 be a subgroup of G such that $S \leq G_2$, $O_2(G_2) \neq 1$ and G_2 is minimal with respect to $G_2 \not\leq M$. Such a group exists, as otherwise M would satisfy the conclusion of the theorem. We have that $m_3(G_2) \leq 3$ by 6.1

amalgam1

Lemma 6.4 $O_2(\langle G_1, G_2 \rangle) = 1$.

Proof: This follows from 6.3 and the definition of the uniqueness case. □

2constr

Lemma 6.5 *We may assume that* $C_{G_2}(O_2(G_2)) \leq O_2(G_2)$.

Proof: Suppose that G_2 has a component L. Let x be some involution in $C_{G_2}(L) \cap Z(S)$. Set $H = C_G(x)$. As $S \leq G_2$, we get that H also has a component $L_1 \geq L$. Now $L_1 \in \mathcal{C}_2$ and we may assume that $L_1 \not\leq M$. If $\langle L_1, S \rangle$ is generated by groups X with $S \cap L_1 \leq X$ and $C_X(O_2(X)) \leq O_2(X)$, we are done, as we may choose G_2 in $\langle X, S \rangle$. So we are left with $L_1 \cong L_2(p)$, p a Fermat - or Mersenne - prime, $L_3(3)$ or M_{11}. By 6.3 we have that $C_G(x)$ involves some $T \cong L_2(q)$, $U_3(q)$ or $L_3(2)$, which contains a good p–element. Hence this group cannot centralize L_1. Let $3 \in \sigma(C(x))$, then with 6.1 we get that $m_3(C_G(x)) = 3$. Hence in any case we have that $m_3(C_G(x)) \leq 3$. This now implies that $T \cong \Sigma_3$ and either there are three components of type $L_2(p)$ or T induces an inner automorphism group on $L_3(3)$ or M_{11}. In the latter, as we have a good 3–element, we see that 9 divides the order of $L_1 \cap M$. Hence $L_1 \cap M \cong 3^2GL_2(3)$. But there are no groups of order 9 in M on which $S \cap K$ acts nontrivially. So we have that there are three components permuted by

T. But now $S \cap L_1$ is a maximal subgroup of L_1 and the element ρ of order three in T centralizes some element of odd order in $L_1^{\langle\rho\rangle}$, which then shows $L_1 \leq M$, a contradiction. $\square$

Set $Z_2 = \langle \Omega_1(Z(S))^{G_2} \rangle$. Then by 6.5 $Z_2 \leq Z(O_2(G_2))$.

$nontriv1$

Lemma 6.6 *If* $[Z_2, O_2(G_1 \cap K)] = 1$, *then* $C_G(Z_2) \leq M$

Proof: Suppose $C_G(Z_2) \not\leq M$. As $m_p(K) \geq 2$, we get $C_{Z_2}(K) = 1$. In particular $O_2(M) = 1$. As $R \cap Z(S) \neq 1$, we see with 6.3 that $C_R(E) = 1$. This shows that $K = G(q)$ and $p \mid q-1$. Furthermore $m_p(G_1) = 2$. Hence $m_p(K) \geq 3$. Application of 1.3 shows $K \cong L_4(q), Sp_6(q), U_n(q), 5 \leq n \leq 7$, or $\Omega^-(8,q)$. As $m_p(K) \geq 3$ and $p \mid q-1$, we get a contradiction. $\square$

$nontriv2$

Lemma 6.7 *Let* $[Z_2, O_2(G_1 \cap K)] = 1$. *If* $P \in Syl_p(C_M(Z_2))$, *then* $N_G(P) \not\leq M$, *in particular* $m_p(P) \leq 1$.

Proof: Suppose false. Then $N_G(Z_2) = C_G(Z_2)N_G(P) \leq M$ by 6.6 and the Frattiniargument.

Suppose that $m_p(P) \geq 2$, then we have that $p = 3$, $P \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \wr \mathbb{Z}_3$ is a Sylow 3–subgroup of M. In particular we have that $m_p(C_M(K)) = 0$. With 1.1 we get $K \cong Sp_6(q)$, $\Omega^-(8,q)$, $L_6(q)$, or $L_7(q)$. In all these cases P is contained in the corresponding group over $GF(2)$. Hence one can see that any element of order 3 is centralized by an elementary abelian group of order 27, and so $N_G(\langle\omega\rangle) \leq M$ for all $1 \neq \omega \in P$, a contradiction. $\square$

NS

Lemma 6.8 *We have* $N_G(S) \leq M$.

Proof: Suppose false. Then we can choose G_2 inside of $N_G(S)$, as $C_G(S) \leq S$. But now by 6.3 a Sylow p–subgroup P of $C_M(Z_2)$ is nontrivial, as $\Omega_1(Z(S)) = Z_2$. By 6.7 it is cyclic. Further $N_G(P) \not\leq M$, which contradicts 6.3 $\square$

$nontriv$

Lemma 6.9 *We may choose* G_2 *such that* $[Z_2, O_2(G_1 \cap K)] \neq 1$.

Proof: Suppose false. Let $K = G(q)$. Let $Z(O_2(G_1 \cap K))/Z(K) = R$. Let E be as in 6.3. If $C_{E\cap K}(R) \neq 1$, then also $C_{E\cap K}(Z_2) \neq 1$ and so by 6.7 $E \cap K = \Omega_1(P)$, where P is a Sylow p–subgroup of $C_M(Z_2)$. But then $N_G(P) \leq N_G(E \cap K) \leq M$ by 6.3. Hence $C_{E\cap K}(R) = 1$ and so p divides $q-1$ and $m_p(G_1) = 2, m_p(G_1 \cap K) = 1$. Application of 1.3 yields a contradiction.

So let now $K \cong Sp_{2n}(q)$ or $F_4(q)$ and $(Z_2 \cap K)Z(K)$ not be contained in $RZ(K)$, otherwise we argue as before. If $K \cong Sp_{2n}(q)$, then Z_2 is centralized by $Sp_{2n-4}(q)$. If $n \geq 4$, then $p \mid q^2 - 1$, a contradiction. Hence we have $K \cong Sp_6(q)$. But in this case there is a p-element $\omega \in C_K(Z_2)$ with $m_p(C_M(\omega)) \geq 3$, hence $N_G(\langle \omega \rangle) \leq M$, contradicting 6.7.

So let $K \cong F_4(q)$. Then we have that there is no diagram automorphism induced by S. Otherwise $G_1 \cap K/O_2(G_1 \cap K) \cong Sp_4(q) \times \mathbb{Z}_{q-1}$, and so Z_2 is centralized by some $Sp_4(q)$ and so by some E as in 6.3. But this contradicts 6.7. So we have that $G_1 \cap K/O_2(G_1 \cap K) \cong Sp_6(q) \times \mathbb{Z}_{q-1}$. Now as Z_2 projects onto $Z(O_2(G_1 \cap K))$, we see with 1.4, that Z_2 is centralized by some $Sp_4(q)$ in K, and so by some good E, which contradicts 6.7. $\square$

center

Lemma 6.10 *We have $\Omega_1(Z(S))$ is not normal in G_2.*

Proof: This follows from 6.9 $\square$

strukturG2

Lemma 6.11 *We have $C_{G_2}(Z_2)$ is 2–closed and $C_{G_2}(Z_2)/O_2(G_2)$ is nilpotent. Further $m_3(G_2) \leq 3$. If G_2 is nonsolvable and U is some normal r–subgroup in $G_2/O_2(G_2)$, r a prime, then $m_r(U) \leq 3$.*

Proof: Let $T = S \cap C_{G_2}(Z_2)$. Then $G_2 = C_{G_2}(Z_2)N_{G_2}(T)$. If $N_G(T) \leq M$, then $C_{G_2}(Z_2) \not\leq M$. Hence $G_2 = C_{G_2}(Z_2)S$, and so $\Omega_1(Z(S))$ is normal in G_2, which contradicts 6.10. So we have that $C_{G_2}(Z_2) \leq M$ and $T = O_2(G_2)$.

Let U be a Sylow r–subgroup of $C_{G_2}(Z_2)$, r odd. Then $G_2 = C_{G_2}(Z_2)N_{G_2}(U)$. In particular $N_{G_2}(U) \not\leq M$. So $TN_{G_2}(U) \not\leq M$. But $S \leq TN_{G_2}(U)$. Hence U is normal in G_2/T. This shows that $C_{G_2}(Z_2)/T$ is nilpotent.

Let X be the preimage of $O_2(G_2/C_{G_2}(Z_2))$ and $T_1 = S \cap X$. Then $G_2 = XN_{G_2}(T_1)$. As $X \leq M$, we have that $N_{G_2}(T_1) \not\leq M$ and so T_1 is normal in G_2, which gives $T_1 = O_2(G_2)$.

Let $m_3(G_2) \geq 4$. Then $3 \in \sigma(G)$ and so by 6.1 $3 \in \sigma(M)$. Hence $G_2 \leq M^g$ for some $g \in G$. But $S \leq M^g$ and so by 6.8 $G_2 \leq M$, a contradiction.

Let U be some r–subgroup in G_2 such that $O_2(G_2)U$ is normal in G_2. Then $G_2 = O_2(G_2)N_{G_2}(U)$. Hence US is a subgroup of G_2. Now assume that G_2 is nonsolvable. Then $US \neq G_2$ and so $U \leq M$. If $m_r(U) \geq 4$, then $r \in \sigma(M)$ and so $N_G(U) \leq M$, hence $G_2 \leq M$, a contradiction. $\square$

nono2

Lemma 6.12 *We have $[Z_2, K \cap G_1] \not\leq O_2(G_1)$.*

Proof: Suppose false. By 6.9 we get $[Z_2, O_2(G_1 \cap K)] \neq 1$. Suppose first that $[Z_2, N_K(R)] \leq O_2(N_K(R))$, where in case of $F_4(q)$ R might be one of the two root groups in $Z(S \cap K)$. Now by 1.6 there is a group U of order q in $O_2(G_1 \cap K)$ with $U \cap C(Z_2) = 1$ and $\mid Z_2 : C_{Z_2}(U) \mid \leq q$. Hence Z_2 is an F-module. If $[Z_2, N_K(R)] \not\leq O_2(N_K(R)))$, then $K \cong F_4(q)$ and S induces a diagram automorphism. We see with 1.7 that $\mid Z_2 : C_{Z_2}(Z(O_2(N_K(R)))) \mid \leq \mid Z(O_2(N_K(R))) : C_{Z(O_2(N_K(R)))}(Z_2) \mid$. Hence also in this case Z_2 is an F-module. Now by 6.11 and 4.6 we have that G_2 is solvable or $E(G_2/C_{G_2}(Z_2)) \cong L_2(r), L_2(r) \times L_2(r), r$ even, or A_9.

Assume first that in case of $K \cong F_4(q)$ we have that $Z_2 \leq O_2(N_{G_1}(R))$ for both root groups R in $Z(S \cap K)$

Let first $K = G(q), q > 2$. For every $u \in U^\sharp$ we have $C_{Z_2}(u) = C_{Z_2}(U)$ by 1.6 and so we get $E(G_2/Q_2) \cong L_2(r)$ or $L_2(r) \times L_2(r)$, and $\mid U \mid = r$. Now we see $q = r$. Let $\omega \in M \cap G_2, o(\omega) = t \mid q - 1$, t a prime. As $G_2 = \langle S, N_{G_2}(\langle \omega \rangle) \rangle$ we have that $N_G(\langle \omega \rangle) \not\leq M$.

Suppose there is some $p \in \sigma(M)$ with $p \mid q - 1$. Then we may choose ω with $o(\omega) = p$. Now $m_p(C_M(\omega)) \leq 2$. As ω is in a minimal parabolic of M, we see that ω is either an outer automorphism of K or centralizes a Cartan subgroup C. Hence $m_p(C) \leq 2$ in any case. By 1.3 we see $K \cong U_4(q)$, $U_5(q)$, $G_2(q)$ or ${}^3D_4(q)$. In all cases we have that $m_p(K) = 2$ and so as $m_p(C_M(\omega)) = 2$, we see that $m_p(C_M(K)) = 0$. This shows that we have a field automorphism of order p. Now ω is in a minimal parabolic and so it centralizes an elementary abelian group of order p^3, a contradiction.

So we have that there is no $p \in \sigma(M)$ with $p \mid q - 1$. Hence with 1.3 we get that $K \cong L_4(q)$, $U_n(q)$, $n \leq 7$, $Sp_6(q)$, ${}^3D_4(q)$, $G_2(q)$ or $\Omega^-(8, q)$.

We have $[Z_2, O_2(G_1 \cap K)] = R$, so $[U, O_2(G_1 \cap K)] = R$. Further by 6.3 we have some $x \in R^\sharp$, which is centralized by E, with $\Gamma_{E,1}(G) \leq M$. In particular $C_G(x) \leq M$. This shows that $E(G_2/C_{G_2}(Z_2)) \cong L_2(q)$ and $Z_2 = A \times B$, with $R \leq B$, B the natural module. Now let $g \in G_2 \setminus M$, with $B = RR^g$ and $[U, R^g] = R$. There is some $a \in R^g$, $aR \in Z(S/R)$. Let $a = uv$, $u \in C_M(K)$, $v \in K$. Then $v \neq 1$. Suppose that $v \sim x$ in K. Then we have some $\nu \in K$, $o(\nu) = p$, $[\nu, v] = 1$, such that $N_G(\langle \nu \rangle) \leq M$. Suppose next that $v \not\sim x$ in K. Then $K \not\cong G_2(q)$, ${}^3D_4(q)$, $\Omega^-(8, q)$. Let $K \cong L_4(q)$. Then $E(C_K(v)/O_2(C_K(v))) \cong L_2(q)$. As $m_p(K) = 2$, we see that all p–elements in K are good, so also $N_G(\langle \nu \rangle) \leq M$, where $o(\nu) = p \mid q + 1$. Let $K \cong U_n(q)$. Then $C_K(v)/O_2(C_K(v))$ involves $L_2(q)$ and so v is centralized by some ν, $o(\nu) = p$ again. This ν now is contained in some $U_4(q) \cong \Omega^-(6, q)$. But then it is contained in some $\Omega^-(2, q) \times \Omega^-(2, q) \times \Omega^-(2, q)$. Hence $N_G(\langle \nu \rangle) \leq M$. Let finally $K \cong Sp_6(q)$. By normal form it is easy to see that all involutions in K are centralized by some $L_2(q)$ and so also by some ν, $o(\nu) = p$. As $\Omega^-(6,q)$ and $Sp_6(q)$ have the same Sylow p–subgroup, we get $N_G(\langle\nu\rangle) \leq M$.

Hence in any case a is centralized by some ν, $o(\nu) = p$ and $N_G(\langle\nu\rangle) \leq M$. As $C_G(x) \leq M$, we have $C_G(a) \leq M^g$, so $\nu \in M^g$. Set $W = N_{M^g}(\langle\nu\rangle)$. Then $W \leq M$ and $m_p(W) \geq 2$. Assume now that $N_G(P) \leq M$ for any p–subgroup P of M with $m_p(P) \geq 2$. Then M and M^g share a Sylow p–subgroup T. Hence we have $M^g = M^h$ for some $h \in N_G(T) \leq M$, so $M = M^g$. So $g \in N_G(M)$ and then $\langle M, g\rangle = MN_{\langle M,g\rangle}(T) = M$. So we have $g \in M$, contradicting the choice of g.

So we have that there is some p–subgroup $P \leq M$ with $m_p(P) = 2$ and $N_G(P) \not\leq M$. This gives $p = 3$ and $|P| = 9$. Further a Sylow 3–subgroup of G is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. As $e(G) = 3$ and there is no $p \in \sigma(M)$ with $p \mid q-1$ and $3 \mid q+1$, we see that $K \not\cong L_4(q)$, $Sp_6(q)$, $\Omega^-(8,q)$, $U_6(q)$ or $U_7(q)$, as in all these cases there is a 2–local whose order is divisble by $(q-1)^3$. If $O_2(M) \neq 1$, then $(q+1)^3$ does not divide $|K|$, as $q+1 \neq 3$. Hence $K \cong G_2(q)$ or ${}^3D_4(q)$. But in none of these case $\mathrm{Aut}_M(K)$ has a Sylow 3–subgroup $\mathbb{Z}_3 \wr \mathbb{Z}_3$, a contradiction. So $O_2(M) = 1$. In $G_2(q)$ and ${}^3D_4(q)$ we have $m_{2,3}(K) = 1$, but we must have $m_{2,3}(K) = 3$, a contradiction. So we have that $K \cong U_4(q)$ or $U_5(q)$. But then the Sylow 3–subgroup is in some $U_4(2)$, as $3 \mid q+1$ and so all elements of order 3 in $U_4(2)$ are centralized by some elementary abelian subgroup of order 27, i.e. $N_G(P) \leq M$, a contradiction.

So we are left with $K = G(2)$. Now U induces transvections and so G_2 is solvable or $E(G_2/C_{G_2}(Z_2)) \cong L_2(4)$, $L_2(4) \times L_2(4)$ or A_9.

Let first $3 \in \sigma(M)$. If $m_3(M) \geq 4$, then all elements of order 3 are good. Let $m_3(M) = 3$. Then we see that $K \cong L_n(2)$, $4 \leq n \leq 7$, $U_4(2)$, $\Omega^-(8,2)$, $Sp_6(2)$, $G_2(2)$ or ${}^3D_4(2)$. In any case all 3-elements are good, as either $m_3(C_M(K)) \neq 1$ or any element of order 3 in K is centralized by some elementary abelian group of order 27. Let G_2 be nonsolvable. As $G_2 = \langle G_2 \cap M, N_{G_2}(\langle\nu\rangle)\rangle$, for ν a 3–element in $G_2 \cap M$, we get a contradiction.

Hence G_2 is solvable. If $m_3(G_2) > 1$, then we have that $G_2 \leq M^g$ for some $g \in G$. But then $S \leq M \cap M^g$ and so we have $g \in N_G(S) \leq M$ by 6.8, a contradiction. So we have shown that $G_2/C_{G_2}(Z_2) \cong \Sigma_3$ and $|[G_2, Z_2]| = 4$. Let g be as before with $x^g = a$. Then as all elements of order 3 are good, we see that a cannot be centralized by a 3–element. By 1.9 we see that $m_3(K) \leq 3$ and so by 1.3 $K \cong L_n(2)$, $4 \leq n \leq 7$, $U_4(2)$, $\Omega^-(8,2)$, $Sp_6(2)$, $G_2(2)'$ or ${}^3D_4(2)$. But in $L_n(2)$, $U_4(2)$, $\Omega^-(8,2)$, $Sp(6,2)$ or $G_2(2)'$ all involutions are centralized by some 3-element, a contradiction. In ${}^3D_4(2)$ the elements in $[G_2, Z_2]$ are conjugated in K, which shows that we also have a

3-element centralizing $t \in Z_2$ with $[t, O_2(G_1)] \neq 1$, a contradiction.

So we are left with $K \cong F_4(q)$ and $Z_2 \not\leq O_2(N_{G_1}(R)), R$ a root group. Now choose $x \in Z(O_2(N_K(R))), [x, Z_2] \neq 1$. Then $\mid [x, Z_2] \mid \geq q$. Let $y \in O_2(N_K(R)) \setminus Z(O_2(N_K(R)))$ with $[y, Z_2] \neq 1$, then $\mid [y, Z_2] \mid \geq q^2$.

Let first $q > 4$. Then we get $|[y, Z_2]| \geq 64$ and so by 4.6 we see that $E(G_2/C_{G_2}(Z_2)) \cong L_2(r)$ or $L_2(r) \times L_2(r)$ and just natural modules are involved.

If $q \leq 4$, we get $3 \in \sigma(M)$ and $m_3(M) \geq 4$. Hence all elements of order 3 are good. If G_2 contains an elementary abelian group of order 9, then $G_2 \leq M^h$ for suitable $h \in G$. But now as M and M^h contain S, we get $M = M^h$ by 6.8, a contradiction. This shows that $G_2/C_{G_2}(Z_2)$ is an automorphism group of $L_2(r)$ and $3 \not| |G_2 \cap M|$. This shows that Z_2 is the natural module.

We have that $[[y, Z_2], O_2(N_K(R))] = R$ and further that $|O_2(N_K(R)) : C_{O_2(N_K(R))}([y, Z_2])| \geq q^2$. Let $yC_{G_2}(Z_2)$ be in $E(G_2/C_{G_2}(Z_2))$, or arbitrary for solvable G_2, then we see that $[y, Z_2]$ is centralized by a Sylow 2–subgroup of this group. Hence we have that $|O_2(N_K(R)) : C_{O_2(N_K(R))}([y, Z_2])| \leq 4$, hence $q = 2$ and $|[y, Z_2]| = 4$. Suppose that no $yC_{G_2}(Z_2)$ is contained in $E(G_2/C_{G_2}(Z_2))$, then $q^2 \leq |O_2(N_K(R)) : (O_2(N_K(R)) \cap C_{G_2}(Z_2))Z(O_2(N_K(R)))| \leq 4$. This again shows that $q = 2$. In both cases we have that either $|[Z_2, y]| = 4$ or Q contains a foursgroup which intersects $E(G_2/C_{G_2}(Z_2))$ trivially. Hence in both cases G_2 contains an elementary abelian subgroup of order 9. As $q = 2$ and $m_3(F_4(2)) = 4$, we see that all elements of order three are good. So $G_2 \leq M^h$ for some h, a contradiction. □

Now we have $Z_2 \not\leq O_2(G_1)$. By 4.1 we get that there is some $g \in G_1$ such that for $X = \langle Z_2, Z_2^g \rangle$ we either have

(1) $X/O_2(X) \cong D_{2u}$ (u odd), $L_2(q_1)$ or $Sz(q_1)$, q_1 even

(2) $Y = (Z_2 \cap O_2(X))(Z_2^g \cap O_2(X) \trianglelefteq X$

(3) $Y \neq Z_2 \cap O_2(X)$

(4) $\mid Z_2 : C_{Z_2}(Y/C_Y(Z_2)) \mid \leq \mid Y : Y \cap O_2(G_2) \mid^2$

or $1 \neq [Z_2, Z_2^g] \leq Z_2 \cap Z_2^g$, with $g^2 \in N(Z_2)$.

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Lemma 6.13 *If Z_2 is not an F–module, then $\mid Z_2 : C_{Z_2}(Y) \mid < \mid Y : Y \cap O_2(G_2) \mid^2$.*

Proof: This is 4.2(3) $\Box$

If Z_2 is not an F–module, then G_2 is as in 4.4 with $Y_Y = Z_2$. If Z_2 is an F–module, we have that G_2 is solvable or $E(G_2/C_{G_2}(Z_2)) \cong L_2(r), L_2(r) \times L_2(r), r$ even, or A_9. by 4.6.

Let us first assume that we have in G_2 a group of Lie type over $GF(r)$. Let t be a primitive prime divisor of $r-1, t = 9$ in case of $r = 64$, and $\omega \in G_2 \cap M$, $o(\omega) = t$. Hence in all cases we have that $N_G(\langle\omega\rangle) \not\leq M$.

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Lemma 6.14 $[K, \omega] \neq 1$.

Proof: Otherwise by 5.3 we have $N_G(\langle\omega\rangle) \leq M$. But this contradicts the structure of G_2. $\Box$

borel

Lemma 6.15 *ω normalizes a Borel subgroup of K.*

Proof: Suppose false. As $\langle S, \omega\rangle$ is a $\{2, t\}$ - group, we have that K has to have a solvable minimal parabolic. This now implies $K = G(2)$ and $t = 3$ or 5. In particular $r = 4, 64$ or 16 and $t = 3$, while $t = 5$ would imply $K \cong {}^2F_4(2)$, a contradiction. So $t = 3$. By 6.2 we have $3 \in \sigma(M)$ or $K \cong {}^3D_4(2)$ and $\sigma(M) = \{7\}$. As $O_2(M) \neq 1$, we see that $m_3(M) = 2$. In particular we get that $3 \not| |M : K|$. This gives $\omega \in K$. But then ω centralizes a good E, $E \cong E_{49}$. Now 5.3 shows that $N_G(\langle\omega\rangle) \leq M$, as $C_{O_2(M)}(\omega) \neq 1$. So let $3 \in \sigma(M)$. Then $N_G(\langle\omega\rangle) \leq M$, if $m_3(M) \geq 4$, again a contradiction. We also have that $m_3(C_M(K)) = 0$, as otherwise $|\Omega_1(Z(U))| \geq 9$ for U a Sylow 3–subgroup of M and so also $N_G(\langle\omega\rangle) \leq M$. Let $m_3(K) = 3$. Now with 1.1 we have $K \cong L_6(2)$, $L_7(2)$, $U_4(2)$, $Sp(6, 2)$ or $\Omega^-(8, 2)$. But it is easy to see that in these groups all 3–elements are centralized by some elementary abelian group of order 27, a contradiction. So let $m_3(K) = 2$. Then K possesses an outer automorphism of order 3 and so $K \cong {}^3D_4(2)$. But $m_{2,3}(K) = 1$, a contradiction. $\Box$

small

Lemma 6.16 *G_2 is solvable or $E(G_2/C_{G_2}(Z_2)) \cong L_3(2)$, A_6, $3 \cdot A_6$, A_9, $L_3(2) \times L_3(2)$ or $3A_6 * 3A_6$.*

Proof: Assume false. Then there is some element ω as before. Let $r \leq q$. By 6.15 ω normalizes a Borel subgroup of K. This implies

$O_2(G_1 \cap K)C_{G_2}(Z_2)/C_{G_2}(Z_2) \leq E(G_2/C_{G_2}(Z_2))$. Now we see $\mid O_2(G_1 \cap K)/R : C_{O_2(G_1 \cap K)/R}(t) \mid \leq r^2 \leq q^2$, for $t \in Z_2$, and $\mid O_2(N_K(R)/R) : C_{O_2(N_K(R)/R)}(t) \mid \leq r^2 \leq q^2$ for $K \cong F_4(q)$ and M involves a diagram automorphism.

This now implies with 1.8 that $q = r$ and $K \cong (S)L_n(q)$, $(S)U_n(q)$, $Sp_{2n}(q)$ or $G_2(q)$. Furthermore $\mid O_2(G_1 \cap K)/R : C_{O_2(G_1 \cap K)/R}(t) \mid = q^2$. Inspection of the groups in 4.4 shows that we have $E(G_2/C_{G_2}(Z_2)) \cong (S)L_3(q), Sp_4(q)$ or $L_2(q) \times L_2(q)$, where Z_2 is the $O^+(4,q)$ - module in the latter.

Let $E(G_2/C_{G_2}(Z_2)) \cong (S)L_3(q)$ or $Sp_4(q)$. Then we may assume that $Z_2^{(1)} \not\leq O_2(G_1 \cap K)$, where $Z_2^{(1)}$ is one of the two natural modules in Z_2. We have $O_2(G_1 \cap K) \leq N(Z_2^{(1)})$. So $1 \neq [Z_2^{(1)}, O_2(G_1 \cap K), O_2(G_1 \cap K)] \leq R$. This gives $R \cap Z_2^{(1)} \neq 1$. Let $u \in R^\sharp \cap Z_2^{(1)}$, then $C_{E(G_2/C_{G_2}(Z_2))}(u)$ involves a minimal parabolic of $E(G_2/C_{G_2}(Z_2))$ and so $C_G(u) \not\leq M$. This shows that $C_M(u)$ does not contain a good E. With 6.3 we see that $p \mid q-1$. Further as no good E centralizes R, we see that $K \cong U_4(q)$, $G_2(q)$ or ${}^3D_4(q)$. Hence in any case $m_p(K) = 2$ and we see that p does not divide $|C_M(K)|$. This shows $e(G) = 3$ and some field automorphism of order p is induced on K. But then again any $u \in R^\sharp$ is centralized by a good E, a contradiction.

We are left with the $O^+(4,q)$ - module. Let $p \in \sigma(M)$, $p \mid q-1$. Let U be a Sylow p–subgroup of $M \cap G_2$. Then $|\Omega_1(U)| = p^2$ and so U contains some ω with $N_G(\langle\omega\rangle) \leq M$, as either U contains a p–central element from M or U is centralized by some p–central element not in U. But we have that $G_2 = \langle M \cap G_2, N_{G_2}(\langle\omega\rangle)\rangle$, a contradiction. So $p \not| q-1$ for $p \in \sigma(M)$. This implies $K \cong L_4(q)$, $Sp_6(q)$, $(S)U_n(q), n \leq 7$ or $G_2(q)$ by 1.3. As $M \cap G_2$ has a factorgroup isomorphic to $Z_{q-1} \wr Z_2$, we see that in the cases of $K \cong Sp_6(q)$ or $G_2(q)$ there is some $\omega_1 \in M \cap K, o(\omega_1) = t, [\omega_1, K] = 1$. Now we get with 5.3 $N_G(\langle\omega_1\rangle) \leq M$, a contradiction.

Let $K \cong (S)U_n(q)$. Then there is $p \in \sigma(M)$ such that $\Omega_1(Z(S))$ is centralized by an elementary abelian group of order p^2 in K. This implies $Z(G_2) = 1$ and so $\mid Z_2 \mid = q^4$. Now we see that $R \leq [Z_2, O_2(G_1 \cap K)]$ and so $\Omega_1(Z(S)) \leq R$. Hence $F^*(M) = K$. Now we see $\mid O_2(G_1) : C_{O_2(G_1)}(Z_2) \mid = q^2$. Hence $C_{O_2(G_1)Z_2}(Z_2) = Z_2 \cdot T$, where T is a special group of order $q^{1+2(n-2)}$. In particular Z_2 contains a conjugate $R^g, g \in K, R^g \neq R$. As $N_G(R) \leq M$, we have that $N_G(R^g) \leq M$. But $\langle N_{G_2}(R), N_{G_2}(R^g)\rangle = G_2$, a contradiction.

So we are left with $K \cong L_4(q)$ and some element in S induces a diagram automorphism on K. Now we get that $p \in \sigma(M)$ divides $q+1$. Further all p–elements are good. This implies that $G_2 \leq M^h$ for some $h \in G$. But then we may assume that $h \in N_G(S)$. By 6.8 we have $N_G(S) \leq M$, a contradiction.

So assume $r > q$. By 1.10 $K \cong (S)U_n(q), {}^2E_6(q), \Omega^-_{2n}(q), r = q^2$, or $K \cong {}^3D_4(q)$ or $\Omega^+(8,q)$. Let $K \not\cong {}^3D_4(q)$. Then by 1.10 in all cases $3 \mid r-1$. If $3 \in \sigma(M)$, then we see that in all cases elements of order 3 are centralized by an elementary abelian group of order 27 and so they are good. As $3 \mid |G_2 \cap M|$, we see that $3 \notin \sigma(M)$. Let $K \not\cong \Omega^+(8,q)$. Then in any case there is $p \mid q^2-1, p \in \sigma(M)$. This implies that there is some $\omega_1 \in G_2 \cap M, o(\omega_1) = p$.

Let $K \cong {}^2E_6(q)$, then ω_1 normalizes some parabolic P with $P/O_2(P) \cong L_3(q^2) \times L_2(q)$ and so $m_p(C_M(\omega_1)) \geq 3$, a contradiction.

Let $K \cong \Omega^-_{2n}(q)$, then ω_1 normalizes a parabolic P with $P/O_2(P) \cong L_2(q) \times L_2(q^2)$. If $\omega_1 \notin K$, we have again that $m_p(C_M(\omega_1)) \geq 3$. If $\omega_1 \in K$ we get the same, as all p–elements in K are good.

So we are left with $K \cong (S)U_n(q)$. Let $n > 5$, then ω_1 normalizes a parabolic P with $P/O_2(P)$ contains $L_2(q) \times L_2(q^2)$, or $L_2(q) \times (S)U_3(q)$, respectively. Again we get $\omega_1 \in K$, but all p–elements in K are good, a contradiction. So we are left with $K \cong U_4(q)$ or $U_5(q)$. We now get that p does not divide $|C_M(K)|$, otherwise any p–element is good. In case of $U_4(q)$ we have P with $L_2(q) \times \mathbb{Z}_{q^2-1}$ and so we are done again. So let $K \cong U_5(q)$. If $p \neq 5$, we can look at the parabolic P with $L_2(q^2) \times \mathbb{Z}_{(q^2-1)/5}$. Otherwise we have $q = 4$ and then ω_1 normalizes P with $P/O_2(P)$ contains $SU_3(4)$. Now ω_1 has to be in P again, but these elements of order 5 are all good.

Let next $K \cong \Omega^+(8,q)$. Then by 1.10, $o(\omega_1) = 3$ or 9 and $q \leq 16$. Hence we get $r = 64$ or $r = 4$. As $3 \notin \sigma(M)$, we get $O_2(M) = 1$ and $e(G) = 4$. As $m_{2,p}(M) = 4$ for some $p \in \sigma(M)$, we see that $q > 2$. This shows $r = 64$. As $\mathrm{Out}(\Omega^+(8,q))$ does not contain a cyclic group of order 9, we see that $\omega^3 \in K$. By 6.15 ω^3 normalizes a Borel subgroup of K and so $3 \mid q-1$. But then the normalizer of R contains an elementary abelian subgroup of order 3^4, a contradiction.

So we are left with $K \cong {}^3D_4(q)$. Then by 1.4 we see that $|Z_2C_M(K)/Z_2 \cap O_2(G_1)C_M(K)| \leq q$, as Z_2 acts quadratically on $O_2(G_1 \cap K)$. As ω acts on this group, we see with the action of $L_2(q^3)$, that if ω induces an inner automorphism, then $o(\omega)$ divides $q-1$, a contradiction. So we are in 1.10(ii). Then we have that $o(\omega) = 3$ or 9 and $q \leq 32$. If $o(\omega) = 3$, then $q = 2$ and so $3 \notin \sigma(M)$. So by 6.2 we have $7 \in \sigma(M)$. Suppose that $M \cap G_2$ contains some subgroup U isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then by 1.10 we have that U normalizes a Borel subgroup of K. Hence we have some element of order three, which centralizes K. But by 6.2 we have that $O_2(M) \neq 1$ and $e(G) = 3$. As now $m_3(M) = 3$, we see that $3 \in \sigma(M)$, a contradiction. Hence we have that $r = 4$ and $E(G_2/C_{G_2}(Z_2)) \cong L_2(4)$. But we have that

$|O_2(G_1 \cap K) : C_{O_2(G_1 \cap K)}(Z_2)| \geq 16$ by 1.4, a contradiction. So we have that $o(\omega) = 9$ and $r = 64$.

If $q = 2$, then as before we have that ω^3 centralizes K. But then $m_3(M) = 3$ and as M does not contain elementary abelian subgroups of order p^4 for p odd, we get $3 \in \sigma(M)$, a contradiction. So we have $4 \leq q \leq 32$. Now choose $\nu \in M \cap G_2$, with $o(\nu) = 7$. If ν does not normalize the Borel subgroup, we must have a $\{2,7\}$–parabolic in K, different from the Borel subgroup, which is not the case. So ν normalizes a Borel subgroup of K. Suppose that ν centralizes K. As $m_p(K) \geq 2$ for $p \in \sigma(M)$, we get with 5.3 that $N_G(\langle \nu \rangle) \leq M$. But we may choose ν such that $G_2 = \langle G_2 \cap M, N_{G_2}(\langle \nu \rangle) \rangle$, a contradiction. As ν normalizes $Z_2 O_2(G_1 \cap K)$ and by quadratic action we have that $|Z_2 O_2(G_1 \cap K)/O_2(G_1 \cap K)| \leq q$, we see that $o(\nu) \mid q - 1$. This shows $q = 8$. Now ω also acts on K and centralizes ν. This implies that ω^3 centralizes K. Application of 5.3 shows $N_G(\langle \omega^3 \rangle) \leq M$, a contradiction again. □

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Proposition 6.17 *If M is a uniqueness group in G then $F^*(M) = O_2(M)$.*

Proof: Suppose false. Then we have the set up of this section for some M. In particular we can start with the result of 6.16 .

We first investigate the case of a solvable G_2. Suppose that M covers $O(G_2/O_2(G_2))$. Then we have that this group normalizes $O_2(G_1 \cap K)$, a contradiction as $O_2(G_1 \cap K) \not\leq O_2(G_2)$. So we have that $G_2 = LS$, where L is a preimage of $O(G_2/O_2(G_2))$. Now let U be some Sylow r–subgroup of L such that $[O_2(G_1 \cap K), U] \not\leq O_2(G_2)$. Then we may assume that US is a subgroup of G_2 and by the same reason as before it is not in M, hence $US = G_2$. Now by 2.1 there is some subgroup U_1 of $G_2/C_{G_2}(Z_2)$ with $U_1 \cong D_{2r} \times \cdots \times D_{2r}$ where the D_{2r} are dihedral groups of order $2r$ and A is a Sylow 2–subgroup of U_1, where A is the offender as F–module or as $2F$–module as given by 6.13. In any case we have $|Z_2 : C_{Z_2}(A)| < |A|^2$. Now U_1 is generated by two conjugates of A and so $|Z_2 : C_{Z_2}(U_1)| < |A|^4$. This shows $r = 3$. Hence if G_2 is solvable it is a $\{2,3\}$–group. By 6.2 we have $m_3(G_2) \leq 3$.

Assume next that $E(G_2/C_{G_2}(Z_2)) \cong A_9$. Then we have $3 \not\in \sigma(M)$. Further $M \cap G_2/C_{G_2}(Z_2) \geq A_8$. If $(M \cap G_2)^{(\infty)} \leq C(K)$, then $m_3(K) = 1$. But then we have a contradiction to 1.1 (i),(ii). Hence we have that K possesses a parabolic P with $O^{2'}(P/O_2(P)) \cong A_8$. Then $K \cong L_n(2), Sp_{2n}(2), \Omega^+_{2n}(2)$, or $E_n(2)$. But then by 6.2 we get $3 \in \sigma(M)$, a contradiction.

Let next $K \cong F_4(q)$ and assume that S induces a diagram automorphism on K. If this is realized by some element $t \in Z_2$, then we see that $[t, S]$ is not elementary abelian, a contradiction. Hence in any case we have that Z_2

centralizes R. Let $Q = O_2(C_K(R))$. Then we have that $QO_2(G_2)/O_2(G_2)$ is elementary abelian. So by 6.16 and as $m_3(G_2) \leq 3$ for solvable G_2 and 2.1, we see that $|Q : C_Q(Z_2)| \leq 16$.

Let $E(G_2/C_{G_2}(Z_2)) \cong 3A_6 * 3A_6$. Then $|Q : C_Q(t)| \leq 16$ and so by 1.8 $|R| \leq 4$. Further we have that $3 \not\in \sigma(M)$ as otherwise $G_2 \leq M^g$ for some $g \in G$ and so again by 6.8 $M = M^g$, a contradiction. Hence by 6.3 we have $C_G(x) \leq M$ for all $x \in R^*$. Now by 4.4 we have that $E(G_2/C_{G_2}(Z_2)), Z_2] = V_1 \oplus V_2$, where any element in V_1 centralizes a component in $E(G_2/C_{G_2}(Z_2))$. Hence $R \cap V_1 = 1$. But $1 \neq [Q, V_1] \leq V_1$ and so $V_1 \cap R \neq 1$. So we have seen that $E(G_2/C_{G_2}(Z_2)) \not\cong 3A_6 * 3A_6$. This then implies that $|Q : C_Q(Z_2)| \leq 8$ and equality can just hold for G_2 solvable.

Now application of 1.8 again yields $K \cong L_n(2)$, $U_n(2)$, $Sp(2n, 2)$, or $G_2(2)'$, or $G_2(4)$. Further we have that $|Q : C_Q(Z_2)| \geq 4$. If $K \not\cong G_2(4)$, we have by 6.2 that $3 \in \sigma(M)$. Let $K \cong G_2(4)$. Then just for $p = 3, 5$ we have that $m_p(K) \geq 2$. Assume that $5 \in \sigma(M)$. As Sylow 5–subgroups of 2-locals in K are cyclic and $m_p(C_M(K)) \leq 1$, we see that there is no 2-local H in G with $m_5(H) \geq 3$. This also shows that $3 \in \sigma(M)$ in the case of $K \cong G_2(4)$.

Suppose that there is an element of order three in M whose centralizer in G is not contained in M. Then we have that $m_3(C_M(K)) = 0$, further $m_3(M) = 3$. This shows $K \cong L_6(2)$, $L_7(2)$, $Sp_6(2)$, or $U_4(2)$. But in all these cases any element of order three is centralized by an elementary abelian subgroup of order 27, a contradiction. So we have that all elements of order 3 are good. If now G_2 contains some elementary abelian subgroup of order 9, we have that $G_2 \leq M^g$, for some $g \in G$. But now $S \leq M \cap M^g$ and so by 6.8 $M = M^g$, a contradiction. This shows that Sylow 3–subgroups of G_2 are cyclic. Application of 6.16 shows that G_2 is solvable or $E(G_2/C_{G_2}(Z_2)) \cong L_3(2)$ and then $G_2/C_{G_2}(Z_2) \cong PGL_2(7)$. As $QC_{G_2}(Z_2)/C_{G_2}(Z_2)$ is an elementary abelian normal subgroup of $SC_{G_2}(Z_2)/C_{G_2}(Z_2)$ of order at least 4, we see that G_2 must be solvable. But then $O_3(G_2/O_2(G_2))$ is cyclic and $QO_2(G_2)/O_2(G_2)$ is elementary abelian of order at least 4 and acts faithfully on $O_3(G_2/O_2(G_2))$, a contradiction. $\square$

7 The centralizers of involutions in Y_M

In this chapter we fix a uniqueness group M and a Sylow 2-subgroup S of M. If H is a subgroup of G with $C_G(O_2(H)) \leq O_2(H)$ then set $C_H = C_H(Y_H)$. We will show that $O_2(C_G(x)) = F^*(C_G(x))$ for all $x \in Y_M^\sharp$. Remember that by 6.17 we have that $F^*(M) = O_2(M)$. We fix the following notation. Set $M_0 = N_M(S \cap C_M)$ and $Q = O_2(M_0)$.

central

Lemma 7.1 *Let $g \in G$ with $[Y_M, Y_{M^g}] \leq Y_M \cap Y_{M^g}$, then $[Y_M, Y_{M^g}] = 1$.*

Proof: Suppose false. Then we may assume that Y_M is an F–module with offender Y_{M^g}. Let first M be exceptional. Then by 5.9 we have that Q centralizes Y_M. But then there is some $\omega \in Q^g$ with $C_{[Y_M, Y^g_M]}(\omega) \neq 1$. Then we have that $\omega \in M$ and as a Sylow p subgroup of M is abelain, we have $Q \leq M^g$, which shows $M = M^g$, a contradiction. Hence M is not exceptional.

Set $\bar{M} = M/C_M$. Let R be a component of $\bar{M}$ such that $Y_{M^g} = Y C_{Y_{M^g}}(R)$ and Y is an F–module offender for R on $[R, Y_M] = V$. Recall that by 3.24 we have that Y_{M^g} normalizes all components of $\bar{M}$. Now the group R is one of the groups from 3.16.

Suppose that for all choices of R we have some W, a nontrivial RY_{M^g}–submodule of $[V, R]$, such that for any $x \in W^\sharp$ we have that x is centralized by some good $E \cong E_{p^2}$. Let $R_1 = R^g$ be the corresponding component in $M^g/O_2(M^g)$. Then we have that $[W, R_1] = 1$, as otherwise there is some $1 \neq x \in [W, W^g]$, which is centralized by some good E in M and M^g as well, which would give $M = M^g$. Hence R_1 acts on $[W, Y_{M^g}]$. But we have that M is the unique maximal 2–local containing $C_G(x)$ for any $1 \neq x \in [W, Y_{M^g}]$. So we see that R_1 is in M. This now implies that R_1 does not contain a good $F \cong E_{p^2}$ in M^g. Hence we have $m_p(R) \leq 1$. As by 3.24 $R \geq [Y, R] \neq 1$, we see that $RR_1 \cong R * R_1$. In particular $m_r(RR_1) \leq 3$ for all odd primes r. This now shows that $R \cong L_2(q)$, q even, $L_3(2)$, $SL(3,4)$, $3A_6$ or $3A_7$. Further $[V, R]$ involves at most two nontrivial modules the natural ones or $R \cong L_2(4)$ and we have the orthogonal module. Suppose that also W acts as an F–module offender on Y_{M^g}. Hence there is some component R_2 in $M^g/O_2(M^g)$ which is not centralized by W and induces an F–module in Y_{M^g} or it acts on a Sylow group of $F(M^g/C_{M^g})$. By assumption we have some element in $[W, Y_{M^g}]$ which is normalized by a good E in M and M^g as well, or W just induces F-module offenders on $F(M^g/C_{M^g})$. In the latter we get that $E(M^g/C_{M^g})$ is in M, a contradiction. So we now get $M = M^g$, a contradiction.

If two modules are involved, then W always acts as an F–module offender. So $R \not\cong SL(3,4)$ and if 3 divides the order of $Z(F)$, then $Z(F)$ acts trivially. But then as W is not an F–module offender, we see that $R \cong L_3(2)$ and $[V,R]$ involves exaxtly one irreducible module and $|Y| = 4$. But then 3–elements in R_1 are centralized by a good E in M^g and so by 5.3 their centralizers are in M^g, which implies $R \leq M^g$, contradicting $[R, Y_{M^g}] \not\leq O_2(M^g)$.

So we have that we may choose R in such a way that for any nontrivial submodule W for RY_{M^g} there is some $x \in W^\sharp$ which is not centralized by a good E. Hence we have one of the cases from 3.42(4)(i) - (viii). Set again $R_1 = R^g$.

Let first $m_p(R) \leq 1$. Then we have $R \cong L_2(q)$ and W is the extension of the trivial module by the natural module. Assume that Y_{M^g} normalizes R. Let $x \in W$, $[x, Y_{M^g}] \neq 1$ and $[x, R_1] = 1$. As $|[x, Y_{M^g}]| = q$, we see that R_1 centralizes this group. Now $[x, W^g] = 1$. As $C_{Y_{M^g}}(x) = C_{Y_{M^g}}(W)$, we see $[W, W^g] = 1$ and so $[R_1, W] = 1$, which implies that $[R_1, [W, Y_{M^g}]] = 1$. But $C_W(R)$ is centralized by a good E in M, so we get that R_1 is in M. So we have $RR_1 \cong L_2(q) \times L_2(q)$. But then R_1 contains a p–element ρ whose normalizer is in M. But the normalizer of ρ in M^g contains a good E, so M contains a good E from M^g, yielding the contradiction $M = M^g$. So we may assume that W acts as an F–module offender on R_1. Now $[W, W^g]$ contains $C_W(R)$ and $C_{W^g}(R_1)$. Further $[W, W^g]/C_W(R)]$ is a 1–dimensional subspace in $W/C_W(R)$. So we have that $[W, W^g]$ is normalized by a good E in M and M^g as well, so $N_G([W, W^g]) \leq M \cap M^g$ and so $M = M^g$, a contradiction.

So assume that there is some $y \in Y_{M^g}$ with $RR^y = R \times R^y$. But by quadratic action we get $|W \cap W^y| = q^2$, which is not possible.

So we have $m_p(R) \geq 2$. Let W be as in 3.42(4). Then any $x \in W$ is centralized by some p–element ρ with $N_G(\langle \rho \rangle) \leq M$. As $[W, Y] \neq 1$ and Y_{M^g} normalizes R and acts quadratically we have that $[Y_{M_g}, W] \leq W$. Assume first that there is some $x \in W$ with $[x, Y_{M^g}] \neq 1$ and $[R_1, x] = 1$. Then first R_1 acts on $[Y_{M_g}, x]$, which is of size smaller than W and so $[R_1, [Y_{M_g}, x]] = 1$. This now shows that $C_G(y) \leq M^g$ for all $1 \neq y \in [Y_{M^g}, x]$. Hence we may assume $\rho \in M^g$. If $m_p(C_{M^g}(R_1)) \geq 1$, we have that the center of a Sylow p–subgroup contains an elementary abelian subgroup of order p^2 and so $C_{M^g}(\rho)$ contains an elementary abelian subgroup of order p^3, which implies that we have a good E in M and M^g as well, a contradiction. So we have that $m_p(C_{M^g}(R_1)) = 0$. Inspection of the groups in 3.42 now shows that $m_p(C_{R_1\langle \rho \rangle}(\rho)) \geq 3$, and so again we have a contradiction.

So we have that W acts nontrivially on W^g. Hence we may assume that

W^g induces an F–module offender on W. Suppose that we do not have (4)(vi) - (vii), then there is some $1 \neq x \in [W, W^g]$ which is centralized by some p–element τ in M^g with $C_G(\tau) \leq M^g$ and some good E in M. Hence we get the same contradiction as above.

So we have to handle (4)(vi)-(viii). Suppose we have $R \cong L_4(2)$. Set $W_2 = [Y_M, R] = W \oplus W_1$. Then we have that $[W_2, W_2^g] \leq W_2 \cap W_2^g$. Suppose that $|[W, W_2^g]| = 4$, then $[W_2, W_2^g]$ is normalized by some good E in M and M^g as well, a contradiction. So we have that $|[W, W_2^g]| = 8$. Now W_2 induces a fours group on W_2 which centralizes a subgroup of index eight, which is not possible.

Suppose next $R \cong Sp(6,2)$ or $U_4(q)$. Then $W = [Y_M, R]$. Now $|[W, W^g]| = 16$ or q^4, respectively. There are elements in $x \in W^g$ such that $C_W(x) = [W, W^g]$. Hence all elements in $[W, W^g]x$ are conjugate under W. As $N_{R_1}([W, W^g])$ acts transitively on $[W, W^g]^\sharp$ and $(W^g/[W, W^g])^\sharp$ as well, this would imply that R_1 acts transitively on $(W^g)^\sharp$, which is not the case.

So we may assume that there is a normal r–group R in $\bar{M}$ on which Y acts faithfully and induces an F–module offender and further there is no component of this type. In particular $r = 3$. Hence by quadratic action and 4.5, there is some $x \in Y_M$ inducing a transvection on Y_{M^g}. If $3 \notin \sigma(M)$, then $m_3(R) \leq 3$ and so by 2.3(a), we have that R is centralized by some good E. But as there are transvections we have by 4.5 some $x \in Y_M$ and $y \in Y_{M^g}$ such that $[x, y] = [x, Y_{M^g}] = [y, Y_M]$ is of order two. Hence there is some 3–element $\rho \in R$ and $\rho_1 \in R_1$, such that $[x, y] \leq [Y_M, \rho] \cap [Y_{M^g}, \rho_1]$ and $|[Y_M, \rho]| = |[Y_{M^g}, \rho_1]| = 4$. But as R is centralized by a good $E \cong E_{p^2}$, $p > 3$, we have that $[x, y]$ is centralized by some good E in M and M^g as well, a contradiction. So we have $3 \in \sigma(M)$. Let C be a characteristic subgroup of R, $C = \Omega_1(C)$. Assume $m_3(C) > 1$. We may assume that C is elementary abelian or extraspecial. If $m_3(C) > 2$, there is a good E in $C_C(x)$ and so it centralizes $[x, y]$ and we have a contradiction as before. So C is elementary abelian of order 9 or extraspecial of order 27. As $C_M(C)$ acts on $[x, y]$, we may assume that $C \geq \Omega_1(C_M(C))$. But as $m_3(M) > 2$, we get C is extraspecial. As x centralizes in C an elementary abelian subgroup C_1 of order 9, we have that C_1 cannot be good. So not all elements of order 3 in M are good. Let P be a Sylow 3–subgroup of M containing C. Then $Z(C) = \Omega_1(Z(P))$ and there is a subgroup F of order 9, which is in $Z_2(P)$. Hence F is normal in P and as $m_3(P) > 2$, we have that F is good. As not all subgroups of order 9 are good, this shows that M induces on C either Σ_3 or $\mathbb{Z}_2 \times \Sigma_3$. In both cases F is normal in $\bar{M}$ and the other subgroups of order 9 are conjugate. Hence F is normalized by x. As $[F, x] \neq 1$, we see that $|[F, Y_M]| = 4$ and so as $F \leq F_1$, F_1 elementary abelian of order 27, we see that there is a good E in F_1, which centralizes $[x, y]$, a contradiction again. Hence we are left with

R cyclic. Now $|[R, Y_M]| = 4$, and so M acts on this group, in particular it is centralized by a good E, a contradiction. $\square$

wc

Lemma 7.2 *Suppose that M_0 contains a good $E \cong E_{p^2}$. Then Q is weakly closed in S with respect to G.*

Proof: Let $Q \neq Q^g \leq S$. Then there is also $T \leq S$ such that $Q \leq T$ and there is some $h \in N_G(T)$ with $Q \neq Q^h$. Then we consider the group $\langle M_0, N_G(T)\rangle = H$. Assume $H \leq M$. Then $N_G(T) \leq M$. Hence $Q^h \leq C_M$, as C_M is normal in M. But then Q and Q^h are Sylow 2–subgroups of C_M and as QQ^h is a 2–group we have $Q = Q^h$, a contradiction. Hence $H \not\leq M$. As M_0 contains a good E, we now have that $O_2(H) = 1$. So we have an amalgam with $G_1 = M_0$ and $G_2 = N_G(T)$. Let $\Gamma = \Gamma(G_1, G_2)$ be the coset graph and $\beta \in \Gamma$ of minimal distance from 1 such that $Y_M \not\leq O_2(G_\beta)$. As $Q \leq T$, we get $\beta \sim 1$. As Q contains all 2–elements centralizing Y_M we see that $Y_M \cap Y_\beta \geq [Y_M, Y_\beta] \neq 1$, which contradicts 7.1. $\square$

Snormalizer

Lemma 7.3 $N_G(S) \leq M$

Proof: Suppose that C_M contains a good E. As $\Omega_1(Z(S)) \leq Y_M$ by 3.4, we have $C_M \leq C_G(\Omega_1(Z(S)))$. Hence $N_G(S) \leq N_G(\Omega_1(Z(S))) \leq M$. So we may assume that M_0 contains a good E. By 7.2 now Q is weakly closed in S. So we get that $N_G(S) \leq N_G(Q)$ and so Q is normal in $\langle M_0, N_G(S)\rangle$ which gives $N_G(S) \leq M$ as M_0 contains a good E. $\square$

Let

$$\mathcal{H} = \{C_G(x) \mid x \in Y_M^\sharp\}$$

Tnormalizer

Lemma 7.4 *Let $H \in \mathcal{H}$. Then $H \cap M$ contains a Sylow 2–subgroup of H.*

Proof: This is clear if $H \leq M$. So assume that $H \not\leq M$. As $C_M \leq H$, we see that C_M does not contain a good E. Then M_0 contains a good E by 2.5 and 5.11. Let $Q \leq T \leq H \cap M$, T be a Sylow 2–subgroup of $H \cap M$. Then by 7.2 we have that $N_H(T) \leq N_G(Q) \leq M$ and so T is a Sylow 2–subgroup of H. $\square$

From now on we choose $H = C_G(x) \in \mathcal{H}$ with $F^*(H) \neq O_2(C_G(x))$ and $x \in Y_M^\sharp$. Then $H \not\leq M$. In particular we have that C_M does not contain a good E. By 2.5 and 5.11 M_0 contains a good E. Further by 7.2 we may assume that $S \cap H$ is a Sylow 2–subgroup of H.

Recall that in the exceptional case p does not divide $|C_M|$ or C_M covers $O_p(M/O_2(M))$.

unique

Lemma 7.5 *H is not contained in some uniqueness group. In particular $m_p(H) \leq 3$ for all odd p.*

Proof: Suppose $H \leq K$, K some uniqueness group. Let $R = O_2(K)$. Set $L = E(H)O_{2'}(H)$. Then we see that $[C_R(x), L] = 1$. So by the $A \times B$–lemma, we get $[L, R] = 1$, a contradiction to 6.17. □

comp1

Lemma 7.6 *We have $[L, Y_M] \leq L$ for any component L of H.*

Proof: Suppose $[L, Y_M] \not\leq L$. As Y_M is normal in $S \cap H$, we get that $L/Z(L)$ has abelian Sylow 2–subgroups, so $L \cong L_2(q)$, or $L \cong SL_2(5)$. Suppose the former. Then there is a group of order q in S, which induces transvections to a hyperplane on Y_M. As $q > 2$, we get that there is a component $K \cong L_n(2)$ in M/C_M. Let $y \in Y_M$ with $[y, L] \neq 1$. Then $C_{L \times L^y}(y) \cong L_2(q)$ and so $L \leq \langle S \cap (L \times L^y), C_{L \times L^y}(y)\rangle$. In particular there is no good E in M which centralizes y. So K is not centralized by some good E. This shows $m_p(C_{M/C_M}(K)) \leq 1$ and so $m_p(K) \geq 2$. Suppose $m_p(C_{M/C_M}(K)) = 1$, then we have that p does not divide the order of $C_K(y)$ and so of $L_{n-1}(2)$. But this is not possible. So we have that $m_p(C_{M/C_M}(K)) = 0$ and then $m_p(K) \geq 3$. This then shows $p = 3 \in \sigma(M)$ and so $n \geq 6$. But then $L_{n-1}(2)$ contains a good E, a contradiction.

So we have $L \cong SL_2(5)$. Then $|Y_M : C_{Y_M}(L)| = 8$. Let $x \in Y_M \setminus N(L)$. Then x centralizes some element ρ of order 5 in LL^x. As ρ does not normalize Y_M and $|Y_M : C_{Y_M}(\rho)| \leq 4$, we see that $\sigma(M) = \{3\}$. Let now P be a Sylow 3–subgroup of LL^x, which is contained in some M^g. Then P is not good, as otherwise $H \leq M^g$, contradicting 7.5. Hence we have that a Sylow 3-subgroup of G is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. Now let $\nu \in M \cap LL^x$ be an element of order 3. Then we have that $|[Y_M, \nu]| = 4$ and $C_G(\nu) \not\leq M$. Hence there is a subgroup P_1 in M, $|P_1| = 9$ and $\nu \in P_1$. In particular P_1 is generated by M–conjugates of ν. This shows $|Y_M : C_{Y_M}(P_1)| \leq 16$. Suppose that P_1 centralizes some element $v \in C_{Y_M}(L)$. Then as 9 not divides the order of LL^x, we see that $C_G(v)$ contains $\langle LL^x, P_1\rangle$ and so an elementary abelian group of order 27, in particular $C_G(v)$ contains some good E, contradicting 7.5. So we have that $C_{Y_M}(P_1) \cap C_{Y_M}(L) = 1$ and then $|Y_M| \leq 32$. Now some 3-central element in M. But then C_M contains a good E, a contradiction. □

comp2

Lemma 7.7 *We have $[L, Y_M] = L$ for any component L of H.*

Proof: Suppose $[L, Y_M] = 1$ for some L. Then $L \leq M$. So $[L, O_2(M)] \leq O_2(M)$. As $O_2(M) \leq H$ we get $[L, O_2(M)] = 1$, a contradiction. □

3strukL

Lemma 7.8 *Let* $3 \in \sigma(G)$ *and* T *be a Sylow 3-subgroup of* L. *Then* $|\Omega_1(Z_2(T))| \leq 9$.

Proof: Let P be a Sylow 3–subgroup of G with $T \leq P$ and R be a uniqueness group for the prime 3 with $P \leq R$. If there is a good E in T, then we get $H \leq R$, contradicting 7.5. So suppose that $|\Omega_1(Z_2(T))| \geq 27$. Let F be normal in P, F elementary abelian of order 9. Then $C_{\Omega_1(Z_2(T))}(F)$ contains an elementary abelian group of order 9. Hence, as there is no good E in T, we have $F \leq T$. But $m_3(C_P(F)) \geq 3$, a contradiction.

complie

Lemma 7.9 *Let* L *be a component of* H *then* $L/Z(L)$ *is a group of Lie type in characteristic two.*

Proof: Assume otherwise. Then $L/Z(L)$ is some sporadic group in $\mathcal{C}_2$ or $L_3(3)$, $U_3(3)$, $U_4(3)$, $G_2(3)$ or $L_2(p)$, p some Fermat- or Mersenne prime. As $L \not\leq M$, we see that $|Y_M : C_{Y_M}(L)| \geq 4$, as any hyperplane in Y_M contains some element centralized by a good E by 5.8. As Y_M is normal in a Sylow 2–subgroup of H, we get that $L/Z(L) \not\cong L_2(p)$ or $L_3(3)$. Recall that we consider $L_2(5)$ as $L_2(4)$ and $L_2(7)$ as $L_3(2)$.

Let next $L/Z(L) \cong G_2(3)$, $U_4(3)$ or $U_3(3)$. By 7.5 we have $L/Z(L) \not\cong U_4(3)$. By 7.8 we have that $3 \not\in \sigma(G)$. Now by 5.8 any subgroup of index 4 in Y_M contains some element v such that $C_M(v)$ contains some good E and so $C_G(v) \leq M$ and so $|Y_M : C_{Y_M}(L)| = 8$. But now for any $t \in Y_M \setminus C_{Y_M}(L)$ we have that $C_L(t) \leq M$. As L is generated by such centralizers we get $L \leq M$, a contradiction.

So we have that $L/Z(L)$ is sporadic. By 7.5 we have that $m_3(L) \leq 3$. This shows $L/Z(L)$ is some Mathieu group, some Janko group, HS or Ru. Further in any case we have that $|Y_M : C_{Y_M}(L)| \geq 4$ by 5.8, so $L/Z(L) \not\cong M_{11}$. If $L/Z(L) \cong J_2$ or J_3, then we get that $|Y_M : C_{Y_M}(L)| = 4$, recall that Y_M is normal in $S \cap H$. By 7.8 we have that $3 \not\in \sigma(M)$. But then $C_L(y) \leq M$ for all $y \in Y_M^\sharp$, a contradiction.

By 5.8 we always have at least one fours group V in Y_M all of whose elements have a centralizer which is in M. Hence $U = \langle C_L(v), S \cap L \mid v \in V^\sharp \rangle$ is a proper subgroup of L. Application of [CCNPW] gives $L/Z(L) \cong M_{22}$, M_{23} or M_{24} and $U/Z(L) \cong 2^4A_6$, 2^4A_7, 2^4A_8 or $2^6 3\Sigma_6$, respectively. Now we see that $Q \cap L$, which is the preimage of $U_2(U)$ in L is elementary abelian and so as $\Phi(Q)$ is normal in M, and $L \leq C_H(\Phi(Q))$, we get that Q is elementary abelian and then $Q = Y_M$. Let first $3 \in \sigma(M)$. Then by 7.8 we have $L \not\cong M_{24}$. Now U contains a 3–element ρ with $C_G(\rho) \leq M$. This now implies $L/Z(L) \cong M_{22}$. Further we see that for $P \leq U$, $|P| = 9$, we have $N_L(P) \not\leq U$, hence we get $e(G) = 3$ and a Sylow 3–subgroup of G is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. Now $N_U(P)/C_U(P) \cong \mathbb{Z}_4$ and $N_M(P)/C_M(P)$ contains $\mathbb{Z}_3$. So we see that $N_M(P)/C_M(P)$ must contain $SL(2,3)$. But then all 3–elements in P are conjugate in M and so $\Gamma_{G,1}(P) \leq M$, which gives that P is good, a contradiction to 7.5. So $3 \notin \sigma(G)$. Now as U induces an F–module on Y_M and $U/O_2(U)$ cannot act faithfully on a 3–group of rank at most 3, we get that there is some component K in M/C_M involving $U/O_2(U)$. As $m_3(K) \leq 3$ we get with 1.1 and 3.16 that $K \cong L_n(2)$, $n \leq 7$, $Sp(2n,2)$, $n = 2,3$, $\Omega^{\pm}(2n,2)$, $n \leq 4$, or A_n, $n \leq 11$. As $[L \cap M, Y_M]$ involves just one nontrivial irreducible module, we also have that $[Y_M, K]$ involves just one irreducible module. Further there is no good E in $C(K)$ as this group has to centralize $[K, Y_M]$. This shows $m_p(K) \geq 2$ and so $K \cong L_6(2)$, $L_7(2)$, A_{10}, or A_{11}. In any case any element in $[Y_M, K]$ is centralized by some good E. This gives $[Y_M, K] = [Y_M, L \cap M]$ and so $|[Y_M, K]| = 2^6$ and $K \cong L_6(2)$. But now we have $C_{Y_M}(K) = C_{Y_M}(U)$. Hence any $x \in C_{Y_M}(U)$ is centralized by a good E in K, which shows $H \leq M$, a contradiction. □

Now we choose H and L such that L is maximal. Hence by the $A \times B$–lemma we have that L is a component for any $x \in C_{Y_M}(L)^{\sharp}$.

root

Lemma 7.10 *Let* $L \cong G(q)$. *Set* $V_0 = C_{Y_M}(L)$. *Then* $V_0^g \cap V_0 = 1$ *for* $V_0^g \neq V_0$ *and* $g \in M$.

Proof: Let first $[Q, L] \not\leq L$. Then we have $L^Q = L \times L^t$ for some $t \in Q$, as $m_p(L^Q) \leq 3$ for all odd p. Further $[V_0, L^Q] = 1$, as $V_0 \leq Z(Q)$. So by abuse of notation we identify L with $L \times L^t$ in that case.

Let $g \in M$ with $V_0^g \cap V_0 \neq 1$. Then there are $v, w \in V_0$ with $v^g = w$. Hence we get that both L and L^g are components of $C_G(v)$ and $C_G(w)$.

Suppose $L \neq L^g$, then we see that $m_p(L) = 1$ for all odd primes p, as $O_p(L) = 1$ by definition of $\mathcal{C}_2$. This shows $L/Z(L) \cong L_2(q)$, $Sz(q)$ or $L_3(2)$. In particular we see that $[L, Q] \leq L$. Let Y be the projection of Y_M into L. We first show $Y = Y_M \cap L$. Suppose false. Set $T = S \cap L$. We have that T is a Sylow 2–subgroup of L. As $Y_M = \Omega_1(Z(Q))$ by 3.4 we see that $Y_M \cap L = \Omega_1(C_T(Q))$. Suppose there is $uv \in Y_M$, $u \in L$, $v \in C(L) \setminus L$ and $[uv, Q] = 1$. As $[u, Q] \in L$ and $[v, Q] \in C(L)$, we see that $u \in Y_M$ or $Z(L) \neq 1$. So assume the latter. Then we have that $L \cong SL_2(5)$, $SL_2(7)$ or $Sz(8)$.

In the first two cases we have that $|Y_M : V_0| \leq 4$. By 7.5 we get that $\sigma(M) = \{3\}$. Further we have that $m_3(M) = 3$. We now see that M induces a fours group of transvections to a point on Y_M. Let K be a component of

M/C_M which realizes this fours group. Then $K \cong L_n(2)$. So we see that $n \leq 6$. This shows $|Y_M| = 2^6$, as $m_3(M) = 3$. Now all involutions are conjugate and so we have that $C_G(i) \leq M$ for all $i \in Y_M^\sharp$. So this group acts on a 3–group. But then we see with 2.1 that 3 divides the order of C_M, which contradicts $C_M \cap L \leq Z(L)$.

So we have that $L/Z(L) \cong Sz(8)$. As $\Omega_1(S \cap L)$ centralizes Y_M, we see that this group is in Q and so it is $\Omega_1(Q \cap L)$. Hence $Y = Y_M \cap L)$.

Let $q_1 = |Y|$. Also $F^*(C_G(y)) \neq O_2(C_G(y))$ for $y \in L \cap Y_M$. We have that L^g is a component of $C_G(L)$. Let $\{L^h \mid h \in M\}$ be the set of components of $C_G(L)$ which are conjugate to L by some element in M. Then we have a group $U = LL^{h_1}L^{h_2} \cdots L^{h_r}$, with $h_i \in M$, which is a central product of its components. Suppose now $q_1 < |V_0|$. Then we have that $L^h \in \{L, L^{h_1}, \ldots, L^{h_r}\}$ for all $h \in M$. In particular U is normalized by M. This now shows that we have at most three components and so there is $M_1 \leq M$ such that M/M_1 is a subgroup of Σ_3 and M_1 normalizes L. Then M_1 also normalizes V_0. But M_1 contains a good E and so $N_G(V_0) \leq M$, a contradiction.

Hence we have that $|Y_M| = q_1^2$. But then $C_{Y_M}(L \times L^g) = 1$, a contradiction.

Suppose now $L = L^g$. Then $V_0 = V_0^g$. Hence we have that $V_0 \cap V_0^g = 1$ if $V_0 \neq V_0^g$. □

$root1$

Lemma 7.11 *Let $L \cong G(q)$ then Y_M does not project into a long root subgroup R of L.*

Proof: Assume false. Let first $L/Z(L) \cong L_2(4)$, $L_3(2)$, A_6, $L_3(4)$ or $Sz(8)$. By 7.10 we get $|Y_M| \leq 2^6$. If there is a good p–element ω centralizing Y_M, then this element normalizes L and $U = \langle S \cap L, N_L(\langle \omega \rangle) \rangle \leq M$. Hence $L/Z(L) \cong L_3(2)$ and $U/Z(L) \cong \Sigma_4$, $p = 3$. But Y_M projects into $Z(L)$ as ω centralizes Y_M, a contradiction. Hence we have that $L/Z(L) \cong Sz(8)$, $p = 3$ and $|Y_M| = 2^6$. But there is just one class of elementary abelian subgroups of order 27 in $GL(6, 2)$, which shows that there is some $\omega \in M$, with $C_G(\omega) \leq M$ and $|C_{Y_M}(\omega)| = 16$. Hence ω normalizes V_0. But again $L = \langle S \cap L, C_L(\omega) \rangle \leq M$, a contradiction.

As there are no outer automorphisms of L which centralize a Sylow 2–subgroup modulo R, we get that $|Y_M : C_{Y_M}(L)| \leq |R|$. Hence by 5.8 we get that $q > 2$. We have that $O_2(M) = O_2(C_M)$. Let Q_1 be the preimage of $O_2(C_{L/Z(L)}(R))$. Suppose that $Q_1 = O_2(M) \cap L$ and $O_2(M) = Q_1(C_{O_2(M)}(L))$. Then $N_L(Q_1) \leq M$. Hence there is a group W of order $q - 1$ acting transitively on $R = [Y_M, W]^\sharp$. By 7.10 we have that $V_0 = C_{Y_M}(L)$ is a

TI-set in M. We have that V_0 is not normalized by $O^2(M)$. So by O'Nan's lemma [GoLyS2, (14.2)] we get that $|Y_M| = 8$. But then C_M contains a good E, a contradition.

So suppose first that $Q_1 \neq O_2(M) \cap L$. Then as $C_L(R) = C_L(Y_M)$, we see that $Z(L) \neq 1$ and $Z(L) = Z(Q_1)$. In particular there is a subgroup of index q in $O_2(C_{L/Z(L)}(R))$, which is invariant under $C_{L/Z(L)}(R)$. With 1.4 we now see that $L/Z(L) \cong L_3(q)$, $L_2(q)$ or $Sz(q)$. This now, as $Z(L) \neq 1$, shows $L/Z(L) \cong L_3(4)$, $L_2(4)$ or $Sz(8)$, a contradiction.

So suppose now that $O_2(M)$ induces outer automorphisms on L. As $[O_2(M), C_M \cap L] \leq O_2(C_L(R))$, we see that $L \cong L_2(q)$ or $U_3(q)$. In all cases $C_L(O_2(M))$ is of the same shape, i.e. $L_2(r)$ or $U_3(r)$, where $r > 2$. But now we can argue as before as long as no element of Y_M induces an outer automorphism. As $[S \cap L, Y_M] \leq R$, we see that we have $L \cong L_2(q)$, $q = r^2$. Now in particular we have a group of transvections on Y_M to some hyperplane. Suppose first $r > 2$, then we get a foursgroup of transvections and so M/C_M has a component $K \cong L_n(2)$. We have that no element in $[K, Y_M]^\sharp$ is centralized by a good E as otherwise some y which induces a field automorphism on L would be centralized by a good E, which gives $L \leq M$. So we have that $m_p(K) \geq 2$ for $p \in \sigma(M)$ and if p divides the order of $C_K(y)$ for some $y \in [Y_M, K]$, we even get $m_p(K) \geq 3$. Now we see that we always get some good E in $C_M(y)$, a contradiction. So we are left with $L \cong L_2(4)$. But this we have handled at the beginning.

So what is left is $[L, O_2(M)] \not\leq L$. Then $L^{O_2(M)}/Z(L^{O_2(M)}) = L_1 \times L_2$, with $L_1 \cong L_2(q)$, $L_3(2)$ or $Sz(q)$. If $Z(L) \neq 1$, then we have $L_1 \cong L_2(4)$, $L_3(2)$ or $Sz(8)$, a contradiction. So we have $Z(L) = 1$. By 7.6 we have $[L, Y_M] \leq L$. If Y_M induces some outer automorphism on L_1, then $L_1 \cap Y_M \neq 1$ and as $[O_2(M), Y_M] = 1$, we get the contradiction $[L_1, O_2(M)] \leq L_1$. Hence Y_M induces just inner automorphism on L, then we have that $S \cap L$ centralizes Y_M. Now there is some $L_3 \cong L$, $L_3 \leq L_1 \times L_2$, $L_3 \cong L_2(r)$ or $Sz(q)$ and $O_2(M) = (O_2(M) \cap L_3)C_{O_2(M)}(L_3)$. Further $C_{Y_M}(L_3) = V_0$. But then we get a contradiction as above, applying O'Nan's Lemma and $|Y_M : V_0| > 2$. □

NT

Lemma 7.12 *Let T be a Sylow t–subgroup of C_M, t odd. Then $N_G(T) \leq M$.*

Proof: We have that $M = C_M N_M(T)$. Hence by 2.5 we have that $N_M(T)$ contains a good E. But then we get that $N_G(T) \leq M$ with 5.3. □

O2

Lemma 7.13 *Let U be the projection of Y_M onto L. Let R be a root subgroup. Then $U \not\leq O_2(C_L(R))$.*

Proof: Suppose false. Let first $[Q, L] \leq L$. Assume further that no element from Y_M induces an outer outomorphism on L. As by 7.1 we have that

$V = \langle Y_M^{C_L(R)} \rangle$ is abelian, we have that there is a normal abelian subgroup W of $C_L(R)$ which by 7.11 is not contained in R. This shows $L \cong Sp(2n,q)$, $F_4(q)$, ${}^2F_4(q)$, or $L_n(q)$.

Let $L \cong Sp(2n,q)$. By 7.5 we have that $m_p(L) \leq 3$ for all odd primes p. So $n \leq 3$. Let $L \cong Sp(6,q)$. Then $|W| = q^3$. We have $Sp(2,q)$ involved in $C_M \cap L$. Let t be a prime which divides $q+1$. Let $1 \neq T$ be a Sylow t–subgroup of $C_M \cap L$. Then we have with 7.12 that $N_L(T) \leq M$, which then shows $L = \langle C_M \cap L, N_L(T) \rangle \leq M$, a contradiction.

Let next $L \cong Sp(4,q)$, $q \neq 2$, or $L \cong L_n(q)$, $(n,q) \neq (3,2)$. Assume further that $N_L(W) \leq M$. We have $N_L(W)/W \cong GL(n-1,q)$ in case of $L \cong L_n(q)$. Hence in both cases we see that $W = Y_M \cap L$. Now $Y_M \cap L = Q \cap L$. This shows $\Phi(Q) \leq C_G(L)$. Hence $\Phi(Q) = 1$ and then $Y_M = Q = O_2(M)$. Further M induces an F–module on Y_M. Let $N_L(W)$ act on a nontrivial p–group P p odd, in M/C_M. Then we see that $p = 3$, as $N_L(W)$ induces an F–module offender on Y_M. Further, as $N_L(W)/W \not\cong L_2(2)$, we see that $m_3(P) \geq 4$. Hence $3 \in \sigma(M)$ and for all 3–elements $\rho \in M$ we have $N_G(\langle \rho \rangle) \leq M$. This with 7.5 shows $m_3(L) = 1$ and then $L \cong L_3(q)$, $3 \mid q+1$. Let $\rho \in N_L(W)$, $o(\rho) = 3$. Then $C_{Y_M}(\rho) = V_0$. As ρ is centralized by an elementary abelian group of order 27 in M, we see that also V_0 is normalized by such a group and then $L \leq N_G(V_0) \leq M$, a contradiction. Hence we may assume that $N_L(W)'/W$ is involved in some component K of M/C_M. With 3.16 we get $K \cong L_m(r)$, $Sp(2m,r)$, $\Omega^{\pm}(2m,r)$, $G_2(r)$ or A_m.

As $N_L(W)$ induces just one nontrivial irreducible module in Y_M the same holds for K. Let $V_0 \cap [K, Y_M] = 1$. Then we have that $[K, Y_M] = [N_L(W), Y_M] \leq L$. This shows that $C_{Y_M}(K) \neq V_0$ and so $C_W(N_L(W)') \neq 1$. Hence we have that $L \cong Sp(4,q)$. Now there is some $\nu \in N_L(W)$, $o(\nu) = q-1$, $[W,\nu] = W$ and $[N_L(W)/W, \nu] = 1$. This now shows $[K,\nu] \leq K$. Hence $[C_{Y_M}(K), \nu] \leq C_{Y_M}(K)$. So we get $C_{Y_M}(K) \neq V_0$, or $C_{Y_M}(K) = V_0 \oplus C_W(N_L(W)')$. By the choice of L, we have that $K \leq H$ and so $K \cong L_2(q)$. Now some element ρ of order $q+1$ in $N_L(W)$ is normalized by some good E in M, which with 5.3 shows that $\langle N_L(W), N_L(\langle \rho \rangle) \rangle = L \leq M$, a contradiction.

So we have that there is $1 \neq t \in [K, Y_M] \cap V_0$. Then $E(C_K(t)) \neq 1$, as $L \leq E(C_K(t))$. This shows that $K \cong A_m$. Let $3 \in \sigma(M)$ and $m_3(L) \geq 2$. Then we have that $m_3(M) \leq 3$ by 7.5. Hence $m \leq 11$. If $m_3(L) = 1$, then $L \cong L_3(q)$ and as $N_L(W)$ is a component in K, we get $q \leq 4$, a contradiction. If $3 \notin \sigma(M)$, then also $m \leq 11$. Then we have that $L \cong Sp(4,4)$, $L_3(4)$ or $L_5(2)$. But on the natural module for A_m these groups induce a permutation module, which they do not induce in L, a contradiction.

So we have that $N_L(W) \not\leq M$. Let first $L \cong Sp(4,q)$, $q > 2$. If Y_M does not project into $Z(S \cap L)$, then $W = C_{S \cap L}(Y_M)$. But then we have that W is normal in C_M, which then again shows $Y_M \cap L = W$, a contradiction. So we have that Y_M projects into the center of a Sylow 2–subgroup of L. We have that $J(O_2(M)) = J(C_{O_2(M)}(L)(S \cap L))$ and so the Cartan subgroup of L is in M. Hence $Y_M \cap L = Z(S \cap L)$. Now $[S \cap L, Y_M] = 1$ and so $S \cap L = Q \cap L = O_2(M) \cap L$. Now let $g \in M_0$ with $V_0 \cap V_0^g = 1$. As $\Omega_1(Z(C_Q(L))) = V_0$ and $C_Q(L^g) \cap V_0 = 1$, we see that $C_Q(L) \cap C_Q(L^g) = 1$. Hence also $C_{O_2(M)}(L) \cap C_{O_2(M)}(L)^g = 1$. As a Cartan subgroup of L is in M, we have that $O_2(M) = (O_2(M) \cap L)(O_2(M) \cap C(L))$. This now shows that $C_{O_2(M)}(L)^g$ is isomorphic to a subgroup of $S \cap L$ and so also $C_{O_2(M)}(L)$ is. This shows that $O_2(M)$ possesses at most 4 elementary abelian subgroups of maximal order. Let E_1 be one of them. Then we have that $|M/N_M(E_1)| \leq 4$. In particular there is some good E normalizing E_1 and then $N_G(E_1) \leq M$. Hence we have that $N_L(E_1) \leq M$, but $M \cap L$ was $N_L(S \cap L)$, while $N_L(E_1)$ involves $L_2(q)$. So we have that $M_0 \leq N_M(V_0)$. Then a good E normalizes V_0 and so $L \leq M$, a contradiction again.

Let now $L \cong L_n(q)$, $(n,q) \neq (4,q)$, $(3,2)$. As some elements in Y_M are centralized by some good E in M, there is some $t \in Y_M \setminus V_0$ such that $C_L(t) \leq M$. As all elements in U are conjugate in L, we may assume that the centralizer of some root element in L is in M. Hence $Y_M \cap L = W$ and so $N_L(W) \leq M$, a contradiction.

So let $L \cong L_4(q)$. Suppose that $C(R)$ is not in M. As there is some $y \in L$ with $C_L(y) \leq M$, and $S \cap L$ is a Sylow 2–subgroup of L, we get that the normalizer of an elementary abelian subgroup of order q^4 is in M. So $Y_M \cap L$ is of order q^4, a contradiction to $Y_M \cap L \leq O_2(C(R))$. So we always have that $C(R) \leq M$ and so $Y_M \cap L$ is of order q^3 and centralized by a graph automorphism of L. Again M induces an F-module on Y_M. Let $q > 2$. As $Y_M \cap L \leq L_1 \cong Sp(4,q)$, we may argue as in the case before to see that $N_L(W) \leq M$ and $Y_M = Q$. Let $t \in Y_M$ such that t projects onto some element in $Y_M \cap L \setminus R$. As $L = \langle C_L(R), C_L(t) \rangle$, we see that $C_G(t) \not\leq M$. In particular $R^M \leq R \times V_0$. Let now as before K be a component of M/C_M involving $N_L(W)'$. As $[N_L(W)', R^M] = 1$, we get that $[K,R] = 1$. Now $[O_2(N_L(W))), Y_M] \leq R$ and so $O_2(N_L(W)) \leq C_M$ as $[Y_M, K] \not\leq R$, a contradiction. So we are left with $L \cong A_8$. There is a foursgroup V_1 the centralizers of all of its elements are in M. Obviously $V_1 \cap L$ contains the root group. But then V_1 contains both types of involutions L. But then again $L = \langle C_L(t) \mid 1 \neq t \in V_1 \rangle \leq M$, a contradiction.

Let next $L \cong A_6$, or $L_3(2)$. We have that $|V_0| \leq 4$, as it has to be a TI-set and $|Y_M \cap L| \leq 4$. Hence we have that $|Y_M| \leq 2^4$. Let $p \in \sigma(M)$. If $p > 3$, then a good E centralizes Y_M, a contradiction. So $p = 3$, $e(G) = 3$

and $|Y_M| = 16$. Now C_M contains a 3–element ρ with $C_G(\rho) \leq M$. As such an element is not contained in L, we see that $[\rho, L] = 1$, a contradiction.

By 7.5 and 1.2 we see $L \not\cong F_4(q)$. So let finally $L \cong {}^2F_4(q)$. Then we get $|W| = q^5$. As $W = \Omega_1(C_L(W))$, we see that there is some conjugate W^h, where h is in the other minimal parabolic P of L containing $S \cap L$ such that $[W, W^h] \neq 1$. This shows with 7.1 that $Y_M \cap L \leq W \cap W^h$. In particular $N_L(W) \not\leq M$. Further $|W \cap W^h| = q^3$ and $Z(O_2(P)) \leq W \cap W^h$, where all elements of $Z(O_2(P))$ are root elements and $[W \cap W^h, O_2(P)] = Z(O_2(P))$.

If $q = 2$, then we have that $|Y_M \cap L| \leq 8$. As there is a fours group in Y_M intersecting V_0 trivially, which is centralized by a good E, we get that for some root element $y \in L$, $C_L(y) \leq M$. But then $Y_M = W$, a contradiction.

So we have that $q > 2$ and then $|Y_M \cap L| \leq q^3$. Let first $Y_M \cap L \leq Z(O_2(P))$. Then all elements in $Y_M \cap L$ are conjugate. As there is some $t \in Y_M \setminus V_0$, which is centralized by a good E in M, we get that $C_L(t) \leq M$ and so we have that $C_L(R) \leq M$, which would give the contradiction $Y_M = W$ again. So $Y_M \cap L \not\leq Z(O_2(P))$. Then we have that $Z(O_2(P)) \leq Y_M$, as Y_M is normal in $S \cap P$. Hence $N_L(Y_M)$ contains a subgroup of index $q - 1$ in P. Now we have that Y_M is an F–module for M and $N_L(Y_M)$ induces $q^2L_2(q)$. As above we get some component K of M/C_M, where K is isomorphic to $L_m(r)$, $Sp(2m, r)$, $\Omega^{\pm}(2m, r)$, $G_2(r)$ or A_m, and contains $N_L(WW^h)C_M/C_M$. As $C_{Y_M}(N_L(WW^h)) = V_0$, we get that $C_{Y_M}(K) \leq V_0$. and so $V_0 \leq [K, Y_M]$. Let now $t \in V_0$, then $C_K(t)$ has a normal subgroup isomorhic to $q^2L_2(q)$. Inspection of the groups K shows that we must have $K/Z(K) \cong L_3(q)$ and $|[K, Y_M]| = q^3$. Now K acts transitively on $[K, Y_M]^\sharp$ and so any such element in $[Y_M, K]$ is centralized by some good E. Hence there is also some element in $V_0^\sharp$ with this property, a contradiction.

So we are left with the case that Y_M induces an outer automorphism on L. Then $[Y_M, C_L(R)] \leq O_2(C_L(R))$. Hence Y_M does not induce a field automorphism. So we get that $L \cong L_4(q)$, $L_3(q)$, or A_6. In case of $L_3(q)$, we have that $[Y_M, S]$ is not abelian.

Let $L \cong L_4(q)$. Then we have that $|Y_M \cap L| = q^3$. Further we have some fours group in Y_M all of whose elements have centralizers in M. Hence we have such an element t in L. As $S \cap L \leq M$ we see that $M \cap L$ is in some parabolic, which has to be $N_L(Y_M \cap L)$. This shows that $t \in R$. Now we have that $C_L(t) \leq M$ for some element in $R^\sharp$. But there must be some outer automorphism $x \in Y_M$ such that $x \in O_2(C_{\mathrm{Aut(L)}}(t)))$. This shows $q = 2$. Then $|Y_M : V_0| = 16$. Now we see that Y_M contains some s with $C_G(s) \leq M$ and $C_L(s) \cong \Sigma_6$. Then $L = \langle C_L(t), C_L(s) \rangle \leq M$, a contradiction.

So let now $L \cong A_6$ and Y_M induces Σ_6. Then as V_0 is a TI–set, we get $|Y_M| \leq 2^6$. Let $\mu \in C_M$, $o(\mu) = p \in \sigma(M)$. As Sylow p–subgroups of C_M are cyclic, we have that $N_G(\langle\mu\rangle) \leq M$. But $C_L(Y_M)$ is a 2–group and so $[\mu, L] = 1$, a contradiction. Hence $\sigma(M) \cap \pi(C_M) = \emptyset$. This now shows $\sigma(M) = \{3\}$ and $|Y_M| = 2^6$. Further we have that $e(G) = 3$. As M/C_M is a subgroup of $GL(6,2)$ we get with 7.5 that Sylow 3–subgroups of G are isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. We have that $N_L(Y_M)/Y_M \cap L \cong \Sigma_3$. Let $\rho \in N_L(Y_M)$ be an element of order three. Then $N_L(\langle\rho\rangle) \not\leq M$. Let P be a Sylow 3–subgroup of M with $\rho \in P$. Then $|C_P(\rho)| = 9$. But all these elements in $GL(6,2)$ satisfy $|[Y_M, \rho]| = 16$, contradicting $[V_0, \rho] = 1$ and $|V_0| = 8$.

We are left with $[L, Q] \not\leq L$. As $L \in \mathcal{C}_2$, we have that $O(L) = 1$. Hence $m_p(L) = 1$ for all odd primes p. This shows with 7.9, 7.5 and 1.2 $L \cong L_2(q)$, $Sz(q)$ or $L_3(2)$. By 7.11 we have that $L \cong L_3(2)$ and then by 7.10 we get that $|Y_M| \leq 16$. Hence in any case C_M contains some p–element ρ, $p \in \sigma(M)$, with $C_G(\rho) \leq M$. But as $C_L(Y_M)$ is a 2–group, we get $[L, \rho] = 1$, a contradiction. □

constrain

Proposition 7.14 *If* $1 \neq x \in Y_M$, *then* $O_2(C_G(x)) = F^*(C_G(x))$.

Proof: Suppose false. Then we may choose H as before. Let L again be as before and R be a long root subgroup. Then by 7.13 for the projection U of Y_M onto L we have that $U \not\leq O_2(C_L(R))$. Then in any case we get $R \leq Y_M$ and in particular $[L, Q] \leq L$. Further with 4.1 we have that Y_M is a $2F$–module. Let M be exceptional, we see with 5.10 that offenders act quadratically. But then 4.1 shows that we have an F-module, which contradicts 5.9. So we have that

(i) M is not exceptional.

Let $q > 2$. Then by 1.2 and we have that $L \cong L_4(q)$, $Sp(6,q)$, $\Omega^-(8,q)$, $U_4(q)$, $G_2(q)$, ${}^3D_4(q)$, or ${}^2F_4(q)$. In ${}^3D_4(q)$ we easily see that $V = \langle Y_M^{N_L(S \cap L)} \rangle$ is not abelian which contradicts 7.1. Let $L \cong G_2(q)$ or ${}^2F_4(q)$ and $t \in Y_M$ which does not project into $O_2(C_L(R))$. Then we have $|[t, O_2(C_L(R))/R]| \geq q^2$, q^4, respectively. As $R \leq Y_M$, we see that Y_M projects onto a group of order at least q^3, q^5 in $O_2(C_L(R))$. But as L does not possesses an elementary abelian subgroup of order greater than q^3, q^5, respectively, we get a contradiction.

Let $q = 2$. Then we have at least that $m_3(L) \leq 3$. Hence $L \cong L_n(2)$, $4 \leq n \leq 7$, $Sp_6(2)$, $\Omega^-(8,2)$, $U_4(2)$ by 1.2, as $G_2(2)$, ${}^3D_4(2)$ and ${}^2F_4(2)$ are not possible by the same reason as for $q > 2$. So we have

(ii) $L \cong L_4(q)$, $L_5(2)$, $L_6(2)$, $L_7(2)$, $Sp_6(q)$, $\Omega^-(8,q)$, or $U_4(q)$.

Next we determine $Y_M \cap L$. Let $W \leq Y_M$ be of maximal order such that $C_G(w) \leq M$ for all $w \in W^\sharp$. Let first $L \cong L_6(2)$ or $L_7(2)$. Then we have that $3 \notin \sigma(G)$ by 7.5 and so also $e(G) \geq 4$. Hence we get that $|W| \geq 2^6$. We have that $C_G(r) \not\leq M$ for $r \in R^\sharp$ as Y_M does not project into $O_2(C_L(R))$. Let W_1 be the projection of W onto $\mathrm{Aut}_H(L)$. As $S \cap L \leq M$, we see that $W_1 \leq L$.

Let first $L \cong L_6(2)$ and U be the transvection group, U normal in $S \cap L$. If $C_L(u) \leq M$ for some $u \in U^\sharp$, we get that $L \cap M = N_L(U)$ and $O_2(N_L(U)) = U \leq O_2(C_L(R))$, a contradiction. So we have that $U \cap W_1 = 1$. As $N_L(U)/U \cong L_5(2)$ there are just two possibilities for W_1 and then $|C_U(W_1)| = 8$ or $|C_U(W_1)| = 4$. In both cases we see that $C_L(Y_M)$ is elementary abelian and so $Q = Y_M$. Hence $|Y_M \cap L| = 2^8$ or 2^9 and $N_L(Y_M \cap L)/Y_M \cap L \cong L_4(2) \times \Sigma_3$ or $L_3(2) \times L_3(2)$, respectively. Let $p \in \sigma(M)$ and $t \in C_M$ with $o(t) = p$. As Sylow p–subgroups of C_M are cyclic we have that $C_G(t) \leq M$. But we have that $[L, t] = 1$, as $C_L(Y_M \cap L) = Y_M \cap L$, and so $L \leq M$, a contradiction. So we have that $m_p(C_M) = 0$. As $|W_1| = 64$, we get that $p = 7$ and $m_7(M) = 4$. By 7.10 $|Y_M| \leq 2^{18}$ and so Sylow 7–subgroups of $GL(18, 2)$ are abelian. Hence all groups of order 49 are good. In particular L contains a good E, which contradicts 7.5.

Let now $L \cong L_7(2)$. Let $U = O_2(C_L(R))$. Assume $W_1 \cap U = 1$. Then we see again that the projection of Y_M onto L is of order 2^{12}, as Y_M is normal in S. Hence we get $|Y_M| \leq 2^{24}$ and $N_L(Y_M)/Y_M \cap L \cong L_4(2) \times L_3(2)$. Again $m_p(C_M) = 0$ and we get $p = 7$. Let now P be a Sylow 7–subgroup of $L \cap M$. Then we have that $N_G(P) \leq M$. But $N_L(P) \not\leq N_L(Y_M)$. So we have that $W_1 \cap U \neq 1$. In $U \setminus R$ we have three $C_L(R)$–conjugacy classes of involutions. Two of them are in L conjugate to $r \in R$. If one of these involutions is in W_1, we get that $L \cap M/O_2(L \cap M) \cong L_6(2)$ and so $W_1 \leq O_2(L \cap M)$, a contradiction. So $W_1 \cap U$ contains just involutions t with $C_{C_L(R))}(t)/O_2(C_{C_L(R)}(t)) \cong L_3(2)$. Let U_1 be the projection of Y_M onto L, then we see that $|U_1 : U_1 \cap U| \leq 16$ and so $|U_1| \leq 2^{10}$. Hence with 7.10 we get that $|Y_M| \leq 2^{20}$. Now as above we get $p = 7$ and Sylow 7–subgroups of M are abelian. But this contradicts $m_7(L) = 2$ and 7.5.

Let $L \cong L_5(2)$. We now have $|W_1| \geq 4$. If W_1 contains a root element, we get that $L \cap M/O_2(L \cap M) \cong A_8$. But then Y_M projects into $O_2(C_R)$. Hence W_1 does not contain root elements. Hence $L \cap M$ is contained in the normalizer of an elementary abelian subgroup of order 64 and then $L \cap M/O_2(L \cap M) \cong \Sigma_3 \times \Sigma_3$ or $L_3(2) \times \Sigma_3$. In particular we get $|L \cap Y_M| = 16$ or 64. Suppose the former. As in $L \cap Y_M$ any subgroup of order 8 contains some root element, we get $|W_1| = 4$ and so $3 \in \sigma(M)$. Let P be a Sylow 3–subgroup of $L \cap M$. Then $N_L(P) \not\leq L \cap M$. So we have that $\mathbb{Z}_3 \wr \mathbb{Z}_3$ is a Sylow 3–subgroup of G and P contains a unique subgroup P_1 of order 3 with $N_G(P_1) \leq M$. Hence we see that $N_L(P_1) \cong \Sigma_3 \times L_3(2)$, which is not in $L \cap M$. So we have that $M \cap L/O_2(M \cap L) \cong L_3(2) \times \Sigma_3$ and either $p = 3$ or $p = 7$. If $p = 3$, we argue as before to see that $N_L(P_1)O_2(L \cap M) = L \cap M$. There is a Sylow 3–subgroup P_2 of M which normalizes P_1. Then as $O_2(L \cap M) = [Y_M, P_1]$, we see that P_2 acts on $C_L(P_1) = V_0$, a contradiction. So let $p = 7$. We have with 7.10 that $|Y_M| \leq 2^{12}$ and so Sylow 7–subgroups are abelian. Again a Sylow 7–subgroup acts on $V_0 = C_{Y_M}(\mu)$, μ some 7–element in $M \cap L$.

So we are left with

(iii) $L \cong L_4(q)$, $Sp_6(q)$, $U_4(q)$ or $\Omega^-(8, q)$.

Let t be some element in Y_M which projects not into $O_2(C_L(R))$. If $L \cong L_4(q)$ or $U_4(q)$, then $|[t, O_2(C_L(R))]| = q^3$. In both cases we see that $C_{S\cap L}(\langle t, Y_M \cap O_2(C_L(R))\rangle)$ is elementary abelian. Hence $Q = Y_M$. In both cases we get $|Y_M \cap L| = q^4$ and

(iv) $M \cap L/Y_M \cap L \cong L_2(q) \times L_2(q) \times \mathbb{Z}_{q-1}$, or $L_2(q^2) \times \mathbb{Z}_{q-1}$.

Let $L \cong \Omega^-(8, q)$, then we get $|[t, O_2(C_L(R))]| = q^5$. We have that $C_L(R)/O_2(C_L(R)) \cong L_2(q^2) \times L_2(q)$. Let t project nontrivially onto the $L_2(q^2)$, then $\langle Y_M^{N_L(S\cap L)}\rangle$ is nonabelian as $L_2(q^2)$ induces $\Omega^-(4, q)$ on $O_2(C_R)$. Hence by 7.1 we see that t projects trivially on the $L_2(q^2)$ and nontrivially onto the $L_2(q)$. This now shows that $C_{S\cap L}(\langle t, [t, O_2(C_L(R))]\rangle)$ is elementary abelian of order q^6. Again $Q = Y_M$ and so

(v) $N_L(L \cap Y_M)/L \cap Y_M \cong \Omega^-(6, q) \times \mathbb{Z}_{q-1}$.

Let $L \cong Sp(6, q)$. If $t \in C(Z(O_2(C_L(R))))$, then we see that $|[t, O_2C_L(R)]| \geq q^3$. Then $C_{S\cap L}(\langle t, [t, O_2(C_L(R))]\rangle)$ is elementary abelian of order q^6 and $N_L(Y_M \cap L)^\infty/Y_M \cap L \cong SL(3, q)$. If $[t, Z(O_2(C_L(R)))] \neq 1$, then $|[t, O_2(C_L(R))]| = q^4$. We have $C_L(R)/O_2(C_L(R)) \cong L_2(q) \times L_2(q)$. Suppose that t does not centralize some component or Σ_3 in case of $q = 2$. Then we see that $\langle Y_M^{N_L(S\cap L)}\rangle$ is not abelian as it covers $S/O_2(C_L(R))$ for $q > 2$. If $q = 2$, then t acts as a c_2–element and so $[t, O_2(C_L(R))]$ is not abelian. Hence we have that t centralizes one of the components or one Σ_3. So we get that $|Y_M \cap L| = q^5$ and

(vi) $M \cap L/Y_M \cap L \cong Sp(4, q) \times \mathbb{Z}_{q-1}$.

Let $3 \in \sigma(M)$. Then by 7.5 we get that $L \cong L_4(q)$. Let further P be a Sylow 3–subgroup of $L \cap M$. Then we see that $N_L(P) \not\leq M$. This now shows that $\mathbb{Z}_3 \wr \mathbb{Z}_3$ is a Sylow 3–subgroup of G. In particular P contains exactly one subgroup P_1 of order three with $N_G(P_1) \leq M$. As $L \cap M/Y_M \cap L \cong L_2(q) \times L_2(q) \times \mathbb{Z}_{q-1}$, we see that P_1 is in one of the components and $[Y_M, P_1] = Y_M \cap L$. Now we have that P_1 is normal in a Sylow 3–subgroup P_2 of M and so P_2 normalizes $C_{Y_M}(P_1) = V_0$, a contradiction. Hence we have shown that

(vii) $3 \notin \sigma(M)$.

Further in all cases we have that $C_L(Y_M \cap L) = Y_M \cap L$. Let $\mu \in C_M$ with $o(\mu) = p$, $p \in \sigma(M)$. Then $[\mu, L] = 1$. As $m_p(C_M) = 1$, we have that $C_G(\mu) \leq M$, a contradiction. So we have that $m_p(C_M) = 0$ for all $p \in \sigma(M)$.

Let next $q = 2$. Then by 7.10 we have that $|Y_M| \leq 2^{12}$. Let $e(G) \geq 4$. Hence by the structure of $GL(12, 2)$, we see that $\sigma(M) = \{7\}$, and $|Y_M| = 2^{12}$. Further all 7–elements are good. We have $L \cong Sp(6, 2)$ or $\Omega^-(8, 2)$. Let $\mu \in M \cap L$, $o(\mu) = 7$. Then we must have $N_L(\langle\mu\rangle) \leq M$. But in case of $Sp(6, 2)$ we have $L \cap M \cong 2^6 L_3(2)$, where an element of order 7 is inverted in L.

Let $e(G) = 3$, then we see by 7.5 that $L \cong L_4(2)$ and so $|Y_M| \leq 2^8$. But $GL(8, 2)$ contains no elementary abelian subgroup of order p^3 for $p > 3$.

So we have shown

(viii) If $q = 2$, then $L \cong \Omega^-(8, 2)$, $e(G) = 4$ and $\sigma(M) = \{7\}$.

Let now first K be a component of M/C_M which induces an $2F$–module on Y_M and involves $N_L(Y_M)C_M/C_M$. We will show that

(ix) $[Y_M, K] = Y_M \cap L = [Y_M, L \cap M]$

Let $Y_M \cap L > [Y_M, K]$ then we may assume without loss that $Y_M = [Y_M, K]$. We may apply 3.43. We cannot have $(\alpha)(1), (3)$ or (4). Suppose we have $(\alpha)(2)$. If we do not have $L \cong Sp_6(q)$, we have that $N_L(R) \leq M$, a contradiction. Hence we have $L \cong Sp_6(q)$ and $|Y_M \cap L| = q^5$. Then even Y_M is an F–module and so 3.42 applies. In particular we may have 3.42(4)(ii), (iii), (v), (vi), (vii) or (viii). As $m_p(L) = 3$ for p a prime dividing $q^2 - 1$, we have that $e(G) \geq 4$ by 7.5. Further $[L \cap M, Y_M]$ involves exactly one nontrivial irreducible module. This shows that we must have $K \cong Sp_4(r)$. But $C_K(t)$ has a component $Sp(4,q)$ for any $t \in V_0^\sharp$. But $[K, Y_M]$ is an extension of the trivial module by the natural module and so $r = q$. Now we see that K contains a good E, a contradiction.

Hence we have one of 3.43(5). Suppose that $[Y_M, K]$ involves exactly one nontrivial irreducible module, we get (5)(i)-(xvi). If $[V, K]$ contains a trivial module, we have that $[N_L(Y_M), Y_M]$ contains a trivial module and so as before we get that $L \cong Sp(6,q)$ and we have the same contradiction as before. So we are left with (5)(ii), (vii) - (xvi). Suppose $e(G) = 3$. Then by 7.5 we have that $q = 2$ and $L \cong L_4(2)$, a contradiction.

So we have that $e(G) \geq 4$. But then Y_M is centralized by some good p–element according to 3.43(5). As $C_L(Y_M \cap L) = Y_M \cap L$, we see again that $L \leq M$, a contradiction.

So we have that $[Y_M, K]$ involves two nontrivial irreducible modules. Then we have $L \cong Sp(6,q)$, $|Y_M \cap L| = q^6$ and $(L \cap M)'/Y_M \cap L \cong SL(3,q)$. Now we have (xvii) - (xxi). For $t \in V_0$, we see that $C_K(t)$ involves $SL(3,q)$. This shows that we have (xviii), (xx) or (xxi). In the first two cases $L \cap M$ had to induce at least four nontrivial irreducible modules on Y_M, a contradiction. So we have (xxi) and $K \cong SL(3,r)$. But then $[t, K] = 1$, a contradiction.

So we have that (ix). Again as $C_G(R) \not\leq M$, we do not have 3.43(α)(1),(3) or (4). Suppose first that we have (2). Then as before we have $L \cong Sp(6,q)$ and Y_M is an F–module for K. Hence we have 3.42 with $e(G) \geq 4$. So we might have 3.42(iii) or (iv). But there is no $Sp(4,q)$ in $G_2(r)$, hence we have $K \cong Sp_4(r)$, $[Y_M, K]$ involves just one nontrivial irreducible module the natural one and $p \mid r-1$ for $p \in \sigma(M)$. As there is some $t \in L \cap M$, $o(t) = q-1$ and $[t, Y_M \cap L] = Y_M \cap L$, we see that t normalizes K and acts on $C_{[K,Y_M]}(K)$, which is nontrivial by (iii). So $C_{[K,Y_M]}(K) = C_{Y_M \cap L}((L \cap M)')$. This gives $r = q$. But then $m_p(L) = 2$ for some $p \in \sigma(M)$ and contains a good E, contradicting 7.5.

So we have 3.43(5). Let L be defined over $GF(2)$. Then $|[Y_M, K]| = 2^6$, $\sigma(M) = \{7\}$, $L \cong \Omega^-(8,2)$ and $e(G) = 4$. But then again $[Y_M, K]$ is centralized by a good E, a contradiction. So we have that L is not over $GF(2)$.

Assume first that in $[Y_M, K]$ there is exactly one nontrivial irreducible K–module involved. As $3 \notin \sigma(M)$, we have 3.43(5)(ii), (iii), (iv), (viii), (x), (xi), (xiv), (xv) or (xvi). As L is not defined over $GF(2)$ we see that (x) and (xi) are not possible. Further as $q > 2$ we see that $m_3(L) = 3$ for $L \not\cong L_4(q)$ and $m_p(L) = 3$ for $p \mid q-1$. So we have $e(G) \geq 4$. Then we just can have (iii), (iv), (xiv) or (xvi). Then $K \cong Sp(4,r)$ or $G_2(r)$. Suppose $C_{Y_M}(K) \neq 1$, then we have that $L \cong Sp(6,q)$. So we must have $Sp(4,q) \subseteq Sp(4,r)$ or $G_2(r)$. This now shows that we must have $q = r$ and $K \cong Sp(4,q)$ as above. But now L contains a good E, a contradiction.

So let now $[Y_M, K]$ involve more than one nontrivial irreducible module. This gives $L \cong Sp(6,q)$ and $|Y_M \cap L| = q^6$. Further $e(G) \geq 4$. As before we are not in 3.43(α)(1), (3), (4). If we are in (2), then as $Y_M \cap L$ is an indecomposable module for $L \cap M$ we get a root element in $C_{Y_M}(T)$ for a Sylow 2–subgroup T of K, a contradiction. So we have 3.43(5)(xviii) - (xxii), (xxiv). As there are exactly two such modules we get (xviii), (xix) or (xx). As $\Omega^+(8,2)$ does not involve $SL(3,q)$, $q > 2$, (xx) is possible. Also $SL(3,q) \not\leq Sp(4,r)$ for $q > r$. So we are left with $K \cong Sp(6,r)$. Now $|Y_M \cap L| \geq r^{14}$ and so $q^6 \geq r^{14}$. This gives $q > r^2$. But then $SL(3,q) \not\leq Sp(6,r)$.

So we are left with the possibility that $L \cap M/Y_M \cap L$ does not involve in some component. There are two possibilities for this. One is that it splits into two components K_1K_2. Then $L \cong L_4(q)$. Now as $Y_M \cap L$ is an irreducible $L \cap M$–module we see that $[K_1K_2, Y_M]$ is a tensor product module. Then we get that $[K_1K_2, Y_M] = [L \cap M, Y_M]$ and so either there is a good E in K_1K_2 and then in L, or $\sigma(M) \cap \pi(K_1K_2) = \emptyset$. So K_1K_2 is normal in M/C_M and so M acts on $C_{Y_M}(K_1K_2) = V_0$, a contradiction.

So we are left with the case that $L \cap M/Y_M \cap L$ acts nontrivially on some p–group P of M/C_M, and so also on some p–group of $M/O_2(M)$. Let $q > 2$. We see that $m_p(P) \geq 4$. So $p > 3$, as $3 \notin \sigma(M)$. Now $S \cap L$ contains a subgroup U of order q such that $C_{Y_M}(U) = C_{Y_M}(u)$ for all $u \in U^\sharp$. By 2.1 there is a subgroup W of $M/O_2(M)$ which is a direct product of dihedral groups of order $2p$ with U as a Sylow p–subgroup. But then the $A \times B$– lemma shows that $[O_p(W), Y_M] = 1$, which contradicts $q > 2$ and $m_p(C_M) \leq 1$.

So $q = 2$ and $L \cap M/Y_M \cap L \cong U_4(2)$. We again see that $m_p(P) = 4$ and so $p = 7$. But as now $|Y_M| = 2^{12}$ we have that M/C_M is a subgroup of $GL(12,2)$. But there is no extension of an elementary abelian group of order 7^4 by $U_4(2)$ in $GL(12,2)$. □

8 M contains 2-central centralizers

In this chapter we fix a uniqueness group M and a Sylow 2-subgroup S of M. If H is a subgroup of G with $C_G(O_2(H)) \leq O_2(H)$ then set $C_H = C_H(Y_H)$. We will show that M contains $C_G(x)$ for all $x \in \Omega_1(Z(S))^\sharp$.

Let
$$\mathcal{H} = \{C_G(x) \mid x \in \Omega_1(Z(S))^\sharp\}$$
We will show that all members of $\mathcal{H}$ are contained in M. By 7.14 we have $O_2(H_1) = F^*(H_1)$ for all $H_1 \in \mathcal{H}$. So assume there is some $H_1 \in \mathcal{H}$ with $H_1 \not\leq M$. We have $C_M S \leq H_1 \cap M$. Hence C_M does not contain an elementary abelian subgroup E of order p^2 with $\Gamma_{E,1}(G) \leq M$. Set $M_0 = N_M(S \cap C_M)$. Then M_0 contains some elementary abelian subgroup E of order p^2 with $\Gamma_{E,1}(G) \leq M$ by 2.5, 5.2 and 5.11. Now choose $H \leq H_1$ minimal with respect to $S \leq H$ and $H \not\leq M$. We are going to investigate the amalgam (M_0, H) (recall $O_2(H) = F^*(H)$). As usual (see 3.5) we have the parameter b.

CH

Lemma 8.1 *Let $Y_H \neq \Omega_1(Z(S))$, then $C_H \leq M$.*

Proof: Suppose false. Then $H = C_H S$, but then $\Omega_1(Z(S))$, which is in Y_H, would be normal in H. □

$CH1$

Lemma 8.2 *Let $Y_H \neq \Omega_1(Z(S))$, then $O_2(H)$ is a Sylow 2- subgroup of C_H.*

Proof: By 8.1 we have that $C_H \leq M$. As $H = C_H N_H(S \cap C_H)$, we get that $S \cap C_H$ is normal in H, so $S \cap C_H = O_2(H)$. □

$Hstruk$

Lemma 8.3 *Let $\bar{H} = H/C_H$. Then either $\bar{H}$ is solvable, or $\bar{H} = L\bar{S}$, where L is a product of isomorphic quasisimple groups K on which $\bar{S}$ acts transitively. If $KN_{\bar{S}}(K)$ induces an F–module on some $\bar{H}$–module V with offender A in $O_2(C_{M_0})$, then we have that $K \cong L_2(q)$, q even. In this case, if $L \neq K$, there are exactly two components.*

Proof: Let K be a component of $E(\bar{H})$. If $K \not\leq H \bar{\cap} M$, we have $\bar{H} = \langle K, \bar{S}\rangle$ and we are done. So we may assume that $E(\bar{H}) \leq H \bar{\cap} M$. Let $T = S \cap E(\bar{H})$. Then $\bar{S}$ normalizes T and $N_{\bar{H}}(T)$ is not in M. By the minimal choice of H we have $E(\bar{H}) = 1$. Now we have that $F(\bar{H}) \not\leq \bar{H} \cap \bar{M}$, otherwise $O_2(M_0) = O_2(H)$ and so as $Y_M = \Omega_1(Z(O_2(M_0)))$ by 3.4, we get that $Y_H = Y_M$ is normal in $\langle M, H\rangle$, a contradiction. So $\bar{H} = F(\bar{H})\bar{S}$ is solvable.

Assume now that $KN_{\bar{S}}(K)$ induces an F–module. Then K is a group of Lie type in characteristic two or alternating. The minimality of H shows that $\langle K, S\rangle$ is a minimal parabolic. Hence K is a minimal parabolic group and we get with 3.16, that $K \cong L_2(q)$.

Suppose now $L \neq K$. The number of conjugates of K is a power of two and the Cartan subgroup of K is in M by minimal choice. As $H \not\leq M$, we have that $m_p(H \cap M) \leq 3$ for each odd prime p. Hence we get that there are exactly two components. □

beven

Lemma 8.4 *We have that b is odd.*

Proof: Suppose false. Let first $b = b_{M_0}$. Hence there is M_α, with $Y_M \leq M_\alpha$ but $Y_M \not\leq H_\beta$ for at least one neighbour β of α. But by 7.1 $Y_M \leq O_2(M_\alpha) \leq H_\beta$ for all neighbours β of α, a contradiction.

So we have $b = b_H$. Now $[Y_H, Y_\alpha] \leq Y_H \cap Y_\alpha$. By 8.2 we have $[Y_H, Y_\alpha] \neq 1$. In particular Y_M is an F–module. Now we have that $Y_H \leq C_{\alpha-1}$. But the choice of M_0 shows that any 2–element of C_{M_0} is in $O_2(M_0)$. The structure of H is given by 8.3. Now make the notation such that M_0 is of distance $b+1$ from α and $Y_\alpha \not\leq M$. We have that $Y_M Y_H$ acts on Y_α.

Assume first that H/C_H has one or two components A_5 which induce a permutation module. We have that some A_4 in K is in M. Now $K \leq \langle O_2(M_0), Y_\alpha\rangle C_M/C_M$. Hence K is generated by elements which centralize a subgroup of index four in Y_M. This shows that $p = 3$ for $p \in \sigma(M)$ and $|Y_M : C_{Y_M}(Y_\alpha)| = 4$. As $Y_M \leq O_2(M_{\alpha-1})$, we see that $Y_M \leq C_{Y_M Y_H}(O_2(M_{\alpha-1}))$, which is the centralizer of a Sylow 2–subgroup of A_5 in the permutation module and so is centralized by some A_4 in K. Hence we see that there is some 3–element ρ in $C_M \cap K$ and so as C_M cannot contain a good E, we see that $N_G(\langle\rho\rangle) \leq M$, a contradiction.

So we have $L_2(q)$ on the natural module and we have that $O_2(M_0)C_H/C_H$ is a Sylow 2–subgroup of $E(H/C_H)$. Further we see that $[Y_H, Y_\alpha] = [Y_H Y_M, Y_\alpha]$ and then $Y_H Y_M$ is normal in H. In particular $O_2(M_0) \cap O_2(H)$ is normal in H and $C_H(O_2(H)/(O_2(H) \cap O_2(M_0))$ covers $E(H/C_H)$. Hence we have that $O_2(M_0)$ is a Sylow 2-subgroup of $U = \langle O_2(M_0)^H\rangle$. Let U_1 be the preimage of one of the components of $U/O_2(U)$ and $U_2 = U_1 O_2(M_0)$. Now applying [Ste, Theorem 3], we get a normal subgroup $1 \neq C$ of U_2 which is normalized by all odd order automorphisms of $O_2(M_0)$. But then there is also some good E in $N_M(C)$, and so $H \leq N_G(C) \leq M$, a contradiction.

Hence we are left with H to be solvable. As $Y_M Y_H$ acts quadratically on Y_α, we get with 4.5 that Y_α is generated by elements centralizing a hyperplane in Y_M. But then all these elements are in M, a contradiction. □

triv

Lemma 8.5 *We have that $Y_H = \Omega_1(Z(S))$.*

Proof: Suppose false. By 8.4 we have that $b = b_{M_0} = b_H$. So we have that $[Y_M, Y_\alpha] \leq Y_M \cap Y_\alpha$, where Y_α is conjugated to Y_H. By 8.2 we get that $[Y_M, Y_\alpha] \neq 1$. Hence one of both is an F–module. We are going to prove that Y_M is an F–module. Otherwise Y_H is an F–module. Now we may apply 8.3. We see that we cannot have transvections on Y_M, so H is not solvable. But then we have a component $L_2(q)$. Hence in all these cases we also have an F–module Y_M.

Now we are going to show $b = 1$. So assume $b > 1$. Choose $\beta \in \Delta(\alpha)$, with $Y_M \not\leq M_\beta$. We adopt notation of 3.42. Assume first we have some submodule U_M where every element from U_M is centralized by some good E. Then $[Y_\alpha, U_M]$ contains such an element and so as $b > 1$, this is centralized by Y_β. In particular $Y_\beta \leq M$. As $M = M_0 C_M$, we see that U_β acts on U_M and as $U_M \not\leq M_\beta$, we see that $C_{U_\beta}(U_M) = 1$, which is not possible. By 5.9 we have that M is not exceptional. So we have one of the cases in 3.42(4). As in any case Y_α contains an offender on Y_M, we also see with 3.42(4) that we do not have the cases (ii) - (v).

Let $K \cong Sp(6,2)$ be a component of M/C_M on the spinmodule. Then $p = 3$ and for all elements of order three, we have that the centralizer is in M. This now shows that we must have that either H is solvable or we have some $L_2(q)$ component $q > 4$. But as there are no transvections on Y_M, we get $L_2(q)$. As an offender has order at least 16, we get $L_2(32)$. But then we have a group of order 32 acting on Y_M, where all elements have the same centralizer, which does not fit with the action on the spin module.

Next let $K \cong U_4(r)$ acting on the natural module. Again there are no transvections. So H is nonsolvable. In that case we have a unique offender, and so $|[Y_M, Y_\alpha]| = r^4$. This group contains a subgroup, which is normalized by some good E in M. So as $b > 1$, we see that $Y_\beta \leq M$. Now $Y_\beta Y_\alpha$ acts on $[Y_M, K]$, and we get that $[[Y_M, K], Y_\alpha Y_\beta] = [[Y_M, K], Y_\alpha]$. This shows that $Y_\alpha Y_\beta$ is normal in H_α. But then we see that Y_α would centralize $Y_H Y_M$, a contradiction.

Let now $K \cong A_8$. Then again $p = 3$ and we have that H has components $L_2(q)$, q of order eight, which then have to induce transvections on each of the two modules in $[K, Y_M]$. Now we have that $[Y_\alpha, Y_M]$ is normalized by some good E, and we get the same contradiction as before.

Asume now that we have that $K \cong L_2(r)$ and $[Y_M, K]$ is a nonsplit extension of the trivial module by the natural module. Now $C_{[Y_M,K]}(K) \leq [Y_M, Y_\alpha]$ and as this is normalized by some good E, we get again the contradiction Y_HY_M is normal in H.

So we may assume that there is a normal r–group R in M/C_M on which Y_α acts faithfully and induces an F–module offender and further there is no component on which Y_α induces some F–module offender. In particular $r = 3$. Hence by quadratic action and 4.5, there is some $x \in Y_M$ inducing a transvection on Y_α. If $3 \notin \sigma(M)$, then $m_3(R) \leq 3$ and so by 2.3, we have that R is centralized by some good E. But then we have $Y_\beta \leq M$, a contradiction. So we have $3 \in \sigma(M)$. Let C be a characteristic subgroup of R, $C = \Omega_1(C)$. Assume $m_3(C) > 1$. We may assume that C is elementary abelian or extraspecial. If $m_3(C) > 2$, there is a good E in C and so it centralizes some element in $[Y_M, Y_\alpha]$ and we have a contradiction as before. So C is elementary abelian of order 9 or extraspecial of order 27. Further we may assume that $C = \Omega_1(C_M(C))$. But as $m_3(M) > 2$, we get C is extraspecial and M induces at least $SL(2,3)$ on C. But then all subgroups of order 9 in C are conjugate and so they are all good, which means that there are elements in $[Y_M, Y_\alpha]$ which are centralized by a good E, a contradiction. Hence we are left with R cyclic. Now $|[R, Y_M]| = 4$, and so M acts on this group, in particular it is centralized by a good E, a contradiction.

So we have shown that $b = 1$. Let first $|Y_M : C_{Y_M}(L)| = 2$ for some component L of $H/O_2(H)$ or H be solvable. By 5.9 M is not exceptional. Then Y_H induces a transvection on Y_M. So we may apply 3.42 and 5.9. Suppose first that we have some submodule W in Y_M, which is not centralized by Y_H such that any $x \in W$ is centralized by some good E in M. Then we have $W \not\leq C_H$. Further also $|W : C_W(L)| = 2$ for some component L, if there are components. Hence there is some $g \in H$ such that $\langle W, W^g\rangle = R$ is some extension of a 2–group by a dihedral group of order $2r$. Obviously R centralizes $W \cap W^g$. As $C_G(W \cap W^g) \leq M$ we see that $W \cap W^g = 1$. Further $O_2(H)$ centralizes $[Y_H, W]$, which is a nontrivial group. Now Y_H induces on W^y, $y \in O_2(H)$ the same transvections as on W. So we get that $O_2(H)$ normalizes W. Now $[W^g \cap O_2(R), W \cap O_2(R)] \leq W \cap W^g = 1$. So $C_W(y) = W \cap O_2(R)$ for any $1 \neq y \in W^g \cap O_2(R)$. In particular $W^g \cap O_2(R)$ is the full transvection group on the hyperplane $W \cap O_2(R)$. But then $R_1 = \langle Y_M, Y_{M^g}\rangle = O_2(R_1)R$ and we have that $Y_{M^g} \cap O_2(R_1)$ normalizes Y_M and then $Y_{M^g} \cap O_2(R_1) = C_{Y_{M^g}}(W)(W^g \cap O_2(R))$. In particular $(Y_{M^g} \cap O_2(R_1))(Y_M \cap O_2(R_1)) = (Y_M \cap Y_{M^g})(W \cap O_2(R))(W^g \cap O_2(R))$. So we have a component $K \cong L_n(2)$ on W. Suppose first $3 \in \sigma(M)$. As the point stabilizer of $L_n(2)$, $n \geq 5$, contains a good E, we have in that case that $\Omega_1(Z(S))$ is centralized by a good E, which contradicts $Z(H) \neq 1$. So we have $n \leq 7$ in any case and $n \leq 4$ if $3 \in \sigma(M)$. Let ω be an r–element in R. Then we see that $|Y_M \cap O_2(R) : C_{Y_M \cap O_2(R)}(\omega)| = 2^{n-1}$. If $e(G) \geq 4$ then ω centralizes some element in Y_M which is centralized by a good E. So we have

$e(G) = 3$ and then $n \leq 5$. Let $n = 5$. Then $p = 5$ or 7. Let ρ be a p–element in K. There is some p–element ν in $C(K)$ which centralizes a group of order 16 or 8 in $C_{Y_M}(K)$, respectively. Hence we have that there is a subgroup of order 32 in $Y_M \cap O_2(R)$ centralized by a good E, a contradiction as before. Let $n \leq 3$, then as in the case of $n = 5$, some 3-element in K centralizes some nontrivial element in W, we see that there is a good E centralizing an elementary abelian group of order 8 or 4, respectively and so we have a contradiction as before. We are left with $n = 4$. If $p \geq 5$, then we can argue as before. So we may assume $p = 3$. Then we get that $|C_{Y_M}(K)| \leq 8$ and so we see that for $x \in Z(H)$ we have that $C_M(x)/O_{3'}(C_M(x)) \cong L_3(2)$. This now shows from the structure of a Sylow 3-subgroup of $C_G(x)$ that $C_G(x)/O_{3'}(C_G(x)) \cong L_3(2)$. If H is nonsolvable we get that $K \cong Sz(q)$. Then $|Y_M : C_{Y_M}(W)| \geq q^2$ and so $|Y_M : C_{Y_M}(W)| \geq 16$, a contradiction. So we have that H is solvable.

Now if $r = 3$, then, as all elementary abelian 3-groups are good, we get that Sylow r-subgroups of H are cyclic. Let $r > 3$, then as $|W \cap O_2(R)| = 8$, we see $r = 7$. Now as $|Y_H : Y_H \cap C_M| \leq 8$ and W inverts the Frattini factorgroup of $F(H/C_H)$, we see that also in that case Sylow r–subgroups of H are cyclic. Hence in any case $Y_M O_2(H) = O_2(M_0)O_2(H)$. As $O_2(H) = (W \cap O_2(H))(W^g \cap O_2(H))C_{O_2(H)}(\omega)$, this now shows that $\Phi(O_2(M_0)) \leq C_{O_2(H)}(\omega)$. Hence $O_2(M_0) = Y_M$, as $\Phi(O_2(M_0))$ is normalized by a good E. But now $(W \cap O_2(R))(W^g \cap O_2(R))$ is normalized by $L_3(2)$ in K and centralized by some 3–element in $C(K)$, a contradiction.

By 3.42 and as we have transvections, we just have to handle the case that we have $K \cong \Omega^-(6, 2)$ on the natural module, in which case $p = 3$. Now let W be the orthogonal module and built R as above. Then we have $|W^g \cap O_2(R)| = 32$, and as the 2-rank of $O^-(6, 2)$ is four we have $W \cap W^g \neq 1$. Hence this intersection corresponds to singular vectors in W. In particular $|W \cap W^g| \leq 4$. Suppose equality, then $W \cap W^g$, would be normalized by some good E in K, a contradiction. So we have $|W \cap W^g| = 2$ and $W^g \cap O_2(R)$ induces a group of order 16 on W, which contains a transvection. But $C_K(Y_H)$ is isomorphic to Σ_6 and W is the permutation module for this group. But there is no elementary abelian subgroup in this group whose commutator with W is of index two.

Hence we are left with the case that Y_H acts on a normal u-group U, and as it induces transvections we have $u = 3$. If $3 \notin \sigma(M)$, then by 2.5 we get that U is centralized by some good E and so again we have some module W where all elements are centralized by a good E. Using R as above, we get that $|W| = 4$ (there are no fours groups of transvections acting nontrivially on U). But now ω centralizes a subgroup of index 4 in Y_M, a contradiction. So $3 \in \sigma(M)$. As Y_M also induces transvections by 4.5, we get that

$F(H/C_H)$ is a 3–group by 8.3 as H is solvable. So $m_3(H/C_H) \leq 2$ and then $|Y_M : Y_M \cap C_H| \leq 4$. Choose $x \in Y_H$, $y \in Y_M$ with $|[x, y]| = 2$. Then in any case $[x, y]$ is centralized by a good E in M. This shows that $m_3(H) = 1$. Now $Y_M O_2(H) = O_2(M_0)O_2(H)$. Further $[x, y] = [Y_M, Y_H]$ is normalized by $O_2(H)$ and so y centralizes a subgroup of index two in $O_2(H)$. This shows that $O_2(H) = V \times C_{O_2(H)}(F(H/C_H))$, where $V = \langle x^H \rangle$ is of order 4. Hence $\Phi(O_2(M_0)) \leq C_{O_2(H)}(F(H/C_H))$, and so we see that $O_2(M_0) = Y_M$. Now $\langle y, Y_M \rangle$ is normalized by some good E in M and so, as there are exactly two elementary abelian sugroups of order $|Y_M|$ in this group, we see that $V(Y_M \cap O_2(H))$ is normalized by some good E. But this group is normal in H, a contradiction.

Now H is nonsolvable and $|Y_M : C_{Y_M}(L)| \geq 4$. We have that H is a minimal parabolic with a quadratic fours group. Now we get by 3.26 and 3.28 that $L \cong L_3(q)$, $L_2(q)$, $Sz(q)$ or $U_3(q)$. Further we may apply 4.2 with the roles of M and H interchanged. Then we get that Y_H either is an F–module or a $2F$–module with non quadratic offender. With 4.8 we get that $L \cong L_2(q)$, $q > 2$. As Y_M is normal in $M \cap H$ we see that Y_M covers a Sylow 2-subgroup of that component and so by quadratic action 3.50 Y_H just involves trivial and natural modules. Suppose first that we have some component K in M and some module W as in 3.42 where all elements are centralized by some good E. Again this module is invariant under $O_2(H)$. Define R and R_1 as above, we see that $\Phi(O_2(M_0))$ is centralized by some element in $H \setminus M$, as $W \cap W^g = 1$. Hence we get $O_2(M_0) = Y_M$ again.

We have that Y_M is a strong F–module, i.e there is a subgroup $X \leq Y_H$ $|X| = q$ and $|Y_M : C_{Y_M}(X)| = q = |Y_M : C_{Y_M}(x)|$ for all $x \in X^\sharp$. We have that $XO_2(M_0)$ is normal in S. Now by 3.17 we have that $K \cong SL_n(r)$, $Sp(2n, r)$, A_7 or $3A_6$, recall as $q > 2$ we do not have $GF(2)$–transvections. Choose ρ to be some generator of a cyclic group of order $q - 1$ in H such that ρ acts transitively on X. As for $SL_n(r)$ we have that $[Y_M, K]$ contains at most $n - 1$ natural modules, while for the other groups we have just one, we see that ρ has to normalize K.

Let $K \cong SL_n(r)$. Then X intersects a root group nontrivially. As $\langle S \cap K, \rho \rangle = (S \cap K)\langle \rho \rangle$, we see that for $r > 2$, we have that X is contained in a root group, while for $r = 2$, we may have $\langle S \cap K, \rho \rangle / O_2(\langle S \cap K, \rho \rangle) \cong \Sigma_3$ and $|X| = 4 = q$.

Let $K \cong Sp(2n, r)$. Then $X \cap Z(S \cap K) \neq 1$. If X does not contain a root element, then $|Y_M : C_{Y_M}(X)| = r^2$. Now as centralizers of elements of type a_2 in K are maximal subgroups, we see that X consists of elements of type c_2. Let R be a long root element with $X \cap O_2(N_K(R)) \neq 1$. As $C_{Y_M}(X) = C_{Y_M}(x)$ for all $x \in X^\sharp$, we see that $|X \cap O_2(N_K(R))| \leq r$. As X is normal in $S \cap K$, we get that X induces $GF(r)$–transvections on $O_2(N_K(R))$. But this is only possible if $r = 2$ and $q = 4$.

Let first $r = q$ and $W^g \cap M$ is the full transvection group of W. This shows $K \cong L_n(r)$ and just one natural module is involved. We have $C_{Y_M}(X) = C_{Y_M}([Y_H, R_1])$. This shows that $\langle X^{R_1} \rangle$ is an extension of the trivial module by the natural module. We have $n \leq 4$, as $\Omega_1(Z(S))$ is not centralized by a good E. As $W^g \cap M$ centralizes a subgroup of index q in Y_M, we get that $W^g \cap M$ projects onto K. So $(W^g \cap O_2(R_1))(W \cap O_2(R_1))$ is unique in $(W^g \cap M)Y_M$, and so it cannot be normalized by a good E. Hence $m_p(K) \geq 2$ and p divides $q - 1$. Then $(W^g \cap M)Y_M$ is normalized by some p–element in K. As $m_p(N(K)/K) \geq 1$, we get that it is normalized by some good E, a contradiction

Let $K \cong SL_n(2)$, $Sp(2n, 2)$, A_7 or A_6. If $p = 3 \in \sigma(M)$, then any 3–element of M is in some elementary abelian subgroup of order 27, so $N_G(\langle \rho \rangle) \leq M$, which than yields $H = \langle M \cap H, N_H(\langle \rho \rangle) \rangle \leq M$. So $p = 3 \notin \sigma(M)$. As $m_p(K) \geq 2$, we get $K \cong SL_n(2)$ and $n = 6$ or 7. As now $m_3(K) = 3$ and so $e(G) \geq 4$, we see that there is some good E centralizing K and so normalizing XY_M, a contradiction.

So assume now that we are in 3.42(4). Still Y_M is a strong F-module and so by 3.17 we have that $K \cong L_2(r)$, $Sp(4, r)$ or $L_4(2)$. Choose ρ as before, then we see that $K \cong L_2(q)$, $Sp(4, q)$. In the case of $L_4(2)$ or $Sp_4(2)'$ we have $q = 4$, but $3 \in \sigma(M)$ by 3.42(4)(vi), which is not possible. Let $K \cong Sp(4, q)$, then by 3.42(4)(iii), there is some power of ρ whose normalizer is in M, a contradiction. So we have $K \cong L_2(q)$ and $[Y_M, K]$ is a nonsplit extension of the trivial module by the natural one. Now we see that $[O^2(R_1), O_2(H)] \leq [R_1, Y_H]$. Hence again $\Phi(O_2(M_0)) \leq C_{O_2(H)}(O^2(R_1))$ and so $Y_M = O_2(M_0)$. Let K not be normal in M/C_M. Then we have at least two conjugates centralizing K, so K is centralized by a good E and as $[Y_H, R_1]$ centralizes all components but K by the strong action, we get that $[Y_H, R_1](O_2(R_1) \cap Y_M)$ is normalized by a good E, a contradiction. So we have that K is normal, but then $m_p(N(K)/K) \geq 2$ and we get the same conclusion that $[Y_H, R_1]Y_M$ is normalized by a good E.

We finally have to treat the case that X acts on a u-group U. But by 4.5 we then get transvections on Y_H, which contradicts $q > 2$. $\square$

zunique

Lemma 8.6 *Let* $1 \neq X \leq \Omega_1(Z(S))$ *then* X *is not normalized by a good* E *in* M.

Proof: Suppose false. Then $C_H(\Omega_1(Z(S))) \leq M$. As $H = C_H(\Omega_1(Z(S)))N_H(S)$, we would get $H \leq M$ with 7.3.

□ $b = 1$

Hypothesis 8.7 *There is* $1 \neq x \in \Omega_1(Z(S))$ *with* $Y_M \not\leq O_2(C_G(x))$.

Assume 8.7. Then we may apply 4.2. By 7.1 we have that 4.2(1) cannot occur. Let L be the group given by 4.2. Then $A = Y_M^g \cap O_2(L)$ is a $2F$–module offender on Y_M. We will study the action of A. for the remainder we will fix the notation A and L.

$Knormal$

Lemma 8.8 *Assume 8.7. Let* K *be a component of* M/C_M *with* $[K, A] \neq 1$. *If* A *is as in 4.2(2) then* $[K, A] \leq K$.

Proof: By 4.3 we have $|A| > 4$ and then $K \cong L_n(2)$. Further $|[Y_M, a]| \geq 2^n$. As $|A| \leq 2^n$, we get equality everywhere and $N_A(K)$ induces the full transvection group on $[Y_M, a]$. This shows that $[Y_M, K]$ just involves the natural module. By 3.36, we get $[Y_M, KK^a] = [Y_M, K] \oplus [Y_M, K^a]$. We see that all elements in $[Y_M, KK^a]$ are centralized by $L_{n-1}(2) \times L_{n-1}(2)$, which contains a good E besides $n = 3$. As $A \cap [Y_M, KK^a] \neq 1$, we would get $Y_{M^g} \leq M$ in the first case. So we have the latter and $3 \notin \sigma(M)$ and we may assume that $7 \in \sigma(M)$. Now $[Y_M, \langle K^A \rangle] = Y_M$, otherwise $\langle K^A \rangle$ would centralize some element in $Y_M \cap Y_M^g$, and so $Y_M^g \leq M$, a contradiction. Further $|\Omega_1(Z(S))| = 2$. Now we have that $C_G(\Omega_1(Z(S))) \cap \langle K^A \rangle A \cong \Sigma_4 \wr Z_2$. This group induces in Y_M the following modules

$$\Omega_1(Z(S)) < V < Y_M$$

where $|V| = 4$ and Y_M/V is irreducible. Let $\Omega_1(Z(S)) = Y_M \cap O_2(C_G(\Omega_1(Z(S))))$. Then we see that $O^2(\langle Y_M^{C_G(\Omega_1(Z(S)))} \rangle)$ centralizes $O_2(C_G(\Omega_1(Z(S))))$, which contradicts 7.14. So $V = Y_M \cap O_2(C_G(\Omega_1(Z(S))))$. This shows that Y_M induces a transvection group of order 16 to a point on $O_2(C_G(\Omega_1(Z(S))))/\Omega_1(Z(S))$. Hence shows there is exactly one component in $C_G(\Omega_1(Z(S)))/O_2(C_G(\Omega_1(Z(S))))$ which is not centralized by Y_M and this component must be some $L_n(2)$, $n \geq 5$. As $M/C_M \cong L_3(2) \wr Z_2$ we get that $O_2(C_G(\Omega_1(Z(S)))) \not\leq C_M$. So we see that $C_M(\Omega_1(Z(S)))/O_2(C_G(\Omega_1(Z(S))))$ is an extension of C_M by $\Sigma_3 \wr Z_2$. But there is no such subgroup in $L_n(2)$, a contradiction. □

$faithful$

Lemma 8.9 *Assume 8.7. Then*
a) If we have 4.2(2) with A *acting cubic but not quadratic, then there is a component* K *of* M/C_M *or a Sylow t-subgroup of* $F(M/C_M)$ *such that* AC_M/C_M *acts faithfully on* K *and induces a* $2F$*–module offender on* $[Y_M, K]$.

b) We do not have $M/C_M \cong \Sigma_3 \times \Sigma_3$ *or* $\Sigma_3 \wr Z_2$, *where* Y_M *is an irreducible 4-dimensional module and* $3 \in \sigma(M)$. *Further* A *acts cubic but not quadratic.*

Proof: By 5.10 we have that M is not exceptional. Choose a component K with $[K, A] \neq 1$, where K always also can be a Sylow subgroup of $F(M/C_M)$. By 8.8 we have $[K, A] \leq K$. If $C_A(K) = 1$, we have a). So assume that $B = C_A(K) \neq 1$. Then there is some further component (or a Sylow subgroup of $F(M/C_M)$) K_1 with $[B, K_1] \neq 1$. Choose K_1 with $|B : C_B(K_1)|$ maximal. Let $C = C_A(K_1)$. If $[C, K] = 1$, then $C \leq B$. Now choose K_2 with $[C, K_2] \neq 1$.

By the choice of B we have that $|B : C_B(K_2)| \leq |B : C_B(K_1)|$. As $C \leq C_B(K_1)$ and $C \not\leq C_B(K_2)$, we have that $C_B(K_1) \neq C_B(K_2)$. In particular $C_B(K_2) \not\leq C_B(K_1)$. Hence there is some $b \in B$ with $[K_1, b] \neq 1$ but $[K_2, b] = 1$. So we may choose two components K_1, K_2 (or Sylow subgroups of $F(M/C_M)$) with $A_i = C_A(K_i) \neq 1$ and $[A_i, K_{3-i}] \neq 1$, $i = 1, 2$.

Let $A = \tilde{A}_1 \times C_A(K_1)$ and V_1 be a quasi irreducible $K_1\tilde{A}_1$–submodule in Y_M. Suppose first that $V_1 \not\leq O_2(L)$, in particular $C_A(V_1) = Y_M \cap Y_M^g$. Let $V_1 \cap Y_M^g \not\leq C_{V_1}(K_1)$. Then for all $a \in A$ we have $V_1 \cap V_1^a \not\leq C_{V_1}(K_1)$. In particular $V_1^A = V_1$. Then $[V_1, A_1] = 1$, which contradicts $V_1 \not\leq O_2(L)$. So we have that $V_1 \cap Y_M^g \leq C_{V_1}(K_1)$. Let $v \in V_1 \setminus O_2(L)$, then we have that $[v, \tilde{A}_1] \cong \tilde{A}_1$. As no element in $A_1^\sharp$ centralizes any element in $V_1 \setminus C_{V_1}(K_1)$, we get that $[v, \tilde{A}_1]C_{V_1}(K_1) = V_1 \cap O_2(L)$ and $|V_1 \cap O_2(L)/C_{V_1}(K_1)| = |\tilde{A}_1|$. Now let $1 \neq a \in A_1$. Set $V_2 = V_1^a$. Then also V_2 is a quasi irreducible $K_1\tilde{A}_1$–module. Further $[V_1, a]$ is also such a module. As $[V_1, a] \leq O_2(L)$, we see that $[[V_1, a], \tilde{A}_1] \leq Y_M \cap Y_M^g$.

We collect some facts about the action on $\tilde{V_1} = V_1/C_{V_1}(K_1)$. We have that $\tilde{A}_1$ acts quadratically on $[V_1, a]$ and so also on $\tilde{V_1}$. Further $\tilde{V_1}$ is an F–module with offender $\tilde{A}_1$. We have that $C_{V_1}(a_1) = C_{V_1}(\tilde{A}_1)$ for all $1 \neq a_1 \in \tilde{A}_1$. Application of 3.17 now gives that K_1 is solvable or $K_1/Z(K_1) \cong L_n(r)$, $Sp(2n, r)$, r even, or A_7 or $3A_6$, or $|\tilde{A}_1| = 2$. Suppose the latter, then we have that $|\tilde{V_1}| = 4$ and so K_1 is solvable. If K_1 is solvable it is a 3–group as it induces an F–module.

As we can look at $\langle \tilde{V_1}^{K_2} \rangle$, we see that there is also some module for K_2 which is not in $O_2(L)$ and so K_2 also has the structure above.

Suppose there is a good E normalizing a nontrivial subgroup U of $[[V_1, a], \tilde{A}_1]]$. As $U \leq Y_M \cap Y_M^g$, we see that $L \leq N_G(U)$, but as $N_G(U) \leq M$, we have a contradiction. In particular we see that there is no good E in K_1 normalizing a nontrivial subgroup of $[\tilde{V_1}, \tilde{A}_1]$.

Let first $K_1 \cong 3A_6$. But then in the 6–dimensional module we see that there is no element v with $[v, \tilde{A}_1] = C_{V_1}(\tilde{A}_1)$.

Let $W = \langle V_1^{K_2} \rangle$. Then W is an irreducible module for $K_1 \times K_2$ with $W \cap Y_M^g \neq 1$ and $W \not\leq O_2(L)$. Let $a \in C_A(K_1 \times K_2)$ with $[a, W \cap Y_M^g] = 1$, so $W = W^a$. As W was irreducible this shows $[a, W] = 1$. But the $W \leq O_2(L)$, a contradiction. Hence A acts faithfully on W.

Let next $K_1/Z(K_1) \cong A_7$. Then $\tilde{V_1}$ is the four dimensional module and $|\tilde{A_1}| = 4$. Now as any fours group in $\tilde{V_1}$ is normalized by some elementary abelian subgroup of order 9 in K_1, we see that $3 \not\in \sigma(M)$. This shows $m_3(K_2) \leq 2$ and $K_2/Z(K_2) \cong L_2(r)$, $L_3(r)$, A_7, or a 3–group. If we have $K_2 \cong L_3(r)$, then W is a tensor product of the natural $SL_3(r)$– module with V_1 and so $|W : C_W(A)| \geq r^8$. As $|A| \leq 4r^2$, we get $K_2/Z(K_2) \cong L_3(2)$. If $K_2 \cong L_2(r)$, we see that $|W : C_W(A)| \geq r^6$, which shows $r = 2$, which is also the case for K_2 to be solvable. If $K_2/Z(K_2) \cong A_7$, we get $|W : C_W(A)| \geq 2^8$. In all cases we have that $|Y_M : C_{Y_M}(A)| = |A|^2$, which contradicts 4.2.

Let next $K_1 \cong Sp(2n, r)$. Then no 1–dimensional module in the natural module is normalized by some good E in K_1, which shows $n \leq 3$. Let first $K_1 \cong Sp(6, r)$. Then we see that $m_p(K_1) \leq 1$ for $p \in \sigma(M)$. In particular $3 \not\in \sigma(M)$, but then $m_3(K_2) = 0$, a contradiction. So we have $K_1 \cong Sp(4, r)$. Now we have that p does not divide $r - 1$ if $p \in \sigma(M)$. Further we have that $|\tilde{A_1}| = r^2$. Suppose first that $K_2/Z(K_2) \cong L_2(t)$, $L_3(t)$ or solvable. Set $s = \max(r, t)$. In the case of $L_3(t)$, we have that $|W : C_W(A)| \geq s^8$. As $|A| = r^2t^2$, we get $|W ; C_W(A)| = |A|^2$, contradicting 4.2. Let $K_2 \cong L_2(t)$, then $|W : C_W(A)| \geq s^6$, again a contradiction to $|A| \leq r^2 t$. If K_2 is solvable we get that $|W : C_W(A)| \geq r^6$, a contradiction again. So we have that $K_2/Z(K_2) \cong L_n(t)$, $n \geq 4$ or $Sp(2n, t)$. If $x \in \tilde{A_1}^\sharp$, then $[\tilde{V_1}, x]$ is normalized by $C_{K_1}(x)$ and so by some $L_2(r)$. As $[\tilde{V_1}, x] = C_{\tilde{V_1}}(x)$ for all $x \in \tilde{A_1}^\sharp$, we see that $[\tilde{A_1}, \tilde{V_1}]$ is normalized by some $L_2(r) \times L_2(t)$. As $3 \in \sigma(M)$ it is normalized by a good elementary abelian subgroup of order 9, a contradiction.

Let now $K_1 \cong L_n(r)$. Then we have that $K_2 \cong L_m(t)$ or K_2 is solvable. Suppose first $r > 2$, then we see that $n \leq 4$, otherwise some one dimensional subspace in the natural module is normalized by a good E. Let first $K_1 \cong L_4(r)$. As $[\tilde{V_1}, \tilde{A_1}]$ contains a 2–dimensional submodule, which is normalized by some elementary abelian subgroup of order 9, we see that $3 \not\in \sigma(M)$. This shows that $K_2/Z(K_2) \cong L_3(t)$, $L_2(t)$ or solvable. Let $GF(\ell)$ be the largest common subfield of $GF(r)$ and $GF(t)$. Let $r = \ell^x$, $t = \ell^y$. Then $W = V_1 \otimes V_2$, V_2 be the natural K_2–module and $U = [V_1, N_A(V_1)] \oplus [V_2, N_A(V_2)] = C_{V_1}(N_A(V_1)) \oplus C_{V_2}(N_A(V_2))$ is contained in a complement of $Y_M \cap Y_M^g$ in $Y_M \cap O_2(L)$ and so of size at most $|A|$. We have that $|A| \leq \ell^{3x+2y}$, ℓ^{3x+y}, $2r^2$, respectively. Further $|U| \geq \ell^{5xy}$, ℓ^{4xy}, r^4. As $r > 2$, we see that K_2 is nonsolvable. Further we get that $r = t = \ell$. But then p divides $r - 1$, for some $p \in \sigma(M)$. So a good E normalizes a 1–space in $\tilde{V_1}$, a contradiction.

Let next $K_1/Z(K_1) \cong L_3(r)$. Then as before we see that $K_2 \cong L_2(r)$ or $r = 4$ and $K_2 \cong K_1 = SL(3,4)$. Let $K_2 \cong SL(3,4)$. Any 1-space in $\tilde{V_1}$ is normalized by some elementary abelian group of order 9. So we see that $3 \notin \sigma(M)$ and so $e(G) > 3$. But then there is a good E centralizing K_1K_2 and so also W, a contradiction. So we have that $K_2 \cong L_2(r)$. Also p does not divide $r-1$ for $p \in \sigma(M)$. So we get that $e(G) > 3$. Further $m_p(K_1 \times K_2) \leq 2$ for $p \in \sigma(M)$. In particular some elementary abelian group of order p^4 normalizes $K_1 \times K_2$ and so a good E normalizes $C_{V_1 \otimes V_2}(T)$, T a Sylow 2-subgroup of $K_1 \times K_2$ which contains A. But $C_{V_1 \otimes V_2}(T) \leq Y_M \cap Y_M^g$, a contradiction.

So we are left with $K_1 \cong K_2 \cong L_2(r)$ and W be the tensor product of two natural modules. Now A is a Sylow 2–subgroup of $K_1 \times K_2$. Then $C_W(A) \leq Y_M \cap Y_M^g$ is normalized by some group $Z_{r-1} \times Z_{r-1}$. Hence if $p \in \sigma(M)$, we have p does not divide $r-1$. Further no good E normalizes a nontrivial 2–group in $K_1 \times K_2$ and so $e(G) = 3$ and $m_p(K_1 \times K_2) = 2$ for all $p \in \sigma(M)$. So we have that $K_1 \times K_2$ is invariant under S. This shows that $C_{Y_M}(K_1 \times K_2) = 1$. Further we have $|A| = r^2$ and $W = Y_M$. Hence we see that $F^*(M/C_M) = K_1 \times K_2$ and p divides $|C_M|$. So we have

$$\text{Either } r = 2 \text{ or } F^*(M/C_M) \cong L_2(r) \times L_2(r), Y_M \text{ is the tensor product module}$$
$$\text{and } p \text{ divides } |C_M|, p \in \sigma(M)$$

Let now $r = 2$. We then have $n \leq 7$. As 3 divides the order of K_2, we even get $n \leq 5$. Let $n = 4$ or 5, then there is a foursgroup in $[\tilde{V_1}, \tilde{A_1}]$ which is normalized by some elementary abelian group of order 9. Hence $3 \notin \sigma(M)$. So we have that $K_2 \cong L_3(2)$ or solvable. In both cases we have a good E, which centralizes $K_1 \times K_2$ as $e(G) > 3$. As $C_A(K_1 \times K_2) = 1$, we see that E acts on W and so has to centralize W, which contradicts $W \cap Y_M^g \neq 1$.

So we may assume that $K_1 \cong L_3(2)$. Suppose K_2 is solvable. Then as no good E can centralize W, we see that $p = 3 \in \sigma(M)$. But then there is some 3–element centralizing W. As in K_1 we have some 3–element normalizing $[V_1, \tilde{A_1}]$, we get some subgroup of $Y_M \cap Y_M^g$, which is normalized by a good E, a contradiction. So we have $K_2 \cong L_3(2)$ and $3 \notin \sigma(M)$, but $7 \in \sigma(M)$. Further $e(G) = 3$ and we get $W = Y_M$ as above and $F^*(M/C_M) = K_1 \times K_2$. So we have

$$F^*(M/C_M) = K_1 \times K_2, K_1 \cong K_2, K_1 \cong L_2(r) \text{ or } L_3(2)$$

Let first $K_1 \cong L_3(2)$. Then $|Y_M| = 2^9$, $|\Omega_1(Z(S))| = 2$. Set $H_2 = C_G(\Omega_1(Z(S)))$. Then we have that $H_2 \cap M/C_M \cong \Sigma_4 \times \Sigma_4$ or $\Sigma_4 \wr Z_2$. In both cases we have that $|Y_M \cap O_2(H_2)| = 32$ and $Y_M/Y_M \cap O_2(H_2)$ is an irreducible module for $M \cap H_2$, as $Y_M \not\leq O_2(H_2)$. Further Y_M acts quadratically on $O_2(H_2)$ and induces an F–module offender on $O_2(H_2)/\Omega_1(Z(S))$. This shows that Y_M has to centralize $F(H_2/O_2(H_2))$. Let R be a component of $H_2/O_2(H_2)$ with $[Y_M, R] \neq 1$. Then by 3.16 we have that $R/Z(R)$ is a classical group, $G_2(q)$ or an alternating group. As $e(G) = 3$, we see that in the latter we just have $R/Z(R) \cong A_7$ or A_6. But as $|Y_M : Y_M \cap O_2(H_2)| = 16$ and R is normalized by $M \cap H_2$, this is not possible. Suppose first that R is not normalized by $M \cap H_2$. Then we have that R has cyclic Sylow 3–subgroups, so $R \cong L_2(q)$ or $L_3(q)$, or 3 divides the order of $Z(R)$ and R has extraspecial Sylow 3–subgroups. As $e(G) = 3$, in the latter we have $R \cong SL_3(4)$, but then $Z(R)$ has to act nontrivially, a contradiction. As Y_M is an F–offender, we get $L_2(4)$ or $L_3(2)$ in the first case. In $H_2 \cap M$ there is some 7–element ν centralizing Y_M, which implies that ν centralizes R. As Sylow 7–subgroups of C_M are cyclic, we have that $M = C_M N_M(\langle \nu \rangle)$. So $N_M(\langle \nu \rangle)$ contains a Sylow 7–subgroup of M and then we have that $m_7(N_M(\langle \nu \rangle)) = 3$, so $C_G(\nu) \leq M$, which cannot be the case as Y_M is not normal in $C_{H_2}(\nu)$. So we have that R is normalized by $H_2 \cap M$. As $|Y_M/Y_M \cap O_2(H_2)| = 16$ and offenders in $G_2(q)$ are of order q^3, we get that $R \not\cong G_2(q)$. Further as $m_3(R) \leq 2$, we get $R \cong L_4(q), L_5(2)$, $Sp_4(q)$ or $\Omega^+(6, q)$. As $Aut(Sp_4(q))$ has no subgroup of type $M \cap H_2/C_M$, we get that $R \not\cong Sp_4(q)$. Let $q > 2$, then 3 does not divide $q - 1$. So, as $|[O_2(H_2)/\Omega_1(Z(S)), x]| \leq 16$ for $x \in Y_M$, we get that we just have $L_4(8)$. But then $m_7(R) = 3$, which contradicts $7 \in \sigma(M)$. So we have $R \cong L_4(2), \Omega^+(6, 2)$ or $L_5(2)$. As $C_M \neq O_2(M)$, we must have $Y_M \leq \Phi(O_2(M))$, so we cannot have $R \cong L_4(2)$ or $\Omega^+(6, 2)$. This shows that $R \cong L_5(2)$. Now choose ν as before, then $[\nu, R] \leq O_2(H_2)$, a contradiction as before.

So let now $K_1 \cong K_2 \cong L_2(r)$. We also include the case of $\Sigma_3 \cong K_1$, which then will give b). Again $Y_M \leq \Phi(O_2(M))$. Let first $r > 2$. Then there is some $\rho \in K_1 \times K_2$ acting fixed point freely on $T = S \cap (K_1 \times K_2)$ and centralizing $\Omega_1(Z(S))$. We see $|Y_M \cap O_2(L)| = r^3$. Now $X = [A, Y_M \cap O_2(L)] = C_{Y_M}(T)$. Hence we have $[X, L] = 1$. Set $H_3 = N_G(X)$, then $Y_M \not\leq O_2(H_3)$. Hence we have that $O_2(M)O_2(H_3) \geq T$ with equality in case of $r > 2$. This gives that $|Y_M \cap O_2(H_3)| = r^3$. Hence in all cases there is some subgroup U in Y_M with $U \cap O_2(H_3) = 1$, $|U| = r$ and some group of order $r - 1$ acts fixed point freely on U. Further $[Z(O_2(H_3)), U] = 1$, $C_{O_2(H_3)/Z(O_2(H_3))}(U) = C_{O_2(H_3)/Z(O_2(H_3))}(u)$ for all $u \in U^\sharp$. Finally $|O_2(H_3)/Z(O_2(H_3)) : C_{O_2(H_3)/Z(O_2(H_3))}(U)| = r^2$.

Suppose that U act nontrivially on $F(H_3/O_2(H_3))$. Then we get $r = 2$ and so it acts on a t–group, $t = 3$ or 5. Let $t = 5$, then we have that $|[U, F(H_3/O_2(H_3))]| = 5$. But as $U \leq \Phi(O_2(M))$, we get that $O_2(M)$ induces a cyclic group of order four. As $[[U, O_2(H_3)], O_2(M)] = 1$, this contradicts the action of a Frobenian group of order 20 on a 4-dimensional $GF(2)$–module. So we may assume $t = 3$ and so we have that $[U, F(H_3/O_2(H_3))]$ is extraspecial of order 27 or elementary abelian of order 9. But in the case of $r = 2$, we have that $3 \in \sigma(M)$. Further $M/C_M \cong \Sigma_3 \times \Sigma_3$ or $\Sigma_3 \wr Z_2$ and so $m_3(M) = 3$. But then all elementary abelian 3-groups of order 9 are good, which contradicts 7.3 and 5.4.

Let R be some component with $[R, U] \neq 1$. As U is normal in $S/O_2(H_3)$, we see that $[R, U] \leq R$. As $e(G) = 3$ and $m_3(M) \geq 2$, we get $m_3(H_3) \leq 2$. Further there is some p–element $\nu \in M \cap H_3$, with $C_G(\nu) \leq M$. In particular $[\nu, R] \neq 1$.

Let first R be alternating, then we may assume $R/Z(R) \cong A_6$ or A_7. In particular R is normal in $H_3/O_2(H_3)$, we see that $\nu \in R$ and so we have that $p = 3$ and $R \cong A_7$, but this contradicts $m_3(R) = 2$. Let next R be sporadic. As R induces a $2F$–module and $m_3(R) \leq 2$, we get with 3.32 that $R \cong M_{12}$, M_{22}, M_{23}, M_{24}, $3M_{22}$ or J_2. In all cases $r > 2$. Now we see that p divides the order of the centralizer of a 2-central involution in R. Further $p \neq 3$. So we get $R \cong M_{23}$, M_{24} or J_2. As ν has to centralize a fours group, we now get a contradiction.

By 3.31 we now see that R is a group of Lie type over a field of characteristic two. This shows with 3.29, as $m_3(R) \leq 2$, that $R/Z(R) \cong L_2(q)$, $L_3(q)$, $L_4(q)$, $L_5(q)$, $Sp_4(q)$, $\Omega^-(4, q)$, $\Omega^+(6, q)$, $U_3(q)$, $G_2(q)$ or $Sz(q)$. Let $R/Z(R) \cong U_3(q)$, or $Sz(q)$, then $r = q$. In the second case ν centralizes R, while in the first we get $m_p(R) = 2$, both is not possible. If we have $R \cong G_2(q)$, then we see that $q \leq r$. But the action of an element of order $r - 1$ and the maximality of the normalizer of some root group shows that we must have $r = q$ and then $m_p(R) = 2$, a contradiction.

Now as ν has to induce an inner automorphism on R which centralizes U, we see that we have $R \cong L_3(q)$, $L_4(q)$ or $L_5(q)$. Finally, as $m_p(R) = 1$, we get that $R \cong L_5(q)$ and p divides $q^3 - 1$ but not $q - 1$. This also shows that U is contained in a root subgroup. In particular we get $r = q$ or $r^2 = q$, as $|[u, V]| = r$ or r^2 for any nontrivial irreducible R–module V involved in $O_2(H_3)$ and any $u \in U^\sharp$. But as p divides $r^2 + 1$ we have that p divides $r + 1$, so it cannot divide $q^2 + q + 1$ at the same time.

So we now have that any quasi irreducible submodule for K_1 is contained in $O_2(L)$ and the same applies for K_2. Let W_1 be the submodule generated by all these submodules for $K_1\tilde{A}_1$ and correspondingly W_2 the one for $K\tilde{A}_2$. As for any V_1, we have that $V_1 \cap Y_M^g \not\leq C_{V_1}(K_1)$, we see that $[V_1, A_1] = 1$, so we have that $[W_1, K_2] = 1$ and also $[W_2, K_1] = 1$. Let now $B = C_A(K_1 \times K_2) \neq 1$. Then we have K_3 with $[K_1 \times K_2, K_3] = 1$ and $[B, K_3] = K_3$. We have

$[W_1, B] \leq Y_M^g$. But then we see that $[W_1, B] \leq C_{W_1}(K_3)$ and so we have that $[K_3, W_1] = 1$ and by the same argument we have that $[K_3, W_2] = 1$. So we see that $[K_3, Y_M] \leq C_{Y_M}(K_1 \times K_2)$. Assume $[K_3, Y_M] \not\leq O_2(L)$. We then have $W_1W_2[K_3, Y_M] \leq [K_3, Y_M](Y_M \cap Y_M^g)$. But then we would have that A and so $K_1 \times K_2$ centralizes $W_1W_2[K_3, Y_M]/[K_3, Y_M]$, a contradiction. So we have that $[K_3, Y_M] \leq O_2(L)$.

In particular there are $K_1 \times K_2 \times \cdots \times K_s$, such that A acts faithfully on $K_1K_2 \cdots K_s$ and there is a faithful module W for $K_1 \cdots K_sA$, which is in $O_2(L)$. Hence A acts quadratically and as an F–module offender on W.

So we may assume that K_1 induces an F-module on W_1 with offender $\tilde{A}_1$. Suppose $m_p(K_1) \geq 2$ for some $p \in \sigma(M)$. As K_1 centralizes W_2 and $W_2 \cap Y_M^g \neq 1$, we get elements in $Y_M \cap Y_M^g$ which are centralized by a good E, contradicting $L \not\leq M$. So we have $m_p(K_1) \leq 1$ for any $p \in \sigma(M)$.

Let K_1 be not normal in M, then all Sylow r–subgroups, r odd, of K_1 are cyclic, or r divides $|Z(K_1)|$. As K_1 induces an F–module, we get with 3.16 that K_1 is solvable, $L_2(q)$, $L_3(2)$, $SL(3,4)$, $3A_6$ or $3A_7$. Now W_1 contains at most two nontrivial irreducible modules. Further there is no good E which centralizes both. If $K_1 \cong L_3(2)$, we must have $p = 3$ and $e(G) = 3$. As $|W_1 : C_{W_1}(A)| = |\tilde{A}_1|$, we get that also K_2 induces an F–module and so we may assume $K_2 \cong K_1$. In particular, $s = 2$. Hence we have that $|A| = 16$. Then $C_{W_1}(A) \leq Y_M \cap Y_M^g$. But now $C_{W_1}(A)$ is normalized by an elementary abelian 3–subgroup of order 9 in K_1K_2, a contradiction. If we have $K_1 \cong SL(3,4)$, $3A_6$ or $3A_7$, then $3 \notin \sigma(M)$. But then a good E centralizes K_1 and so also W_1, a contradiction. So we just have one module involved. Then $K_1 \cong L_2(q)$ and p divides $q-1$. Again we have $K_1 \cong K_2$ and so we get $|A| = q^2$ and then an elementary abelian group of order p^2 normalizes $C_{W_1}(A)$, which is in $Y_M \cap Y_M^g$.

So we may assume that K_1 is normal in M. Suppose first that K_1 is solvable. Then $|W_1| = 4$ and W_1 is centralized by a good E. But $C_{Y_M}(S) \cap W_1 \neq 1$, as W_1 is normal in M. This contradicts 8.6. So we can apply 3.42. Let first $\tilde{W}$ be a submodule such that any element is centralized by a good E. As $1 \neq [\tilde{A}_1, \tilde{W}] \leq Y_M \cap Y_M^g$, we get a contradiction. So we have one of the cases in 3.42(4). As $m_p(K_1) \leq 1$, for $p \in \sigma(M)$, we just have $K_1 \cong L_2(q)$ and W_1 is an extension of a trivial module by the natural module. Now $1 \neq C_{W_1}(K_1)$ is normalized by a good E, which contradicts the fact that $C_{W_1}(K_1) \leq [W_1, A] \leq Y_M \cap Y_M^g$. □

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Lemma 8.10 *Assume 8.7. Let 4.2(2) then A acts quadratically.*

Proof: Assume that A does not act quadratically. Then by 5.10 we have that M is not exceptional. Then by 8.9 there is a component K (maybe solvable) of M/C_M on which AC_M/C_M acts faithfully. Let first K be non-solvable. Then we may apply 3.43. We have that no subgroup of $Y_M \cap Y_M^g$ is normalized by a good E. Suppose we have 3.43(1). Then $[Y_M, K] \cap Y_M^g \neq 1$, a contradiction. Let next 3.43(2). Let $T = S \cap K$ and $\langle K^S \rangle = K_1 \cdots K_s$. Then $C_{Y_M}(S \cap \langle K^S \rangle)$ is centralized by a good E. Hence also $C_{Y_M}(S)$ is centralized by a good E, a contradiction.

Assume next that we have 3.43(3) or (4). Let W be the corresponding module. Suppose $[W, A] \neq 1$. Let $W \leq O_2(L)$, then $W \leq M^g$. This implies now $[W, W^g] = 1$. In particular $[W, C_{Y_{M^g}}(K^g)] \neq 1$. But then as there is some $x \in W^\sharp$ which is centralized by a good E in M and some p–element ν with $C_G(\nu) \leq M^g$, we get $\nu \in M$. As we are not in 3.43(1), we have that p divides $|K|$ and as we may assume that $C_W(K)$ is not centralized by a good E, we have that $m_p(K) = 1$. As ν cannot centralize an elementary abelian subgroup of order p^3 in M, we get that K^ν is a direct product of p conjugates of K. As ν centralizes a group isomorphic to K, we see that Sylow r–subgroups, r odd, of K are cyclic. This shows that $K \cong L_2(q)$ or $L_3(2)$. Further either $p = 3$ and $e(G) = 3$, or $p > 3$. In both cases any odd prime dividing $|K|$ is in $\sigma(M)$. Further as $[W, K]/C_{[W,K]}(K)$ is irreducible we get that (3) or (4) is true for any odd prime dividing $|K|$. So choose first $p > 3$, which gives us more than 3 conjugates, and then choose $p = 3$, which shows that $C_M(\nu)$ contains an elementary abelian subgroup of order 27, a contradiction. Now $W \not\leq O_2(L)$. Then $Y_M^g \cap W \neq 1$, a contradiction. So we may assume that $[W, K] = 1$. Then we have $C_K \leq O_2(L)$ and so $C_{Y_M}(K) \leq M^g$. As A centralizes $C_{Y_M}(K)$ and $C_{Y_M}(K) \not\leq [Y_M, K]$ we see that $Y_M^g \cap C_{Y_M}(K) \neq 1$, a contradiction.

So we have one of the cases in 3.43(5). Let first $[Y_M, K]/C_{[Y_M,K]}(K)$ be irreducible. Let $[K, S] \not\leq K$. Then we see that for $m_p(K) \geq 2$ we get (3) or (4). Then we have that $m_p(K) \leq 1$. Then we have 3.43(5)(i). Now $C_{[Y_M,K]}(K) \neq 1$. As $C_{[Y_M,K]}(K) \leq Y_M^g$ and this group is normalized by some good E, we get a contradiction.

So we have that S normalizes K. We see that $C_{Y_M}(S)$ does not contain subgroups normalized by a good E by 8.6. In particular 3.43(5)(i), (iii), (iv) and (v) are not possible. Further A must not act quadratically. We show next

$(*)$ *If* $[Y_M, K]/C_{[Y_M,K]}(K)$ *is irreducible, we have* $K \cong \Omega^-(6, q)$ *on the natural module, or* $K \cong Sp(6, 2)$ *or* A_9 *on the 8–dimensional module. In all cases* $e(G) = 3$ *and* $3 \in \sigma(M)$.

If we have (viii) or (xii), then an offender A has to induce inner automorphism on K. Hence $[Y_M, A, A]$ contains $C_{Y_M}(S \cap K)$. But this group is normalized by a good E and centralized by L, showing $L \leq M$, a contradiction.

Assume next that we have (x). Let first $K \cong A_7$. We have at most two 4–dimensional modules involved. If $p \neq 3$, then a good E centralizes $[Y_M, K]$. Hence we are in (1), (3) or (4). So we may assume that $p = 3$. As $|Y_M : C_{Y_M}(A)| < |A|^2$, we see that $[Y_M, K]$ involves exactly one nontrivial irreducible module. But then we have (3) or (4), a contradiction.

Let next $K \cong 3A_6$ and the 6–dimensional module involved in $[Y_M, K]$. Then as before we get $p = 3$ and so we are in (3) or (4).

Let $K \cong A_6$ and the 4-dimensional module be involved in $[Y_M, K]$. Then again we have $p = 3$, and we are in (3) or (4).

Let $K \cong A_5$, then we have the natural module and $p = 3$. But this is (3) or (4) or (5)(i).

Suppose (xi), then we get a contradiction with 3.34

If $K \cong U_3(q)$ or $Sz(q)$, we have that A acts quadratically.

If we have (xiv), then $C_{[Y_M,K]}(K) = 1$, otherwise there is some $1 \neq x \in Z(S)$ centralized by a good E, contradicting 8.6. But now as $m_p(K) \geq 2$, we see that any element in $[Y_M, K]$ is centralized by some p–element and we have (3) or (4).

If we have (xv) or (xvi). Then p divides $q^2 - 1$. Further $m_p(K) = 2$. This now shows that $C_{Y_M}(K) = 1$. Now all elements in $C_{Y_M}(S \cap K)$ are conjugate and centralized by $L_2(q^2)$. As $e(G) > 2$, they are all centralized by some good E, a contradiction.

So we have shown $(*)$.

We now go over the three cases in $(*)$. Let first $K \cong \Omega^-(6, q)$. Then $Y_M = [Y_M, K]$ and $C_{Y_M}(K) = 1$. Further $[Y_M, A, A]$ is normalized by $L_2(q^2) \times Z_{q-1}$, which shows $q = 2$. So we have $|Y_M : Y_M \cap O_2(H_2)| = 2$, for $H_2 = C_G(\Omega_1(Z(S)))$. By 5.4 we know that $m_3(H_2) \leq 1$. Suppose first that Y_M acts nontrivially on a Sylow t–subgroup U of $F(H_2/O_2(H_2))$. As $M \cap H_2$ involves $L_2(4)$, we see that $t > 3$. Assume that the $L_2(4)$ centralizes U. Then we may choose $L = \langle Y_M, Y_M^g \rangle$, with $g \in U$ such that the $L_2(4)$ acts on L. But as $Y_M \cap O_2(H_2)/\Omega_1(Z(S))$ is the orthogonal $L_2(4)$–module, we see that $L \times L_2(4)$ acts on a direct sum of two such modules, which is just possible for $t = 3$. So we have that the $L_2(4)$ has to act nontrivially on U. Let C be a critical subgroup, $C = \Omega_1(C)$. As $e(G) = 3$, we have $m_t(C) \leq 3$.

Let first C be elementary abelian. and U_1 be an irreducible submodule which is inverted by Y_M. As $|[Y_M, O_2(H_2)/\Omega_1(Z(S))]| = 16$, we see that $t = 5$ and $|U_1| = 25$. In particular we induce $SL_2(5)$ on U_1. Choose $\nu \in U_1$, $o(\nu) = 5$, which is centralized by some 5–element in $H_2 \cap M$ and inverted by Y_M. Let V be some nontrivial irreducible $U_1(M \cap H_2)$–module involved in $O_2(H_2)$. Then we get that $|[\nu, V]| = 2^8$ and $[Y_M, C_V(\nu)] = 1$. But U_1 is inverted by Y_M, so we get that $[U_1, C_V(\nu)] = 1$, which shows $C_V(\nu) = 1$. Then $|V| = 2^8$, but $GL(8, 2)$ does not contain such a subgroup. So we have that C is extraspecial. In particular it is of order t^3. Again $t = 5$ and $SL_2(5)$ is induced. Then $m_5(H_2) = 3$ and so $5 \in \sigma(H_2)$. We have that $(M \cap H_2)Z(C)$ acts on $Y_M \cap O_2(H_2)$. Hence there is some 5–element ν centralizing this group. In particular $\nu \in M$. But now $M \cap H_2$ contains an elementary abelian subgroup of order 25, which is contained in some elementary abelian sugroup of order 125 in H_2. Let H_3 be a uniqueness subgroup containing H_2, then by 5.4 we have that $M \leq H_3$. But this now shows $M = H_3 \geq H_2$, a contradiction.

So we have seen that $[Y_M, F(H_2/O_2(H_2))] = 1$. Hence there is some component R of $H_2/O_2(H_2)$, which is not centralized by Y_M. As $m_3(R) \leq 1$, we get $R/Z(R) \cong Sz(r)$, $L_2(r)$, $L_3(r)$, $U_3(r)$ or J_1. As $R \not\leq M$, we see that the $L_2(4)$ has to induce an inner automorphism on R centralizing Y_M. Let r be odd. Then by 3.47 we get $R \cong L_2(25)$ and Y_M induces field automorphisms on R. But then some $y \in Y_M$ inverts some element of order 13 in R. So we could have chosen L with $L/O_2(L) \cong D_{26}$. But no element of order 13 acts on a group of order 512, which is the order of $(Y_M \cap O_2(L))(Y_M^g \cap O_2(L))$. If $R \cong J_1$, then y inverts some element of order 11, which gives the same contradiction as before. So we have $R/Z/R) \cong L_2(r)$, $L_3(r)$, $U_3(r)$, $Sz(r)$, r even. As y centralizes some $L_2(4)$ in H_2, we see that that y induces some outer automorphism on R. But $[y, S \cap R] = 1$, a contradiction.

Let next $K \cong Sp(6, 2)$ or A_9 and $[Y_M, K] = Y_M$ be of order 2^8. Let again $H_2 = C_G(\Omega_1(Z(S)))$. Then $M \cap H_2$ involves $L_3(2)$. Again $m_3(H_2) \leq 1$. If Y_M acts nontrivially on a Sylow t–subgroup U of $F(H_2/O_2(H_2))$, we get $t > 3$. Let $\rho \in M \cap H_2$ be of order three. Then $[\rho, U] \neq 1$, as $U \not\leq M$. As $[Y_M, O_2(H_2)/\Omega_1(Z(S))]$ is of order 8 or 64, we get that $t = 7$ or 5 in the latter. If $t = 7$, we get as above that a critical subgroup C of U is elementary abelian of order 7^2 and $M \cap H_2$ induces $SL_2(7)$. Hence with the same arguments as above we see that for a module V involved in $O_2(H_2)$ we have $|[V, C]| = 2^{12}$. But Sylow 7–subgroups of $GL(12, 2)$ are abelian. So we have the case of $t = 5$. Let C be again a critical group with $C = \Omega_1(C)$. We have C is elementary abelian of order at most 125 or extraspecial of order at most 5^5. But in all these cases $L_2(7)$ cannot act on C, a contradiction. So Y_M acts nontrivially on some component R, which has to be $Sz(r)$, $L_2(r)$, $L_3(r)$, $U_3(r)$ or J_1. As the $L_3(2)$ in $H_2 \cap M$ has to induce an inner automorphism group normalizing Y_M, we get a contradiction with 3.47 as above.

So we have seen that $[V, K]$ involves more than one nontrivial irreducible K–module. We are going over the remaining cases in 3.43(5).

Suppose first that we have one of (xvii) - (xxiii). Then in all cases we see $[K, S] \leq K$, otherwise we would have (3) or (4). Now we have two modules involved, where at least one of them has to be an F-module.

If we have (xvii), then F–module offenders are exact, so $|Y_M : C_{Y_M}(A)| = |A|^2$, a contradiction to 4.2(3).

Assume we have (xviii). If we have two spin modules, we get that $|A| > q^4$, otherwise we would have exact offenders. Then A acts quadratically on one of these modules, which gives that it is in $O_2(L)$. Now there is a subgroup in $Y_M \cap Y_M^g$, which is normalized by $Sp(4, q)$, a contradiction. So we have an extension of the spin module by the natural module and A is not an F–module offender on the spin module. So A is better than an offender on the natural module. This shows for the natural module W that $|W : C_W(A)|q^2 \leq |A|$. This implies that $|A| \geq q^5$ and so A stabilizes a 3–dimensional subspace in W and acts cubic on the spinmodule W_1. This shows $W_1 \not\leq O_2(L)$, so we have $Y_M = [W_1, A](Y_M \cap Y_M^g)$ and then A is trivial on Y_M/W_1, a contradiction.

Assume next (xix). Then we get $|A| = q^3$ and A acts quadratically on the natural submodule W and cubic but not quadratically on $[Y_M, K]/W$. Hence $W \leq O_2(L)$ and $[A, W] \leq Y_M \cap Y_M^g$. This shows that there is no p-element in M centralizing W, and then p divides $q - 1$. But $[W, A]$ is normalized by $L_2(q) \times Z_{q-1}$ in K, a contradiction.

Suppose we are in (xx). Then we have $p = 3 \in \sigma(M)$. Let $W = V_1 \oplus V_2$, where the V_i are the two half spin modules. Then we see that $C_G(x) \leq M$ for any $x \in V_i^\sharp$, $i = 1, 2$. As $L \not\leq M$, we get that $A \cap K$ acts quadratically. This shows that A has to interchange the two modules. Now we see that $|W : C_W(A \cap K)| \leq 2|W : W \cap O_2(L)|$. This shows $|W : W \cap O_2(L)| \geq 8$. Then we see that $|W : C_W(A)| \geq 2^{10}$ and so $|A| \geq 2^6$. Hence $|A \cap K| \geq 2^5$, which would give $|W : W \cap O_2(L)| \geq 2^5$. But then we see that A would act quadratically, a contradiction.

In (xxi) and (xxii) any F–module offender acts quadratically, a contradiction.

Assume next (xxiii). Then we have four modules involved. In particular A is better than an F–module offender. If A is a transvection group, we get that A acts quadratically, a contradiction. So we have that $K \cong L_5(2)$ and we may assume that $|W : C_W(A)| = 4$, for the natural module W. Now A

acts quadratically on W. As this cannot be the case for the dual module too, we see that $|Y_M : C_{Y_M}(A)| = 2^{10}$ and then $|A| = 2^6$. But then A acts quadratically on the natural module and the dual one as well, a contradiction.

So we are left with (xxiv) and (xxv). Let $n > 3$. Then $[K, S] \leq K$, as $3 \notin \sigma(M)$. In all cases there is some good E centralizing K. Hence this group cannot normalize A. This shows that $[A, C_{Y_M}(K)] \neq 1$. In particular, we do not have (xxiv), as in that case $C_{Y_M}(K)$ is centralized by a good E and so there are elements in $Z(S)$ centralized by a good E, contradicting 8.6.

Assume that A acts on another component K_1, then this is a $3'$–group and so isomorphic to $Sz(r)$. As there are at most 8 nontrivial modules in $[Y_M, K]$, we get that $[Y_M, K, K_1] = 1$. As A does not act quadratically on $[Y_M, K]$, we see that $|[Y_M, K] : C_{[Y_M,K]}(A)| \geq 64$ and so $|Y_M : C_{Y_M}(A)| \geq 2^{12}$. But $|A| \leq 2^6$, a contradiction. So we have that A acts nontrivially on $F(M/C_M)$. In particular it acts nontrivially on a Sylow p–subgroup. Hence A inverts some p – element ν which acts nontrivially on $[Y_M, K]$. As $p \geq 5$, we get that $|[Y_M, K] : C_{[Y_M,K]}(A)| > 2^{2n}$. This shows $n = 5$ and $|A| = 2^6$. But then A acts quadratically, a contradiction.

Let finally $K \cong L_3(2)$. Then $[Y_M, K]$ is a sum of three natural submodules and so A has to induce a transvection group. In particular A acts quadratically, a contradiction.

So we are left with the case that A acts faithfully on some Sylow t–subgroup of $F(M/C_M)$. Let $|A| = 2^s$. By 2.1 we have a subgroup $D_1 \times \cdots \times D_s$ of dihedral groups with A as a Sylow 2-subgroup. Set $D_i = \langle \nu_i, a_i \rangle$, $i = 1, \ldots, s$, with $a_i \in A_i$ and $o(\nu_i) = t$. As we do not have quadratic action we may assume that $[Y_M, a_1, a_2]$ is nontrivial. Now $[Y_M, a_1, a_2] \leq Y_M \cap Y_M^g$ and so not normalized by a good E. This shows $s \leq 3$.

We may assume for the moment that $O_t(D_1 \times \cdots \times D_s)$ acts faithfully, as there is a similar group in M/C_M. Now a_2 induces a transvection on $[Y_M, a_1]$. This gives that either $t = 3$ or $[Y_M, \nu_1, \nu_2] = 1$. So assume that $t > 3$, then we see that $|[Y_M, O_t(D_1 \times \cdots \times D_s)] : C_{[Y_M, O_t(D_1 \times \cdots \times D_s)]}(A)| \geq 2^{2s}$, a contradiction to $|Y_M : C_{Y_M}(A)| \leq 2^{s+1}$ and $s > 1$. So we have $t = 3$.

We first show $3 \in \sigma(M)$. Assume otherwise. If $|A| = 8$, we get that $e(G) > 3$ and so by 2.3 there is some elementary abelian subgroup F of order p^3, $p \in \sigma(M)$, which centralizes $O_3(D_1 \times D_2 \times D_3)$. Let $|A| = 4$. Then we may assume $e(G) = 3$, otherwise we have the same as before. Now we get with 2.3 again that there is some elementary abelian subgroup F of order p^3, which centralizes $O_3(D_1 \times D_2)$. As A acts faithfully we may again assume that also

$O_3(D_1 \times D_2)$ acts faithfully. But then we have $|[\nu_i, Y_M]| \leq 2^6$. As we may assume that some p–element acts nontrivially on $[\nu_1, Y_M]$, we see that either we have $|[\nu_1, Y_M]| = 2^4$, or 2^6. Now F acts on a 2–dimensional or 3–dimensional module over $GF(4)$. As there is no elementary abelian p–subgroup of order p^2 in $GL(3,4)$, $p > 3$, we get a good E centralizing $[O_3(D_1 \times D_2), Y_M]$, a contradiction.

So we have $3 \in \sigma(M)$. Let first $|A| = 8$. Set $W = [Y_M, O_3(D_1 \times D_2 \times D_3)]$. Then we have $|W| \leq 2^8$. Assume $|[W, a_1]| = 8$. Then as $Y_M^g \cap [W, a_1] \neq 1$, we may assume that D_2 acts nontrivially on $[W, a_1]$. So $|C_{[W,a_1]}(D_2)| = 2$, which then is also centralized by D_3. But $C_{[W_1,a_1]}(D_2) \leq Y_M^g$, a contradiction. So we have $|[W, a_i]| \leq 4$ for all i. This now gives that $D_1 \times D_2$ induces a 4–dimensional tensorproduct module W_1. As $|Y_M : C_{Y_M}(A)| = 8$, we get $W_1 = [Y_M, O_3(D_1 \times D_2)]$ and that $O_3(D_3)$ centralizes this module. But $W_1 \not\leq O_2(L)$ and so $[A, W_1] \leq W_1$ is of order at least 16, a contradiction to $|W_1| = 16$. So we have $|A| = 4$. Let P be a Sylow 3–subgroup of $F(M/C_M)$. Let C_1 be a critical subgroup in P and $C = \Omega_1(C_1)$ and $D = [A, C]$. Then $m_3(D) \geq 2$. Suppose first $m_3(D) > 2$. We have $|[C, a_1]| = |[C, a_2]| = 9$. This shows that DA induces $\Sigma_3 \times \Sigma_3$ on Y_M and so $m_3(C) \leq 3$. Hence we get C elementary abelian of order 27 and so $C = D$. As S normalizes D, we get that $C_{Y_M}(D) = 1$ by 8.6. We have $|[Y_M, a_1]| = |[Y_M, a_2]| = 4$ and DA induces the orthogonal module. We now see that $|[D, Y_M]| = 16$. As $Y_M = [Y_M, D]$, we see that 3 divides $|C_M|$ and M induces the orthogonal module on Y_M, contradicting 8.9(b).

Let $m_3(D) = 2$, then D is elementary abelian of order 9 or extraspecial of order 27. Suppose that D contains no good E. Then we have $D = C$ and so D is normal in a Sylow 3–subgroup of M, which is of rank at least three, a contradiction. Hence D always has a good E. Let first $D = C$. Then $C_{Y_M}(D) = 1$. We see $|[Y_M, D]| \leq 2^6$. As $Y_M = [Y_M, D]$ we get that $|Y_M| \geq 2^4$. Let $|Y_M| > 2^4$, then $|Y_M| = 2^6$. As some element in A inverts $Z(D)$ if D is extraspecial, we see that $[Z(D), Y_M] = 1$. Hence in both cases an elementary abelian group of order 9 is induced. But then there is some element of order three in D, which has to act nontrivially on $[Y_M, a_1]$, otherwise we get $Y_M = [Y_M, a_1]$. As before we see that $[D, Y_M]$ is of order 16. So in any case we have $|Y_M| = 16$, 3 divides $|C_M|$ and M induces an orthogonal module on Y_M, contradicting 8.9(b). Hence we have $C > D$. We have that $C_{Y_M}(A) = Y_M \cap Y_M^g$. Further $C_{Y_M}(D) \cap Y_M^g = 1$. This shows that $C_{Y_M}(D) = 1$. Now we may argue as before.

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Proposition 8.11 *Let S be a Sylow 2–subgroup of M then $Y_M \leq O_2(C_G(x))$ for any $1 \neq x \in Z(S)$.*

Proof: Assume false. By 7.1 we have that 4.2(1) is not possible. Hence 8.7 is satisfied and we may choose $H = C_G(x)$. By 4.2 we get an offender A on the $2F$–module Y_M. Now 8.10 shows that A acts quadratically on Y_M and so by 4.2 $C_{Y_M}(a) = C_{Y_M}(A)$ for all $a \in A^\sharp$. Further A induces an F–module offender. Suppose first that A normalizes any component. By 3.24 this is the case if $|A| > 2$. Suppose further there is some component K with $[K, A] \neq 1$, such that KA induces an F–module on $[Y_M, K]$. By quadratic action and $C_{Y_M}(A) = C_{Y_M}(a)$ for all $a \in A^\sharp$, we see that A acts faithfully on K. Then by 3.17 we have that $K/Z(K) \cong L_n(q)$, $Sp(2n, q)$, A_6 or A_7, or $|A| = 2$.

Let us assume $|A| \geq 4$. Let $K \cong A_7$ or $3A_6$, then $[Y_M, K]$ is the four dimensional module or 6–dimensional module. In both cases we have $|[Y_M, K] : C_{[Y_M,K]}(A)| = |A|$ and any element in $[Y_M, K]$ is centralized by a good E. This shows that there is some element in $C_{Y_M}(S)^\sharp$ which is centralized by a good E contradicting 8.6.

Let next $K/Z(K) \cong L_n(q)$. Then $[Y_M, K]$ just involves natural or dual modules. Let first $q = 2$ and $3 \notin \sigma(M)$. Then we have that $n \leq 7$. Suppose $n > 3$, then $[K, S] \leq K$. Let $n = 6, 7$, then $e(G) > 3$ and so K is centralized by a good E. As we have at most $n-1$ natural modules involved and $[Y_M, K]$ cannot be centralized by a good E, we see $p = 7$. But as 7 divides the order of $C_K(C_{Y_M}(S \cap K))$, we get some element in $C_{Y_M}(S)^\sharp$ centralized by a good E, a contradiction. Let next $n = 5$. Then we have at most 4 modules and so they all have to be of the same type. But then $p = 5$ or 7 and so $C_{Y_M}(S)^\sharp$ again contains elements centralized by a good E. Let finally $n = 4$. Then we have three modules and $p = 7$. But as 7 divides the order of $L_3(2)$, we get hat some element in $C_{Y_M}(S)^\sharp$ is centralized by a good E.

So assume now that $3 \in \sigma(M)$. As $L_4(2)$ contains a good E we have that for $n > 5$ elements in $C_{[Y_M,K]}(S \cap K)^\sharp$ are centralized by a good E, so we get that elements in $C_{Y_M}(S)^\sharp$ are centralized by a good E, a contradiction. So we have $n = 4$ or 5. If $[S, K] \not\leq K$, then $C_{\langle K^S \rangle}(C_{Y_M}(S \cap \langle K^S \rangle))$ involves $\Sigma_3 \times \Sigma_3$, a contradiction. So we have $[K, S] \leq K$. If $n = 5$ we now must have natural and dual modules be involved. But as there is a 3–element centralizing K, we get two natural and two dual modules. But this is not an F–module. So let $n = 4$. Then we get at most three natural modules in $[Y_M, K]$. Further we see that $C_{Y_M}(K) = 1$. Hence we have that $M/C_M \cong L_4(2) \times L_3(2)$ or $L_4(2) \times \Sigma_3$. In the first case $|Y_M| = 2^{12}$ and we see that $C_M(Z(S))$ contains some good E. So we have the second case. Then we have $|Y_M| = 2^8$. Let $|A| = 8$ and V be one of the modules in Y_M with $V \not\leq O_2(L)$. Let $v \in V \setminus O_2(L)$. Then $|[v, A](Y_M^g \cap V)/(Y_M^g \cap V)| = 8$. This shows $Y_M^g \cap V = 1$. But all elements in Y_M, which are not in one of these three $L_4(2)$–modules are centralized by a good E. Hence there are such elements in $Y_M \cap Y_M^g$, a contradiction. So let $|A| = 4$. Then $|Y_M \cap Y_M^g| = 16$. But in this case we have $[V, a](Y_M^g \cap V) = V$, and so again there are elements in $Y_M \cap Y_M^g$ which are not in those three modules. But all these elements are centralized by some elementary abelian group of order 9, a contradiction.

So we are left with $K \cong L_3(2)$. Now we have at most two natural modules involved, which are of the same type. If $3 \in \sigma(M)$ we see that S has to normalize K, otherwise there are elements in $Z(S)$ centralized by $\Sigma_3 \times \Sigma_3$ in $\langle K^S \rangle$. But now there is some elementary abelian group of order 9 centralizing K, which gives a 3–element centralizing $[Y_M, K]$ and so there is some element in $C_{[Y_M,K]}(S)^\sharp$ centralized by a good E. So $3 \notin \sigma(M)$. Then a good E centralizes $[Y_M, K]$ and so $[K, S] \not\leq K$. Now we see that $p = 7$, $K^S \cong L_3(2) \times L_3(2)$ and $|\Omega_1(Z(S))| = 2$. This shows $[Y_M, K]$ is the natural module and $|A| = 4$. But also $|Y_M : C_{Y_M}(A)| = 4$. This shows that $L/O_2(L) \cong L_2(4)$ and so $A \leq O_2(C_G(\Omega_1(Z(S)))C(Y_M)$, since $O_2(L)/Y_M \cap Y_M^g$ is an irreducible module. But $O_2(C_{\langle K^S \rangle}(\Omega_1(Z(S))$ does not contain A, as A is the transvection group to a hyperplane and not to a point.

Assume now $q > 2$. Then we see $n \leq 4$. Let $n = 4$. We see that p does not divide $q - 1$. In particular $e(G) > 3$, further $[K, S] \leq K$. We have at most three natural modules. Let first $m_p(K) \leq 1$ and p not be a divisor of $q^3 - 1$. Then we have some good E which centralizes $[Y_M, K]$, a contradiction. Let next p be a dividsor of $q^3 - 1$. Then there is some p–element in K centralizing $C_W(S \cap K)$, where W is some natural module. As $e(G) > 3$, there is some good E centralizing K and so there is some p–element centralizing $[Y_M, K]$. Hence there are elements in $Z(S)^\sharp$ which are centralized by some good E, or S induces a graph automorphism on K. Then $[Y_M, K] = W \oplus W^*$ and then $[Y_M, K]$ is centralized by a good E. So we may assume that p divides $q + 1$. Again $C_{[Y_M,K]}(S \cap K)$ is centralized by some p–element. Now we may assume that $C_M(K)$ contains no good E. Then we have some p-element, which induces a field automorphism on K, hence normalizes S, which shows that $C_{[Y_M,K]}(S)$ is normalized by some good E, a contradiction to 8.6.

Let next $K \cong SL_3(q)$. Then there are at most two natural modules involved. Suppose first that $[K, S] \leq K$. If we have two modules, then $|A| = q^2 = |Y_M : C_{Y_M}(A)|$. But as A induces transvections to a hyperplane, we get that $A \not\leq O_2(C_G(\Omega_1(Z(S))))$ a contradiction as in the case of $L_3(2)$. So we have that $[Y_M, K]$ involves just one natural module. By 3.36 $[Y_M, K]$ is the natural module. But then we see that there is $1 \neq x \in \Omega_1(Z(S))$ centralized by a good E. So we have that $[K, S] \not\leq K$ and then p divides $q - 1$. Now there is a conjugate of K which centralizes $[Y_M, K]$. So $C_{Y_M}(S)$ contains $L_2(q) \times L_2(q)$, contradicting 8.6.

Let now $K \cong L_2(q)$, then we see that $|[Y_M, K] : C_{[Y_M,K]}(A)| = |A|$. Let $[K, S] \leq K$. If $C_{[Y_M,K]}(K) = 1$, we see that any element in $[Y_M, K]$ is centralized by a good E, a contradiction. So let $C_{[Y_M,K]}(K) \neq 1$. If p does not divide $q-1$, we get the same contradiction. If p divides $q-1$ and there is some good E in $C(K)$, we see that $C_{[Y_M,K]}(K)$ is centralized by a good E, again a contradiction. So we have $e(G) = 3$ and some p–element induces a field automorphism on K. Hence it normalizes S. So it normalizes $C_{Y_M}(S)$. As this group is centralized by some p–element in K, we see that it is normalized by some good E, a contradiction. So let $[K, S] \not\leq K$. Then, as $[Y_M, K]$ is irreducible, we see that $K^S = K_1 \times \cdots \times K_n$ and $[Y_M, K^S] = V_1 \oplus \cdots \oplus V_n$, where $[V_i, K_j] = 1$ for $i \neq j$. Let $K = K_1$, then we see that V_2 contains elements from $Y_M \cap Y_M^g$. Hence V_2 is not centralized by a good E, which gives $n = 2$ in the first place. Now there are two M–orbits of elements in $[Y_M, K_1 \times K_2]$ one of length $2(q^2-1)$ and the rest. Let $v \in Y_M^g \cap V_2$ and P be a Sylow p–subgroup of M. Then $P = C_P(v)(P \cap K_2)$. As $m_p(P) \geq 3$ we get with 2.5 and 5.11 that $C_P(v)$ contains a good E, a contradiction.

Let next $K \cong Sp(2n, q)$. Then we have natural modules involved. Hence we get $n \leq 3$. As $C_{Y_M}(A) = C_{Y_M}(a)$ for all $a \in A^\sharp$, we see that just one module is involved. Let first $K \cong Sp(6, q)$. Then we see that p cannot divide $q^2 - 1$. So we have that $m_p(K) \leq 1$. As $e(G) > 3$, we have that a good E centralizes K and so also $[Y_M, K]$, a contradiction. So we have $K \cong Sp(4, q)$. Now $|[Y_M, K] : C_{[Y_M,K]}(A)| = |A|$. If $[K, S] \not\leq K$, then there is $p \in \sigma(M)$ with p divides $q + 1$. But then there is $1 \neq x \in Z(S)$, which is centralized by a good E. So 8.6 shows $[K, S] \leq K$. This now shows p divides $q-1$ and $e(G) = 3$. Further $C_{[Y_M,K]}(K) \neq 1$. But $C_{[Y_M,K]}(K)$ is centralized by a good E in K and so there is some element in $Z(S)$, which is centralized by a good E, a contradiction to 8.6.

Let now $|A| = 2$. Then $K \cong L_n(2)$, $Sp(2n, 2)$, $\Omega^\pm(2n, 2)$ or A_n. Further we have that Y_M^g centralizes a subgroup of index 4 in Y_M. As $Y_M^g \not\leq M$, we see that $p = 3$ and $e(G) = 3$. This gives $K \cong L_n(2)$, $n \leq 7$, $Sp(2n, 2)$, $n \leq 3$, $\Omega^+(6, 2)$, $\Omega^-(2n, 2)$, $n \leq 4$ or A_n, $n \leq 11$. In any case $[Y_M, K]/C_{[Y_M,K]}(K)$ is the natural module. Now by 3.42 any element in this module is centralized by a good E or we have $K \cong \Omega^-(6, 2)$. Let $[K, S] \leq K$. Then with 8.6 we have $K \cong \Omega^-(6, 2)$ and $[Y_M, K]$ is the natural module. Then $Y_M \cap Y_M^g$ just contains singular vectors and so $|Y_M \cap Y_M^g| \leq 4$. But we have that $|Y_M : Y_M^g \cap Y_M| = 4$, a contradiction. So we have $[K, S] \not\leq K$. This shows $K^S \cong L_3(2) \times L_3(2)$ or $A_5 \times A_5$. But then in any case some $1 \neq x \in \Omega_1(Z(S))$ is centralized by an elementary abelian subgroup of order 9 in K^S, contradicting 8.6.

So we may assume that A acts faithfully on some Sylow t–subgroup of $F(M/C_M)$. As we have an F–module, we get $t = 3$. As $C_{Y_M}(A) = C_{Y_M}(a)$ for all $a \in A^\sharp$, we get that $|A| = 2$. By 5.9 we have that M is not exceptional. Again we see that $3 \in \sigma(M)$ and $e(G) = 3$. Let P be a 3-group of $M/O_2(M)$, with $S = (S \cap C_M)N_S(P)$ such that PC_M/C_M is a Sylow 3–

subgroup of $F(M/C_M)$. Let C_1 be a critical subgroup of P and $C = \Omega_1(C_1)$. We see that a subgroup of index 3 in C centralizes $[A, Y_M]$. Suppose C to be cyclic. Then $[C, Y_M]$ is of order 4 and normal in M, a contradiction with 8.6 again.

So we have that C is not cyclic. Suppose that $[A, Y_M]$ is centralized by S. Then we get that $m_3(C) \leq 2$. Then C is elementary abelian of order 9 or extraspecial of order 27. In the first case C is centralized by an elementary abelian group of order 27. As C contains some element ν with $|[\nu, Y_M]| = 4$, we see again that $[Y_M, A]$ is centralized by a good E. So we have C extraspecial and $Z(C) \leq C_M$. If $[C, A]$ is contained in some elementary abelian subgroup of order 27, we may argue as before. As $C \cap C_M = Z(C)$, we see that $C_C(A) \not\leq Z(C)$. As some element of order three acts nontrivially on $C/Z(C)$ we see that a preimage of $C_C(A)$ is a good E. But this group centralizes $[Y_M, A]$, a contradiction.

So we have that $[[Y_M, A], S] \neq 1$. Let $s \in S$ with $[C, A]^s \neq [C, A]$. Then we have that $|[Y_M, \langle [C, A], [C, A]^s \rangle]| = 16$. We have that $|Y_M : Y_M \cap Y_M^g| \leq 4$. Suppose first that C is elementary abelian. Then we have that any subgroup in C is good and so $|C_{Y_M}(\langle [C, A], [C, A]^s \rangle)| \leq 2$ as $C_{Y_M^g}(\langle [C, A], [C, A]^s \rangle) = 1$, which gives $|Y_M| \leq 2^5$. By 8.6 we get that $|Y_M| = 16$ and so $M/C_M \cong O^+(4, 2)$. Let C be extraspecial, then as above we see that there is some good E in $\langle [C, A], [C, A]^s \rangle$ and again we have that $M/C_M \cong O^+(4, 2)$. As C_M was cyclic, we now see that C is elementary abelian of order 27. In particular all groups of order 9 are good.

Now we have that $|\Omega_1(Z(S))| = 2$. Set $H = C_G(\Omega_1(Z(S)))$. Let $L \leq H$. We have that $O_2(H)$ normalizes $Y_M \cap O_2(L)$ and then also $Y_M^g \cap O_2(L)$, i.e. $O_2(H)$ normalizes A. So we see that $|O_2(H)/C_M \cap O_2(H)| = 4$ and is generated by elements inducing transvections on Y_M. Hence in contrary we see that Y_M is generated by elements inducing transvections on $O_2(H)$. Further we see with 8.6 that $m_3(H) = 1$. This now shows that $\langle Y_M^H \rangle O_2(H)/O_2(H) \cong L_3(2)$, Σ_5 or Σ_3. Further we know that 3 divides the order of $M \cap H$. Let U be a Sylow 3–subgroup of $H \cap M$, then $N_H(U) \leq M$. As $H \cap M$ contains a Sylow 2–subgroup of H, we get a contradiction. □

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Proposition 8.12 *We have* $Y_M \leq O_2(C_G(x))$ *for all* $1 \neq x \in Y_M$.

Proof: By 8.11 we have that the assertion is true for $x \in Z(S)^\sharp$. We may assume that there is some $1 \neq x \in Y_M$ with $Y_M \not\leq O_2(C_G(x))$. Hence there is some L with $C_G(O_2(L)) \leq O_2(L)$ and with $Y_M \not\leq O_2(L)$. We may even assume that $C_S(Y_M) \leq L$. Now we choose L with $|L \cap M|_2$ maximal and then L minimal. Let T be a Sylow 2-subgroup of $L \cap M$. We first show that T is a Sylow 2–subgroup of L. By 7.2 we have that $C_S(Y_M)$ is weakly closed in T with respect to G. Hence $N_L(T) \leq N_G(C_S(Y_M)) \leq M$, as $Y_M = \Omega_1(Z(C_S(Y_M)))$, the assertion.

By the minimal choice of L we have that $L/O_2(L)$ is a minimal parabolic with respect to T. As Y_M is normal in T, Y_M acts quadratically on $O_2(L)$. Hence it normalizes any component by 3.24. Let now P be a proper parabolic of L containing T. The minimal choice of L gives that $Y_M \leq O_2(P)$. By 7.1 we get that $A = \langle Y_M^P \rangle$ is abelian. Now $[Y_L, A] \leq A$ and so A acts quadratically on Y_L.

We next show that $Y_L \neq \Omega_1(Z(T))$. Otherwise $N_G(\Omega_1(Z(T))) \geq L$. By the maximality of $|L \cap M|_2$ we now see that $T = S$ is a Sylow 2–subgroup of M. But we have that $Y_M \leq O_2(N_G(\Omega_1(Z(S))))$ by 8.11, a contradiction.

Let $y \in Y_L$ with $|Y_M : C_{Y_M}(y)| = 2$. By 5.9 we have that M is not exceptional. Let K be a component of M/C_M with $[K, y] \neq 1$. Then we may apply 3.42. Let $[K, S] \leq K$ and $W = [K, Y_M]$. If any element in W is centralized by a good E, we also get that some element in $C_W(S)^\sharp$ is centralized by a good E, contradicting 8.6. So we have 3.42(4). Then we have that $K \cong \Omega^-(6, 2)$ and $p = 3$. As no element in $\Omega_1(Z(S))^\sharp$ is centralized by some good E, we have that $Y_M = [K, Y_M]$ is the natural module. But then we have two conjugacy classes in Y_M, one is centralized by Σ_6 and the other are the 2–central ones. So we have that $Y_M \leq O_2(C_G(t))$ for all $t \in Y_M^\sharp$.

Assume now $[K, S] \not\leq K$. Set $R = S \cap K$. Then we have that $C_{[Y_M,K]}(R)$ does not contain involutions centralized by some p–element in K for $p \in \sigma(M)$. This shows that there are exactly two conjugates of K and that all Sylow r-groups, r odd, of K are cyclic or of type r^{1+2}, where in the latter r divides the order of $Z(K)$, if r divides the order of $C_K(C_{[Y_M,K]}(R))$. Hence $K \cong L_3(2)$ or A_5, $3A_6$ or $3A_7$. In all cases $p \neq 3$. If 3 divides the order of $Z(K)$, we see that $m_p(M) \geq 4$. But then there is a good E which centralizes $[Y_M, \langle K^S \rangle]$, a contradiction. So we have one of the first two cases. Finally we have that $Y_M = [Y_M, \langle K^S \rangle]$ is of order 2^6, 2^8, respectively. In the first case elements of Y_M are either 2-central or centralized by a good E, so we are done. In the second case we see that also Y_M induces a transvections on Y_L. Further some elements in $[Y_M, Y_L]$ are centralized by a good E. As $[y, O_2(L)] = 1$, we have that there is $x \in Y_M$ with $[y, x] \neq 1$ and $|O_2(L) : C_{O_2(L)}(x)| \leq 4$. So we have that

> $(*)$ Let $y \in Y_L$ with $|Y_M : C_{Y_M}(y)| = 2$. Let K be some component of M/C_M with $[y, K] \neq 1$, then $[K, S] \not\leq K$ and $K \cong A_5$.

Assume now that Y_L acts nontrivially on some Sylow r–subgroup R of $F(M/C_M)$. Then of course $r = 3$. And then again Y_M also induces transvections on Y_L by 4.5. Let $W = [Y_M, R]$. Let $3 \notin \sigma(M)$. Then R is centralized by some good E. Let $K = [y, R]$, then $|[Y_M, K]| = 4$, so it is centralized by E. But then also $[Y_M, \langle K^S\rangle]$ is centralized by E, and so there is some nontrivial element in $\Omega_1(Z(S))$, which is centralized by some good E, a contradiction. Hence we have that $3 \in \sigma(M)$. Let first R be cyclic, then $[Y_M, R]$ is centralized by some good E, a contradiction. Hence $m_3(R) \geq 2$ and so $C_{Y_M}(R) = 1$, i.e. $Y_M = [Y_M, R]$. We have $[K, S] \not\leq K$, so there are two conjugates of K, which centralize a subgroup of index 16 in Y_M. Further $[K, Y_M]$ centralizes a subgroup of index two in $O_2(L)$.

So we have

$(**)$ *Let* $y \in Y_L$ *with* $|Y_M : C_{Y_M}(y)| = 2$. *Then* Y_M *also induces transvections on* Y_L, $[y, Y_M]$ *is centralized by some good* E *and there is a subgroup of index 16 in* Y_M *which is centralized by some good* E.

Let K be a component of $L/O_2(L)$, set $Y_M = \tilde{Y} \times C_{Y_M}(K)$. Assume that either $|\tilde{Y}| = 2$ or we have that L is solvable. In the latter we have transvections of Y_L on Y_M by 4.5 and so vice versa transvections on Y_L, which then shows that $L/O_2(L)$ is a dihedral $\{2,3\}$–group. If L is nonsovable, we get that there is some $y \in Y_L$, which induces a transvection on Y_M and so as by $(**)$ $[Y_M, y]$ is centralized by a good E, we have that there is just one component, which is $L_n(2)$, $Sp(2n, 2)$, $\Omega^{\pm}(2n, 2)$, or A_n. Now as $Y_M O_2(L)/O_2(L) \leq Z(T/O_2(L))$, we see that $K \not\cong \Omega^{\pm}(2n, 2)$. Further in all cases we now have that $[Y_M, Y_L]$ is of order two and so is centralized by a good E in M. So we get that $C_K([Y_M, Y_L])$ has to centralize $Y_M O_2(L)/O_2(L))$, as this group is in M. This shows that also $L_n(2)$ is not possible. Further we have that $O_2(M)O_2(L)/O_2(L)$ is elementary abelian.

Now for $x \in Y_M \setminus O_2(L)$, we may assume that $|O_2(L) : C_{O_2(L)}(x)| \leq 4$. Let $x^g \in L$, with $x^g \notin M$. Then we have that $|Y_M \cap O_2(L) : C_{Y_M \cap O_2(L)}(x^g)| \leq 4$ and so $|Y_M : C_{Y_M}(x^g)| \leq 8$. This first shows that we do not have a component A_5 in M/C_M and by $(**)$ there is a subgroup of index 16 in Y_M which is centralized by a good E. So we get that $|Y_M| \leq 2^6$. If we have $|Y_M| > 16$, then the structure of $GL(6, 2)$ yields some element in $\Omega_1(C_{Y_M}(S))^\sharp$, which is centralized by some good E (recall that M is not exceptional). So we have $|Y_M| = 16$. Hence $M/C_M \cong O^+(4, 2)$. Further Sylow 3–subgroups of C_M are cyclic. Now as $C_M \neq O_2(M)$, we see that $\Phi(O_2(M)) \neq 1$. Hence $Y_M \leq \Phi(O_2(M))$. But in all possible cases for L we have that $\Phi(O_2(M)) \leq O_2(L)$, a contradition.

Hence we have shown

$$L \text{ is nonsolvable and } |\tilde{Y}| \geq 4.$$

Let now $W \leq Y_L$ minimal with $1 \neq [K, W] \leq W$.

We first treat the case of $C_W(K) = 1$. As K possesses a quadratic fours group we get with 3.26 that either K is a group of Lie type in characteristic two, A_n, $U_4(3)$ or some sporadic group.

Let $K \cong A_n$. As $\tilde{Y}$ is in $O_2(P)$ for any proper parabolic, we see that $\tilde{Y}$ projects onto $\langle (12)(34), (13)(24) \rangle$. As this group does not act quadratically on the natural module, we have that W is the spin module and so $[x, W] = [\tilde{Y}, W]$ for all $1 \neq x \in \tilde{Y}$. Now we have that Y_M is a strong F-module with offender W and so we get with 3.17 that we have a component R of M/C_M which is $Sp(2n, q)$, $L_n(q)$, $3A_6$ or A_7. Let first $[R, S] \leq R$. Then we may apply 3.42. As no element in $\Omega_1(Z(S))^\sharp$ is centralized by a good E, we have 3.42(4). Now R is either $Sp(4, q)$, $L_2(q)$ or $L_4(2)$. As all commutators are also equal, we cannot have a nonsplit extensions. Hence we have $L_4(2)$ on the sum of two natural modules. But in that case never all commutators are equal. So we have that $[R, S] \not\leq R$. If there are at least 4 conjugates under S, we see that point stabilizers have to be 2-groups, otherwise there is some element in $\Omega_1(Z(S))$ centralized by a good E. Hence we get $R \cong L_2(q)$. If there are exactly two, we must have that for all odd r, which divide the order of the point stabilizer, Sylow r–subgroups are cyclic or of type r^{1+2} where r divides the order of $Z(R)$. This now shows $R \cong L_2(q)$, $SL(3, 4)$, $3A_6$, $3A_7$, or $L_3(2)$. If $Z(R)$ is nontrivial, it is contained in both conjugates, so it has to act trivially on the module, which shows that $SL(3, 4)$ is not possible. In the other two cases we have that $m_p(M) \geq 4$ and so $[Y_M, \langle R^S \rangle]$ is centralized by some good E, a contradiction. So we just have $R \cong L_2(q)$ or $L_3(2)$. But as for a fours group we have equal centralizers and commutators, $R \cong L_3(2)$ is not possible. So we have that $R \cong L_2(q)$. Further $Y_M = [Y_M, \langle R^S \rangle]$ is a direct sum of natural modules for the particular components. As $|\tilde{Y}| = 4$, we get $q = 4$. Further in all cases there is some good E, which normalizes $[W, Y_M]$. Suppose that $3 \in \sigma(M)$. There is always some 3-element ρ in K which normalizes $\langle (12(34), (13)(24) \rangle$, so it also normalizes $[W, Y_M]$, hence is in M. But then also $N_K(\langle \rho \rangle)$ is in M, as any 3–element in M centralizes an elementary abelian group of order 27. But $\langle (12)(34), (13)(24) \rangle$ is not normal in $\langle N_K(\langle \rho \rangle), \langle (12)(34), (13)(24) \rangle \rangle$. So we have that $3 \notin \sigma(M)$. In particular there are just two components $L_2(4)$ and $p = 5 \in \sigma(M)$. This shows that there is a 5–element centralizing Y_M and so all elements are either 2-central or centralized by a good E, a contradiction.

Let next K be sporadic. As $\langle \tilde{Y}^P \rangle$ acts quadratically on W for all proper parabolics, we see with 3.26 that just $K \cong 3M_{22}$ is possible. Further we get that $|\tilde{Y}| = 4$ and W is the 12–dimensional module. Now we see again that we have a strong F-module with $[W, x] = [W, \tilde{Y}]$ for all $1 \neq x \in \tilde{Y}$. As above we see that M/C_M just has components $L_2(4)$. Again we have some 3-element in $Z(K)$ normalizing $\tilde{Y}$ and we can argue as above.

If $K \cong U_4(3)$, then there is always some some parabolic P such that $\langle \tilde{Y}^P \rangle$ is non abelian.

So let finally K be of Lie type in characteristic 2. By 3.28 we have that $\tilde{Y}$ is in a root subgroup. Hence we can embed it into some $L_2(q)$ or $Sz(q)$, which by 3.50, induces a natural module W_1 in W. So as above we see that we just have components $L_2(r)$ in M/C_M. As $|W_1 : C_{W_1}(\tilde{Y}| = q$, we get $r = q$ and so also $|\tilde{Y}| = q$. Hence $\tilde{Y}$ is normalized by some element of order $q - 1$ in K, whose normalizer is not in M. This as above shows that for a uniqueness prime we always have that it has to divided $q + 1$. In particular we just have two conjugates $L_2(q)$ in M and then again as before elements in Y_M either are 2–central or centralized by a good E, a contradiction.

So we have shown

$$C_W(K) \neq 1$$

Assume that T is a Sylow 2–subgroup of G. Then we have shown that L centralizes some element of $Z(T)^\sharp$. As $T \leq M$, we may assume $T = S$, but this contradicts $Y_M \leq O_2(C_G(x))$ for all $1 \neq x \in \Omega_1(Z(S))$. Now let X be either $\Omega_1(Z(T))$ or $J(T)$. Suppose that X is normal in L. As $S > T$, we have that $N_S(X) > T$. But this contradicts $Y_M \not\leq O_2(N_G(X))$ and the choice of L with respect to $|M \cap L|_2$. So we have that neither of the two groups is normal in L in particular Y_L is an F–module for L.

Choose W as before. Then we have that W is a nonsplit extension of a trivial module by some irreducible module. As now $\tilde{Y}$ cannot act as $\langle (12)(34), (13)(24) \rangle$ on the natural module, we see with 3.16 that K has to be a Lie group in characteristic two. Then as before we have that $\tilde{Y}$ is in some root group. Application of 3.36 shows that $K \cong L_2(q)$, $Sp(2n, q)$ or $G_2(q)$. Further we still have that $\tilde{Y}$ is in some $L_1 \cong L_2(q)$ which now induces a module U such that $U/C_U(L_1)$ is the natural module. Again we get a strong F − module, besides now commutators might be different. Hence again we have components R in M/C_M and so we may argue as before besides that we now may have that $R \cong L_3(2)$, $Sp(4, r)$, $L_2(r)$ or $R \cong L_4(2)$ and we have 3.42(4)(vi). Further for $R \cong L_2(r)$ or $Sp(4, r)$ and $[R, S] \leq R$, we also may have the nonsplit extension of the natural module.

Let first $R \cong L_3(2)$, then as $q > 2$, we get $q = 4$ and $[R, Y_M]$ is a direct sum of two natural modules. We have that there are exactly two conjugates of R under S. As $q = 4$, we see that there is some element ρ of order three in K, which normalizes projection of $\tilde{Y}$ and so also $[\tilde{Y}, U]$, which is normalized by a good E in M. If $3 \in \sigma(M)$, then $N_G(\langle \rho \rangle) \leq M$, but $N_K(\langle \rho \rangle)$ does not normalize the projection of $\tilde{Y}$, so we have that $3 \notin \sigma(M)$. Hence $7 \in \sigma(M)$

and so $[U, Y_M]$ is centralized by a good E. But $C_U(L_1) \cap C_W(K) \neq 1$, which implies $L \leq M$, a contradiction.

Let next $R \cong L_2(r)$ and $[R, S] \leq R$. Then $C_{[Y_M,R]}(R) \neq 1$. Further we see that $|\tilde{Y}| = q = r$. We have that $[Y_M, U]$ is normalized by a good E. Hence as before we see that there are no primes in $\sigma(M)$ dividing $q-1$. But then we have that $C_{[Y_M,R]}(R)$ is centralized by some good E, a contradiction as $C_{[Y_M,R]}(R)$ contains elements from $Z(S)^\sharp$.

Let next $R \cong Sp(4, r)$. Then $[R, S] \leq R$. By 3.42(4)(iii) we have a uniqueness prime, which divides $r-1$. But then $C_{[Y_M,R]}(R)$ is centralized by a good E, a contradiction as before.

So we are left with $R \cong L_2(r)$, $[R, S] \not\leq R$. As R just induces the natural module, we see $r = q$. Again there are no uniqueness primes dividing $q-1$. Hence we just have two conjugates. Now $C_{Y_M}(\langle R^S \rangle) = 1$, so we have that Y_M is the direct sum of two natural modules, each for any component. As in any module we have all elements conjugate, we have that the element from each module are centralized by a good E. Further all other elements in Y_M are 2-central. So $Y_M \leq O_2(C_G(x))$ for all $1 \neq x \in Y_M$, a contradiction. This proves the lemma. □

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Proposition 8.13 *We have* $C_G(x) \leq M$ *for all* $1 \neq x \in \Omega_1(Z(S))$.

Proof: Otherwise we may choose H as before. Then by 8.5 we have $Y_H = \Omega_1(Z(S))$. Further we have that $b > 1$ by 8.11. Assume first that $[Y_M, [O_2(H), O^2(H)]] = 1$. Set $V_H = \langle Y_M^H \rangle$ and $W_H = C_{V_H}(O_2(H))$. Then $W_H \not\leq Y_H$. Let $C_W = C_H(W_H)$. Then we have that $O_2(H/C_W) \neq 1$. Let $T_1 \leq S$ such that $T_1C_W/C_W = O_2(H/C_W)$. As $W \leq Z(O_2(H))$ we see that $H \neq N_H(T_1)$. So $H = C_WS$. But then $O^2(H) \leq C_W$ and so the $P \times Q$–lema shows $[V, O^2(H)] = 1$, which gives $[Y_M, O^2(H)] = 1$ and then $H \leq M$, a contradiction. Hence we have that $[Y_M, [O_2(H), O^2(H))] \neq 1$.

Now we may apply 3.10 - 3.14 to the pair (M_0, H). Recall that by 7.1 3.10(3) does not occur. These provide us with Y_M being a strong, or strong dual F–module. Hence by 5.9 we have that M is not exceptional.

Suppose that we do not have 3.11. Assume further that that $V_{\alpha'} \leq M$. Suppose now that for any $\delta \in \Delta(\alpha')$ we have that $Y_M \leq M_\delta$. But then there is also some δ with $1 \neq [Y_M, Y_\delta] \leq Y_M \cap Y_\delta$, which contradicts 7.1. Hence we have

(∗) If $V_{\alpha'} \leq M$, then there is some $\delta \in \Delta(\alpha')$ with $Y_M \not\leq M_\delta$.

Let A be the offender and K be a component of M/C_M on which A acts nontrivially. Suppose first that K is normalized by S, then there is no submodule where every element is centralized by a good E, since otherwise this also applies for some nontrivial element in $Z(S)$ and so $H \leq M$. Hence we are in the situation of 3.42(4). If we have 3.42(4)(iv) or (v) then there are elements in $Z(S)$ which are centralized by a good E, a contradiction. As $[Y_M, K]$ has to be a strong or strong dual F–module, we see with 3.17, 3.22 that $K \cong L_2(q)$, $Sp(4,q)$, $\Omega^-(6,2)$, or $L_4(2)$.

Let first $K \cong L_4(2)$. Then we have two natural modules and so, as there are two modules we cannot have 3.11 or 3.14. So we have a strong module with offender a transvection group to a hyperplane, and we are in the situation of 3.13. Let now W_1 one of the two natural modules, then $[A, W_1]$ is of order 4 and so normalized by a good E. But then we get $V_{\alpha'} \leq M$.Hence there is also some $V_2^g \leq M$, where $V_2^g \cap O_2(H)$ induces A. But $H = \langle H \cap M, V_2^g \rangle$, a contradiction.

Let $K \cong \Omega^-(6,2)$ and $[Y_M, K]$ be the natural module. Now the offender A is of order two. If we have 3.11, then the offender is normal in $S/O_2(M_0)$. Hence we do not have 3.11. So we have 3.13 or 3.14. In both cases we can assume that $V_{\alpha'} \leq M$. As no element in $Z(S)^\sharp$ is centralized by a good E, we see that $M/C_M \cong O^-(6,2)$ and $V_{\alpha'}C_M/C_M$ is contained in some $Z_2 \times \Sigma_6$. Hence any element in $[Y_M, K]$ centralizes a subgroup of index eight in any $Y_\delta \leq V_{\alpha'}$. But any such subgroup contains some element which is centralized by a good E, so $Y_M \leq M_\delta$, contradicting $(*)$.

Let finally $K \cong L_2(q)$ or $Sp(4,q)$. Then $C_{[Y_M,K]}(K) \neq 1$. Hence this group is not centralized by a good E. This shows $K \cong L_2(q)$. But also in that case we get a contradiction with 3.42(4)(i) as either some p–element centralizes $[Y_M, K]$ and so $C_{[Y_M,K]}(K)$ is centralized by some good E, or there is a field automorphism of order p, which then normalizes S and so also $C_{C_{[Y_M,K]}(K)}(S)$.

So we have that K is not normalized by S. Then A normalizes K. We first see that if r is an odd prime which divides the order of the point stabilizer in K, then Sylow r–subgroups of K are cyclic or of type r^{1+2}, where in the latter r divides the order of the center of all the conjugates. So if $C_{[Y_M,K]}(K) \neq 1$, we get that all Sylow r–subgroups, r odd, of K are cyclic. So by 3.16 $K \cong L_2(q)$, $L_3(2)$, or A_5 on the permutation module. If $C_{[Y_M,K]}(K) = 1$ then $[V, K]$ is a strong F–module or dual F–module and so with 3.17, 3.22 we get the additional possibilities $K \cong SL(3,4)$, $3A_6$ or $3A_7$. In the last three cases, we have $3 \notin \sigma(M)$ and so $m_p(M) \geq 4$ for $p \in \sigma(M)$. Hence $[Y_M, \langle K^S \rangle]$ is centralized by a good E, a contradiction.

Let K be one of $L_3(2)$ or A_5, then we have exactly two conjugates. Further we have that $3 \notin \sigma(M)$. In particular $[A, Y_M]$ is centralized by some good E. If we have 3.11, then A is normalized by S. This shows that we have $K \cong L_3(2)$, as in the A_5-case an offender is a transvection, which is not normal in $S/O_2(M)$. But now A is a dual offender and normal in $S/O_2(M)$, which is not possible, as then A has to intersect K nontrivially and so $[A, Y_M]$ has to be contained in $[K, Y_M]$, which is not normalized by S. Hence we have 3.13 or 3.14. As above we get that $V_{\alpha'} \leq M$. Then Y_M is generated by elements centralizing a subgroup of index four in Y_δ, with notation as above. But then again $Y_M \leq M_\delta$, contradicting $(*)$.

So we are left with $K \cong L_2(q)$. Now we have that in $[V, K]$ we just have the natural module over some trivial module. Now first we see that we do not have 3.12, as A cannot be normal in S. Assume that we have at least four conjugates of K, we see that $[A, Y_M]$ is centralized by some good E. Now as before we see that Y_M centralizes in Y_δ a subgroup of index q. As there is a subgroup of order q^2 centralized by a good E in M_δ, we have that Y_M acts on Y_δ, a contradiction as before. So we have exactly two conjugates of K. As $C_{Y_M}(\langle K^S \rangle) = 1$, we see that Y_M is a direct sum of two natural modules one for each component. Let W_1 be the module for the first component, then we may assume that $[A, Y_M] = [W_1, A]$ and this group is normalized by some good E by 3.42. So we just can have the situation of 3.13 or 3.14. Now as all elements in W_1 are conjugate under K, we have that every one is centralized by a good E. Now again Y_M is generated by elements which centralize a subgroup of index q in Y_δ. But each such subgroup contains elements which are centralized by a good E in M_δ, which gives the contradiction $Y_M \leq M_\delta$, again.

So we are left with the case that A acts nontrivially on $F(M/C_M)$. Then this is only possible for $|A| = 2$. Hence A acts on a Sylow 3–subgroup R of $F(M/C_M)$. Let R_1 be an preimage of R. Suppose that R_1 is cyclic. Then we have that $[R_1, Y_M]$ is of order four. But M normalizes $[Y_M, R_1]$ so there is some element in $Z(S)$, which is centralized by a good E. So we have $m_3(R_1) \geq 2$. We have $M = N_M(R_1)C_M$. If $3 \notin \sigma(M)$, then by 2.3 R_1 is centralized by some elememtary abelian p–subgroup E with $\Gamma_{E,1}(G) \leq M$. So also $[R_2, Y_M]$ is centralized by E for any subgroup R_2 of R with $|[Y_M, R_2]| = 4$. Hence as $[R, A]$ is normal in R and of order three, we see that $[Y_M, \langle [R, A]^S \rangle]$ is centralized by E. Then some element in $\Omega_1(Z(S))^\sharp$ is centralized by E, a contradiction. So we have $3 \in \sigma(M)$. If $m_3(R_1) = 2$, we see that we have either a characteristic subgroup isomorphic to E_9 or extraspecial of type 3^{1+2}. In both cases we see that R_1 contains a good E. This shows $[Y_M, R_1] = Y_M$.

Let C be either an elementary abelian or extraspecial characteristic subgroup of R_1 of rank at least two. If $[A, C] \leq C_M$, then $[A, Y_M]$ is centralized by a good E. Suppose now that $[A, C] \not\leq C_M$. If the rank of C is at least three we again have that $[A, Y_M]$ is centralized by a good E. So assume that the rank is at most two. Then $|[Y_M, \langle [A, C]^{\langle S,U \rangle} \rangle]| \leq 16$, where U is a Sylow 3-subgroup with $R_1 \leq U$. But now $U/C_U([Y_M, \langle [A, C]^{\langle S,U \rangle} \rangle])$ is elementary abelian of order 3 or 9, and so again $[A, Y_M]$ is centralized by a good E.

Suppose now that A is normalized by S, then some nontrivial element in $[Y_M, A]$ is centralized by S, a contradiction. So we are not in the situation of 3.12. As $[Y_M, A] \leq V_{\alpha'}$, we get $V_{\alpha'} \leq M$. By quadratic action we have that Y_M is generated by elements centralizing a subgroup of index two in $V_{\alpha'}$. Hence $Y_M \leq M_\delta$ for all $\delta \in \Delta(\alpha')$, contradicting $(*)$. This contradiction proves the lemma. □

eventype

Proposition 8.14 *Let H be some 2-local which contains S, then $F^*(H) = O_2(H)$.*

Proof: We have $O_2(H) \cap Z(S) \neq 1$. Set $H_1 = C_H(O_2(H))S$. Then by 8.13 we have $H_1 \leq M$. By 6.17 we have that $C_{H_1}(O_2(M)) \leq O_2(M)$. Hence also $C_{H_1}(O_2(H_1)) \leq O_2(H_1)$. This shows $F^*(H_1) = O_2(H_1)$. As $E(H)O_{2'}(H) \leq F^*(H_1)$, we get $F^*(H) = O_2(H)$, the assertion. □

9 M is unique

In this chapter we just prove one proposition.

Munique

Proposition 9.1 *There is just one uniqueness group M which contains S.*

Proof: Suppose that H is a second one. By 6.17 we have that $F^*(M) = O_2(M)$ and $F^*(H) = O_2(H)$. By 8.11 we have that $\langle N_G(S), C_G(x) \mid x \in \Omega_1(Z(S))^\sharp\rangle \leq M \cap H$. So we have $C_M C_H \leq M \cap H$, and then $O_2(H)O_2(M)$ is normalized by $C_M C_H$. This shows $O_2(H)O_2(M) \cap C_M = O_2(M)$ and $O_2(H)O_2(M) \cap C_H = O_2(H)$. Let now $\Gamma(M, H)$ be the coset graph for the amalgam (M, H). By 7.1 we get $b = b_\Gamma$ is odd. So let (α, α') be a critical pair. Then we may assume that M is attached to α and Y_M is an F–module with offender $Y_{H'_\alpha}$. In particular by 5.9 we have that M is not exceptional.

Let $(\alpha, \beta, \ldots, \alpha')$ be a path from α to α' of length b. We may choose notation in such a way that H is attached to β. Let $b > 1$. Then $Y_{\alpha'} \leq O_2(H_{\alpha'})O_2(M_{\alpha'-1}) \cap C_{M_{\alpha'-1}} = O_2(M_{\alpha'-1})$. Hence by iterating this we get that $Y_{\alpha'} \leq O_2(H)$. This is obviously true if $b = 1$. So in any case $Y_{\alpha'} \leq O_2(H)$. In particular $Y_{H'_\alpha} \leq O_2(C_G(x))$ for any $x \in \Omega_1(Z(S))^\sharp$.

Let first K be a component of M/C_M such that $Y_{H'_\alpha}$ induces an F–module offender on $[K, Y_M]$. Assume further that K is normalized by S. We see with 3.23 that $K \cong L_n(q)$, $Sp(2n, q)$, $G_2(q)$ or A_n and just one natural module is involved. Further we have that also $Y_{H'_\alpha}$ is an F–module with offender $W = [Y_M, K]$. Finally $C_{Y_M}(K) = 1$.

Now we have that also neither M nor H is excetional by 5.9. We now apply 3.42. Suppose W satisfies 3.42(1), (2) or (3). Then there is some $1 \neq x\mathbb{C}_W(S)$, which is centralized by a good E in M. But then as $C_G(x) \leq H$, we would get $H \leq M$, a contradiction. Hence we have 3.42(4). But then in all cases we would get $C_{Y_M}(K) \neq 1$.

So we have that S does not normalize K. Let $1 \neq x \in C_{Y_M}(S)$. Then the projection of s onto $[Y_M, K]$ is centralized by $N_S(K)$ and so we may apply 3.23 to K and $[Y_M, K]$, which gives us that $K \cong L_n(q)$, $Sp(2n, q)$, $G_2(q)$ or Σ_n and $[Y_M, K]$ is the natural module. Application of 3.42 shows that any element in $W = [Y_M, K]$ is centralized by a good E in M.

Let now first L be a component of $H_{\alpha'}/C_{H_{\alpha'}}$ on which W induces an F–module. But now things are symmetric and so in $[L, Y_{H_{\alpha'}}]$ any element is centralized by a good E in $H_{\alpha'}$. As $[W, [Y_{H_{\alpha'}}]] \neq 1$, this implies $H_{\alpha'} = M$, but $Y_M \not\leq O_2(H_{\alpha'})$.

So we have that W induces an F–module on $[Y_{H_{\alpha'}}, F(H_{\alpha'}/C_{H_{\alpha'}})]$. This gives that it acts on a 3-group and as W satisfies 3.42(3) we have with 2.1 that W induces a group of order 2, i.e. $Y_{H_{\alpha'}}$ induces a transvection on W. If $3 \not\in \sigma(H)$, then with 2.3we get a good E in $H_{\alpha'}$ centralizing $[W, F(H_{\alpha'}/C_{H_{\alpha'}})]$ and then also $[W, Y_{H_{\alpha'}}]$. But this group is also centralized by a good E in M, a contradiction. So we have $3 \in \sigma(H)$. This shows that we have $m_3(K) = 1$ and then we get that $K \cong L_3(2)$. Now $\langle K^S \rangle = L_3(2) \times L_3(2)$, and then $[Y_M, \langle K^S \rangle]$ is of order 2^6. Now $H \cap M$ contains a Sylow 3–subgroup F of M. This shows that $m_3(F) = 2$. Further we have that F contains a 3-central element of H, otherwise all elements in F are good and then $M \leq H$, a contradiction. Hence we have that $F \cap C_M = 1$, otherwise we have the 3-central element in C_M and abain $M \leq H$. So F is elementary abelian of order 9. Now $F = \Omega_1(C_P(F))$ for some Sylow 3-subgroup P of H with $F \leq P$. Now $H \cap M$ induces a dihedral group of order 8 on F and so H induces $GL(2,3)$ on F. But then all elements in $F^\sharp$ are conjugate and so good, which again gives $M \leq H$.

So we have that $Y_{H_{\alpha'}}$ induces an F–module offender on a Sylow 3–subgroup of $F(M/C_M)$. By 4.5 then also Y_M induces an F–module offender on $Y_{H_{\alpha'}}$ and so by symmetry it also acts nontrivially on $F(H_{\alpha'}/C_{H_{\alpha'}})$. Again by 5.9 both M and H are not exceptional. We now have $x \in Y_{H_{\alpha'}}$, $y \in Y_M$ such that $[x, y] = [x, Y_M] = [y, Y_{H_{\alpha'}}]$ is of order 2. Further y acts nontrivially on $F(H_{\alpha'}/C_{H_{\alpha'}})$ and x acts nontrivially on $F(M/C_M)$. If $3 \not\in \sigma(H)$, by 2.3 there is a good E centralizing the Sylow 3–subgroup of $F(H/C_H)$ and so also centralizing $[x, y]$. Hence if the same is true for M, we get a contradiction. This shows that we may assume that $3 \in \sigma(H)$. Hence $C_M([x, y])$ contains a good E. Let F be a Sylow 3–subgroup of $F(M/C_M)$ and F_1 be a preimage. Then $M = C_M N_M(F_1)$. By 2.5 we have that $N_M(F_1)$ contains a good E, as C_M cannot contain an elementary abelian subgroup of order p^2 for $p \in \sigma(M)$ by 5.11. Let C be a critical group in F_1 and $C_1 = \Omega_1(C_1)$. We have $m_3(C_1) \leq 2$ and so we get that C_1 is of order at most 27. Hence a good E in $N_M(F_1)$ centralizes C_1 and so also F_1. If now $m_3(F_1) = 2$, there is some 3-element ρ in F_1 with $C_G(\rho) \leq H^g$ for suitable g. But then H^g contains a good E from M and so $H^g = M$. But then we would have $3 \in \sigma(M)$ as well, a contradiction. So we have that F_1 is cyclic. Then also F is cyclic. Now $|[F, Y_M]| = 4$ and so $[Y_M, x]$ is centralized by S. But then $C_M([x, y]) \leq H$, a contradiction, as $C_M([x, y])$ contains a good E from M. $\square$

10 The 2-locals containing M_0

Let $M_0 = N_M(S \cap C_M)$. In this chapter we are going to prove that there is some 2-local H with $M_0 \le H$ but $H \not\le M$.

For the remainder we assume that there is some minimal parabolic H containing S, such that $O_2(\langle M_0, H\rangle) = 1$. Further we may assume that for any $S \le H_1 < H$, we have that $H_1 \le M$. Let $\Gamma = \Gamma(M_0, H)$ be the coset graph of the amalgam (M_0, H) and $b = b_\Gamma$. Set $Q_H = O_2(H)$ and $Q_0 = O_2(M_0)$. Recall that by 8.14 we have that $F^*(M_0) = Q_0$ and $F^*(H) = O_2(H)$.

Hstr

Lemma 10.1 *We have $O_2(H)$ is a Sylow 2–subgroup of C_H. Further let $\alpha \in \Gamma$, which belongs to a conjugate of H, then any 2-element which fixes all neighbours of α is in $O_2(G_\alpha)$.*

Proof: Let $X = C_H$ or $G_{\Delta(\alpha)}$ and $T = S \cap X$. Then $H = XN_H(T)$. By 8.11 $C_H \le M$, so always $X \le M$. So we have $H = N_H(T)$, i.e. $O_2(H)$ is a Sylow 2–subgroup of X. □

A9

Lemma 10.2 *We have $H/C_H \not\cong A_9$ and*

Proof: Let $H/C_H \cong A_9$ then $H \cap M/C_H \cong A_8$. Let $O_2(M_0) \le C_H$, then $Y_H \le Y_M$. Let (α, α') be a critical pair. Then α belongs to M. By 7.1 α' belongs to H. Then $[Y_\alpha, Y_{\alpha'}] \ne 1$ by 10.1. But then also $[Y_\alpha, Y_{\alpha'-1}] \ne 1$, contradicting 7.1. Hence we have that $O_2(M_0) \not\le C_H$ and so $(C_M \cap H)C_H = M \cap H$. Firts of all we now see that $3 \notin \sigma(M)$. So $m_3(C_M) \le 3$. By 2.3we now get that a Sylow 3–subgroup P of $C_M \cap H$ is centralized by a good E in M. With 5.3 we get $N_G(P) \le M$. But then as $N_H(P) \not\le M \cap H$, we have a contradiction. □

bev

Lemma 10.3 *We have b is odd.*

Proof: Assume that b is even. By 7.1 we have that $b = b_H$. Let (α, α') be a critical pair, where α belongs to H. Further we choose notation such that M_0 belongs to a neighbour β of α with $d(\beta, \alpha') = d(\alpha, \alpha') + 1$. We have $Y_H \le H_{\alpha'}$. By 10.1 we have that $[Y_H, Y_{H_\alpha}] \ne 1$.

Hence we have that Y_H is an F–module. We are going to apply 4.6 and 4.7. By 8.11 we have that $C_{Y_H}(H) = 1$. By 10.2 we have that H does not induce A_9.

Suppose that H induces Σ_5 on a permutation module or $\Sigma_5 \wr Z_2$ on a direct sum of two permutation modules, Σ_3 on 2-dimensional module or $\Sigma_3 \wr Z_2$

on the 4–dimensional module. Assume first additionally that $|Y_M| = 2$. Then $M_0 = N_G(S)$. Let $J(S) \leq O_2(H)$, then we get $J(S)$ is normal in $\langle M_0, H\rangle$, a contradiction. So we see that $J(S) \not\leq O_2(H)$. By 4.6 we see that $J(S)O_2(H)$ is the transvection group. Hence $O_2(H)B(S) = O_2(H)J(S)$, where $B(S) = C_S(\Omega_1(Z(J(S))))$ is the Baumann group. Further we see that $J(O_2(H)) \leq J(S)$. So $\Omega_1(Z(J(O_2(H)))) \geq \Omega_1(Z(J(S)))$. We have $Y_H \leq \Omega_1(Z(J(O_2(H))))$ and as offender are exact on Y_H we see that $[J(S), \Omega_1(Z(J(O_2(H))))] \leq Y_H$. Hence for $X = \langle J(S)^H\rangle$ we get $[X, \Omega_1(Z(O_2(H)))] \leq Y_H$. Hence we even now get that $\Omega_1(Z(J(S)))Y_H$ is normal in H. This shows that $R = C_{O_2(H_1)}(Z(J(S)))$ is normal in H. Then we see that $[J(S), O_2(H)] \leq R$, and so even $[X, O_2(H)] \leq R$. In all cases we now have a subgroup H_1 in H with $O_2(H)J(S)$ a Sylow 2–subgroup of H_1 and $H_1/O_2(H_1) \cong \Sigma_3$. Hence we have that H_1/R is a direct product of a 2-group by Σ_3. As $B(S) \not\leq O_2(H)$, we now see that $B(S)$ is a Sylow 2–subgroup of $H_2 = \langle B(S)^{H_1}\rangle$. Let C be a nontrival characteristic subgroup of $B(S)$ normal in H_2, then it is normal in $\langle H_2, M_0\rangle = \langle H, M_0\rangle$, a contradiction. So we have no such group and then by [Ste, Theorem1] we have that $|[\rho, O_2(H)]| = 4$ for some element of order three in H_1. This shows that $[O^2(H), O_2(H)] = Y_H$. By 3.35 we see that $O_2(H) = Y_H$. Suppose first that $|S| \leq 2^7$, then we have that $|O_2(M)/\Phi(O_2(M))| \leq 2^4$, contradicting $m_p(M) \geq 3$ for some odd p. So we have $|S| = 2^{15}$. Let $p \in \sigma(M)$. Suppose $p = 3$. Then as $M \cap H$ cannot contain a good E, we see that $e(G) = 3$. But then at least $M \cap H$ contains some 3–element ρ with $N_G(\langle\rho\rangle) \leq M$. As $H = \langle M \cap H, N_H(\langle\rho\rangle)\rangle$, we have a contradiction. Hence $3 \notin \sigma(M)$. This gives that $|O_2(M)/\Phi(O_2(M))| \geq 2^9$. As $Y_H \neq O_2(M)$, we have $|O_2(M)Y_H/Y_H| = 2^4$ and so $|[O_2(M), Y_H]| = 2^6$ and $|[O_2(M), Y_H, O_2(M)]|$. Hence we have $|O_2(M)/\Phi(O_2(M))| = 2^6$ or 2^8 a contradiction.

So we have that $|Y_M| \geq 4$. As any subgroup of index two Y_M contains an element centralized by a good E, and $Y_{H_{\alpha'}} \not\leq M$, we get that $Y_{H_{\alpha'}}$ cannot be generated by elements which centralize a subgroup of index two in Y_M. By 7.1 $U = \langle Y_M^H\rangle$ is abelian. So $[Y_M, Y_{H_{\alpha'}}, Y_M] \leq [U, Y_M] = 1$. This shows that Y_M acts quadratically on $Y_{H_{\alpha'}}$. With 4.7 we now see that H is nonsolvable. Further for $|Y_M| = 4$ we have that $Y_M \cap C_{H_{\alpha'}} = 1$. So assume $|Y_M| \geq 8$. Now we find elements in $Y_{H_{\alpha'}} \setminus M$, which centralize a subgroup of index four in Y_M. This now shows $\sigma(M) = \{3\}$. Further as H cannot contain a good 3–element from M, we get $e(G) = 3$ and $H/O_2(H) \cong \Sigma_5$. Further we have that C_M does not contain a good E. As Y_H centralizes $Y_{M_{\alpha'-1}}$ we have that $Y_H \leq O_2((M_{\alpha'})_0)$. Hence we have that $O_2(M_0)$ contains an offender on Y_H and so $O_2(M_0))O_2(H)/O_2(H) \not\leq (H/O_2(H))'$. So we get that $M \cap H = (C_M \cap H)S$. Hence there is some element ρ of order 3 in $C_M \cap H$. But now as C_M does not contain a good E, i.e. a Sylow 3-subgroup contains no normal elementary abelian subgroup of order 9 in C_M, we see that Sylow 3-subgroups of C_M are cyclic. So we have that elements of order three in C_M are centralized by an elementary abelian subgroup of order 27. Then we have $H = \langle C_M \cap H, N_H(\langle \rho \rangle)\rangle \leq M$, a contradiction.

So we are left with $|Y_M| = 4$ and $Y_M \cap O_2(H_{\alpha'}) = 1$. Suppose $Y_M O_2(H_{\alpha'}) \not\leq Y_H O_2(H_{\alpha'})$. Then there is $x \in Y_{H_{\alpha'}}$ with $[x, Y_H] = 1$ but $[x, Y_M] \neq 1$. So $x \in O_2(H) \leq M$, $[x, Y_M] \leq Y_M$ and as Y_M is centralized by a good E, we have $C_G([x, Y_M]) \leq M$. But $[x, Y_M] \leq Y_{H_{\alpha'}}$ and then $Y_{H_{\alpha'}} \leq M$, a contradiction. So we have that $[Y_M Y_H, Y_{H_{\alpha'}}] = [Y_H, Y_{H_{\alpha'}}]$. But $Y_{H_{\alpha'}}$ is generated by elements which centralize subgroups of index two in Y_H, so they now centralize subgroups of index two in Y_M which gives $Y_{H_{\alpha'}} \leq M$, a contradiction.

Now by 4.6 we have $E(H/O_2(H)) \cong L_2(q)$ or $L_2(q) \times L_2(q)$ and just natural modules are involved. Now the action of an offender shows that we have an element x in $Y_{H_{\alpha'}}$ with $x \notin M$ but $[Y_H, x] = [Y_H Y M, x]$. So $Y_H Y_M$ is normalized by $\langle H \cap M, x\rangle = H$. Further $[O^2(H), Y_M Y_H] \leq Y_H$. As $Y_M \cap Y_H$ is a 1-dimensional subspace in each of the 2-dimensional modules for $E(H/O_2(H))$, we see that Q_0 is a Sylow 2–subgroup of $E(H/O_2(H))$. Now $C_{Q_H}(Y_H Y_M) = Q_0 \cap Q_H$. In particular this group is normal in H.

As $q > 2$, there are elements of order $q-1$ which normalize $C_M \cap H$. This shows that $H_1 = \langle Q_0^H \rangle$ has a Sylow 2–subgroup Q_0. Suppose first that $H_1/(C_H \cap H_1) \cong L_2(q)$. Then we have that no nontrivial characteristic subgroup of Q_0 is normal in H_1. Application of [Ste, Theorem 1] shows that $O^2(H)$ acts trivially on $O_2(H)/Y_H$. This shows $Y_H = O_2(H)$ as $Z(H) = 1$ by 8.11 and $O_2(H)$ is the natural module. Now $O_2(H)$ is normalized by any automorphism of Q_0 of odd order, since Q_0 just contains two elementary a abelian subgroups of order q^2. But as $O_2(H)$ is normalized by S, it is normal in M_0, a contradiction. So we may assume that H_1 involves $L_2(q) \times L_2(q)$. Now there is a subgroup H_2 with Sylow 2-subgroup Q_0 such that $H_2/O_2(H_2) \cong L_2(q)$. Again this group by the same argument just induces one module and then again we get that Y_H is the direct sum of two natural modules and $Y_H = O_2(H)$. As $Y_M = \Omega_1(Z(Q_0))$ by 3.4, we now see that $[O_2(M), Y_H, Q_0] = 1$ and so $[O_2(M), Y_H] \leq Y_M$. As $Y_H \leq C_M$, we see that $Y_H \leq O_2(M)$. In particular $O_2(M) = Q_0$. We see that $O_2(M)$ contains exactly four elementary abelian subgroups of order q^4. Hence if F is an elementary abelian subgroup of order p^3 in M then some good E normalizes all these groups and so even $O_2(H)$, contradicting $H \not\leq M$. □

*bodd*1

Lemma 10.4 *We have $b = 1$.*

Proof: By 10.3 we have that b is odd. Let $b > 1$. Again we fix a critical pair (α, α'), where α belongs to M_0. We choose notation such that H belongs to a neighbour β of α with $d(\beta, \alpha') = d(\alpha, \alpha') - 1$. By 10.1 we have that $O_2(H)$ is a Sylow 2–subgroup of C_H. Further $O_2(M_0)$ is a Sylow

2–subgroup of C_M by construction, so we have that $[Y_M, Y_{H_{\alpha'}}] \neq 1$. Suppose that Y_M is an offender on $Y_{H_{\alpha'}}$ as an F–module. Then we may apply 4.6, 4.7 which shows that also Y_M is an F–module with offender $Y_{H_{\alpha'}}$. Hence in any case we have that Y_M is an F–module with offender in $Y_{H_{\alpha'}}$. Further by 5.9 we have that M is not exceptional.

We first show

$(*)\qquad Y_M Y_H \not\trianglelefteq H$

Suppose false. As $b \geq 3$, we have a neighbour δ of β with $d(\delta, \alpha') = d(\beta, \alpha') - 1$. Hence $[Y_{M_\delta}, Y_{H_{\alpha'}}] = 1$. As $Y_M Y H = Y_H Y_{M_\delta}$, we get $[Y_M, Y_{H_{\alpha'}}] = 1$, a contradiction. Tis proves $(*)$.

We now may apply 3.42. Let K be a component of M/C_M on which $Y_{H_{\alpha'}}$ induces an F–module offender on $[K, Y_M]$. Suppose there is a quasi irreducible submodule W_M of $[Y_M, K]$ such that for any $1 \neq x \in W_M$ we have that M is the unique maximal 2–local of G with $C_G(x) \leq M$. As $b > 1$ we see that $[[W_M, Y_{H_{\alpha'}}], Y_{M_\beta}] = 1$ for any neighbour β of α'. Hence we have that $Y_{M_\beta} \leq M$. By 7.1 we have that $\langle Y_{M_\beta}^{H_{\alpha'}} \rangle$ is abelian. So $Y_{H_{\alpha'}} Y_{M_\beta}$ acts quadratically on Y_M and $[Y_{H_{\alpha'}}, W_M] \not\leq C_{W_M}(K)$. This gives with 3.24 that $[Y_{M_\beta}, W_M] \leq W_M$. Now we may choose β such that $W_M \not\leq M_\beta$. Hence $W_{M_\beta} \cap C_M = 1$. Moreover there is some $x \in W_M$ such that $C_{W_{M_\beta}}(x) = 1$. As $\langle x^{W_{M_\beta}} \rangle \leq W_M$ this is not possible.

Hence we have one of the cases in 3.42(4). Now set $W_M = [K, Y_M]$. If we are not in case (vi) or (vii) there is always a subspace in $[Y_{H_{\alpha'}}, W_M]$ which is normalized by a good E. Now choose β as before. Then $Y_{M_\beta} \leq M$. By $(*)$ we have that $[W_M, Y_{M_\beta}] \not\leq [W_M, Y_{H_{\alpha'}}]$. This now shows that we have 3.42(4)(ii) with $q = 2$, (iii), (vi) or (vii).

Suppose that in one of these cases we have that $|Y_{H_{\alpha'}} : C_{Y_{H_{\alpha'}}}(W_M)| > |W_M : C_{W_M}(Y_{H_{\alpha'}})|$. then we see that in all that cases $[Y_{H_{\alpha'}}, W_M]$ contains a subgroup which normalized by a good E. Hence by the remark above we see that $Y_{H_{\alpha'}}/C_{Y_{H_{\alpha'}}}(W_M)$ cannot project on a maximal elementary abelian subgroup of $\mathrm{Aut}(K)$. This shows that we just can have $K \cong Sp(4, q)$ and $|Y_{H_{\alpha'}} : C_{Y_{H_{\alpha'}}}(W_M)| > q^2$.

So we may assume that also $Y_{H_{\alpha'}}$ is an F–module with offender W_M. Let first $K \not\cong Sp(4, q)$. Then we have $\sigma(M) = \{3\}$ and all elementary abelian subgroups of order 9 are good. So by 5.4 we have that $m_3(H) \leq 1$. With 4.6 we get that $E(H/C_H) \cong L_2(q)$ inducing the natural module or $H/C_H \cong \Sigma_5$ inducing the permutation module, or $H/C_H \cong \Sigma_3$ and $|Y_H| = 4$. Suppose we have the $L_2(q)$–case. Then $[x, Y_{H_{\alpha'}}] = [W_M, Y_{H_{\alpha'}}]$ for all $x \in W_M \backslash C_{W_M}(Y_{H_{\alpha'}})$. This is only possible with $K \cong \Omega^-(6,2)$ and $Y_{H_{\alpha'}}$ induces a transvection on W_M. But then W_M does the same, a contradiction. If we have $H/C_H \cong \Sigma_5$, then we have some 3-element in $C_M \cap H$, but the centralizer of such elements is in M, and so we would get $H \leq M$, a contradiction. This shows $|Y_H| = 4$. Then $Y_{H_{\alpha'}}$ induces a transvection on W_M, so again $K \cong \Omega^-(6,2)$. Now $[Y_M, Y_{H_{\alpha'}}]$ is centralized by some good E in K. This shows that $C_{H_{\alpha'}}([Y_M, Y_{H_{\alpha'}}])$ is in M, so M and $H_{\alpha'}$ share a common Sylow 2–subgroup. We have that $[Y_M, Y_{H_{\alpha'}}]$ is contained in the center of a Sylow 2–subgroup of $M_{\alpha'-1}$. So by 8.11 and 9.1 this gives that $M = M_{\alpha'-1}$ But then $Y_M = Y_{M_{\alpha'-1}} \not\leq O_2(H_{\alpha'})$, which contradicts $b > 1$.

So we are left with $K \cong Sp(4,q)$ and W_M is a nontrivial extension of the trivial module by a natural module. Further by 3.42(4)(iii) we have $q > 2$. Now by 3.53 we see that Y_M induces an F–module offender on $Y_{H_{\alpha'}}$. Inspection of the list in 4.6 and 4.7 shows that we have for any quadratic offender A on $Y_{H_{\alpha'}}$ that $[A, Y_{\alpha'}] = |A|$. But this again contradicts 3.53.

So we see that $Y_{H_{\alpha'}}$ induces an F–module offender on $F(M/C_M)$. In particular $[F(M/C_M), Y_{H_{\alpha'}}]$ is a 3–group. Further Y_M induces transvections on $Y_{H_{\alpha'}}$ by 4.7. By 4.6 we see that there are elements y in $Y_{H_{\alpha'}}$ which centralize subgroups of index 4 in Y_M. Suppose $3 \notin \sigma(M)$, then by 2.3 $[F(M/C_M), Y_{H_{\alpha'}}]$ is centralized by a good E $[Y_M, [F(M/C_M), x]]$ is centralized by a good E in M. But then again $Y_{M_\beta} \leq M$ and so by quadratic action of $Y_{H_{\alpha'}} Y_{M_\beta}$ on Y_M we get that Y_M is generated by elements centralizing subgroups of index two in $Y_{H_{\alpha'}} Y_{M_\beta}$ This shows that $Y_{H_{\alpha'}} Y_{M_\beta}$ is normal in $H_{\alpha'}$, a contradiction to $(*)$. So we have $3 \in \sigma(M)$. With 4.6, 4.7 we now get that $H/C_H \cong \Sigma_3$, $\Sigma_3 \wr Z_2$, Σ_5 or $\Sigma_5 \wr Z_2$. In the latter $M \cap H$ contains an elementary abelian subgroup of order 9 and so this group contains at least one element ρ which is good in M. But then $H = \langle H \cap M, N_H(\langle \rho \rangle) \rangle \leq M$, a contradiction.

Let $H/C_H \cong \Sigma_3$. Then we have $|Y_H| = 4$. In particular $|[Y_M, Y_{H_{\alpha'}}]| = 2$. Let $H/C_H \cong \Sigma_3 \wr Z_2$, or Σ_5. By 5.4 we have that $e(G) = 3$ in the former. If $e(G) > 3$, then all 3–elements are good, but 3 divides $|H \cap M|$ and so we also get $e(G) = 3$ in the latter. We now have $|Y_H| = 16$. in both cases

Let $x \in Y_{H_{\alpha'}}$ with $|[Y_M, x]| = 2$. We are going to show that $[x, Y_M]$ is centralized by a good E in M. Let now P be a Sylow 3–subgroup of the preimage of the Sylow 3-subgroup of $F(M/C_M)$, which is normalized by $Y_{H_{\alpha'}}$. Further let U be a Sylow 3-subgroup of M containing P. Let C be an abelian characteristic subgroup of P. Suppose first $|C| \geq 27$. Then we see that $[Y_M, x]$ is centralized by some elementary abelian subgroup of order 9 in C. Assume next that $|C| = 9$. There is some 3–element $\rho \in C$ with $|[Y_M, \rho]| = 4$, with $[x, Y_M] \leq [\rho, Y_M]$. If $\langle \rho \rangle$ is normal in U, we see that $[\rho, Y_M]$ is centralized by a good E. Assume that $\langle \rho^U \rangle = C$. Now $|[C, Y_M]| \leq 16$. Then $U = C_U([C, Y_M])C$. As $U \neq C_U([C, Y_M]) \times C$ we see that $|C_U([C, Y_M]) \cap C| = 3$. Hence $|[C, Y_M]| = 4$ and so again $[x, Y_M]$ is centralized by a good E. So we have that any characteristic abelian subgroup of P is cylic and so P is extraspecial or cyclic. If P is extraspecial of order at least 3^5, then we get for $\langle \rho \rangle = [P, x]$, that $[\rho, Y_M]$ is centralized by an extraspecial group of order 27, and so by a good E, a contradiction. So we have that $|P| = 27$. Let ρ be as before. If $\langle \rho^U \rangle = P$, then again we have $|[P, Y_M]| \leq 16$ and so $U = C_U([Y_M, P])P$, which again shows that $[x, Y_M]$ is centralized by a good E. So we have that $\langle Z(P), \rho \rangle$ is normal in U, but then it is contained in some elementary abelian subgroup of order 27 and so $[x, Y_M]$ is centralized by a good E again. So we are left with P cyclic. Then $|[P, Y_M]| = 4$ and normalized by M, so centralized by a good E.

Now in any case we have that $[x, Y_M]$ is centralized by a good E in M. In particular $C_{H_{\alpha'}}([x, Y_M]) \leq M$. This gives that $Y_{M_\beta} \leq M$ for $\beta \in \Delta(\alpha')$ with $Y_M \not\leq M_\beta$. But then again we see that $Y_{H_{\alpha'}} Y_{M_\beta}$ is normal in $H_{\alpha'}$, contradicting $(*)$. □

$M0$

Proposition 10.5 *There are at least two maximal 2–local subgroups in G containing* M_0.

Proof: Assume false. Then we look at the amalgam before and by 10.4 we have that $[Y_H, Y_M] \neq 1$. Again by 4.6, 4.7 we get that Y_M is an F–module and so by 5.9 M is not exceptional. Let L and g be the subgroup of H given by 4.2 with respect to Y_M. As $L \leq C_L(Y_M \cap Y_M^g)$, we get with 8.12 that $Y_M \cap Y_M^g = 1$. Hence $(Y_M \cap O_2(L))(Y_M^g \cap O_2(L)) = (Y_M \cap O_2(L)) \times (Y_M^g \cap O_2(L))$. Set $A = Y_M^g \cap O_2(L)$. Then A induces an F–module offender on Y_M with $C_{Y_M}(a) = C_{Y_M}(A)$ for all $a \in A^\sharp$. So Y_M is a stong module with respect to A. Further $[Y_M, A] = Y_M \cap O_2(L)$. So $|[Y_M, A]| = |A|$ and $[A, y] = [A, Y_M]$ for all $y \in Y_M \setminus C_{Y_M}(A)$. In particular Y_M is alos a strong dual F–module with respect to A. Let now K be some component of M/C_M on which A induces the F–module offender. By 3.22 we get $K \cong L_n(q)$ or $Sp(2n, q)$ and $W_M = [Y_M, K]$ is an extention of the trivial module by the natural module or $K \cong A_6$ or A_7 and $W_M = [Y_M, K]$ is an extention of the trivial module by the 4-dimensional module. Recall that the third case of 3.22 cannot occur. Otherwise we would have $|A| = 2$ and then $|Y_M| = 4$, which certainly is not possible. In particular in all cases we have that W_M is normal in M.

Assume first that any element in W_M is centralized by a good E in M.

Next we determine the structure of H. Suppose H is nonsolvable and H/C_H has more than one component. By 3.7 we now get that the components are $L_2(r)$, $Sz(r)$, $L_3(2)$, $3A_6$, $SU_3(8)$ or $SL(3, 4)$. In cases of $L_3(2)$,

$3A_6$ and $SL(3,4)$ there is a diagram automorphism involved. So let first one of $L_2(r)$, $Sz(r)$, $SU_3(8)$ or $SL_3(4)$. Then we see that $W_MC_H/C_H = \Omega_1(Z(S \cap E(H/C_H)))$. As W_M acts quadratically on Y_H, we get a submodule $X_1 \times X_2$, where X_i is a module for one of the components and centralized by the other one. Now $[W_M, X_i] \neq 1$ for both i, so M covers $E(H/C_H)$, a contradiction.

So we have the $L_3(2)$ or $3A_6$–case. If W_M intersects both components nontrivially, we may argue as before. So we have $|Y_M : Y_M \cap O_2(H)| = 2$. Now choose a parabolic U in $E(H/C_H)$ containing SC_H/C_H but not be contained in $M \cap H/C_H$ with $Y_MC_H/C_H \leq O_2(U)$. With 7.1 we see that $R = \langle Y_M^U \rangle$ acts quadratically on Y_H and R intersect any component nontrivially. But then we can argue as before.

We have seen that $E(H/C_H)$ is quasisimple or H is solvable. So assume first that H is nonsolvable. As $Y_H \not\leq O_2(M_0)$, we get with 4.2 that Y_H is a 2F–module. Now application of 3.29, 3.30, 3.31 and 3.32 shows that $E(H/C_H) \cong L_2(r)$, $Sz(r)$, $U_3(r)$, $L_3(r)$, $Sp(4,r)$ or $3A_6$. Further in the last three cases diagram automorphisms are involved. In the last three cases we see that there is a natural submodule V in Y_H. Now $[W_M, V]$ always contains a nontrivial element whose centralizer picks up a parabolic in the simple group, but then $E(H/C_H)$ is covered by M, a contradiction. So we have the first three cases. Then, as Y_M acts quadratically, we see with 3.50 that just natural modules are involved. Now also the $2F$–module offender acts quadratically and so we get with 4.2 that Y_H even is an F–module. Hence $E(H/C_H) \cong L_2(r)$.

Now we see that $LC_H/C_H = E(H/C_H)$. Hence $(Y_M \cap O_2(L))(Y_M^g \cap O_2(L))$ is normal in H. Then A contains some element $1 \neq a$ in $Z(SC_M/C_M)$. As $C_{Y_M}(a) = C_{Y_M}(A)$, we see that $K = L_n(q)$ and A is the full group of transvection to a hyperplane or $K = A_6$, A_7 and $|A| = 4$. In the last two cases as $[v, A] = [Y_M, A]$ for all $v \in Y_M \setminus C_{Y_M}(A)$, we see that $|Y_M| = 16$. Now Y_MC_H/C_H is a sylow 2-subgroup of $E(H/C_H)$ and so $Y_M \not\leq \Phi(O_2(M))$. This now shows $Y_M = O_2(M)$. As $m_p(M) \geq 3$ for some odd prime p, this contradicts $Y_M = F^*(M)$.

So we have shown that $K \cong L_n(q)$. Let $C_{W_M}(K) \neq 1$, then by 3.36 we have either $n = 2$ or $n = 3$ and $q = 2$. In the former case we have by 3.52 that $|[A, Y_M]| > q = |A|$, a contradiction. In the latter we have that $|A| = 4$. But a transvection group to a hyperplane does not act quadratically on the nonsplit 4–dimensional module. So we have again that Y_M is the natural module and as before we get that $O_2(M) = Y_M$. Hence we can see that $N_M((Y_M \cap O_2(L))(Y_M^g \cap O_2(L)))$ involves $L_{n-1}(q) \times Z_{q-1}$. Hence this group cannot contain a good E since $(Y_M \cap O_2(L))(Y_M^g \cap O_2(L))$ is normal in H. As

$|Y_MC_H/C_H| = r \geq 4$, we get that $q > 2$. Hence now we have that $n \leq 4$ as otherwise we have uniqueness primes p dividing $q-1$. In case of $n = 3$ or 4, we may also assume that we do not have a uniqueness prime dividing $q-1$ Then for $n = 4$ we have $e(G) > 3$ and so in both cases $m_p(K) \leq m_p(M)-2$. In particular there is a good E normalizing $(Y_M \cap O_2(L))(Y_M^g \cap O_2(L))$, a contradiction. So we are left with $K \cong L_2(q)$. But then all Sylow p–subgroups, p odd, of K are cyclic, and then a Sylow 2–subgroup of K is normalized by a good E and then also $(Y_M \cap O_2(L))(Y_M^g \cap O_2(L))$.

So we may assume that H is solvable. Then by 2.1 we have that the p-rank of H is at most three and so $|Y_M : Y_M \cap O_2(H)| \leq 8$. Now we get that $L/O_2(L)$ is dihedral and so A induces the full group of transvections to a hyperplane on Y_M. This again shows $K \cong L_n(2)$ and $Y_M = O_2(M)$. This gives that $K = M(O_2(M)$. In particular $3 \in \sigma(M)$ and $n \geq 6$. But then $N_M((Y_M \cap O_2(L))(Y_M^g \cap O_2(L)))$ contains $L_5(2)$ and so a good E.

Hence we have one of the cases in 3.42(4). This means that we can have 3.42(4)(i) or (iii), as Y_M is a strong dual F–module. By 3.52 and $|[Y_M, A]| = |A|$ we see that we cannot have (i). So we have $K \cong Sp(4,q)$. As we have $C_{Y_M}(A) = [Y_M, A]$ is of order $|A|$ we must have $|A| = q^3$. But then $K = \langle C_K(a) \mid a \in A^\sharp \rangle$ would act on $C_{Y_M}(A)$, a contradiction.

We are left with the case that A induces an offender on $F(M/C_M)$. As this is a strong dual offender we get with 4.5 that $|A| = 2$. Then $|Y_M| = 4$ and so $M/C_M \cong \Sigma_3$. Now also Y_M induces transvections on Y_H and so as $Y_MO_2(H) \trianglelefteq S$, we get with 4.6, 4.7 that $H/C_H \cong \Sigma_3$. Then $Y_M \not\leq \Phi(O_2(M))$ and so $Y_M = O_2(M)$. Hence $C_M = Y_M$ and $M \cong \Sigma_4$, a contradiction. □

11 The group H

For the next three chapters let H be a subgroup of G with $M_0 \leq H$ and the following properties

(1) $H \not\leq M$

(2) $C_H(O_2(H)) \leq O_2(H)$

(3) Y_H is maximal with respect to (1) and (2)

(4) $M \cap H$ is maximal with respect to (3)

(5) H is maximal with respect to (1) - (4)

Recall that by 10.5 we have such a group H. By 9.1 we have that H is not contained in a uniqueness group. In particular $m_p(H) \leq 3$ for any odd prime p. This we will use freely in the sequel.

YHnormal

Lemma 11.1 *We have that $H = N_G(Y_H)$.*

Proof: We have $H \leq N_G(Y_H)$. Hence $N_G(Y_H)$ satisfies (1) and (2) by 8.14. By (3) and 3.4 we have $Y_H = Y_{N_G(Y_H)}$. Now we have $N_G(Y_H) \cap M = H \cap M$ by maximality. So we get with (5) that $H = N_G(Y_H)$. □

Hfaith

Lemma 11.2 *We have $Y_M \leq Y_H$ and $C_H \leq M \cap H$.*

Proof: By 3.4 we have $Y_M = Y_{M_0} \leq Y_H$. Hence $C_H \leq C_M$, so $C_H \leq M \cap H$. □

Hnormal

Lemma 11.3 *Let $Y \leq Y_M$ with $M \cap H < N_H(Y)$, then $Y = 1$.*

Proof: We have that Y is normalized by M_0 and C_M. As $M = C_M M_0$, we get that Y is normal in $\langle M, N_H(Y)\rangle$, so $Y = 1$. □

component

Lemma 11.4 *Let K be a component of H/C_H which is not covered by $M \cap H$ and which induces an F–module on $[Y_H, K]$ with offender A normalizing K. If K is not of Lie type in characteristic two, then one of the following holds. Further assume that for $K \cong A_6 \cong Sp(4,2)'$ or $K \cong A_5 \cong L_2(4)$ the group K does not act on the natural module. Then there is a subgroup P of H containing M_0 such that one of the following holds*

(i) $E(P/C_P) \cong A_5$, Y_P is the $\Omega^-(4,2)$–module.

(ii) There is a normal subgroup P_1 containing C_PS such that $P_1/C_P \cong \Sigma_3$, Y_P involves natural modules and trivial modules, $P = P_1M_0$ and $Z(P_1) = 1$

(iii) $E(P/C_P) = K_1 \times K_2 \cong \mathbb{A}_5 \times A_5$, Y_P is a direct sum $V_1 \oplus V_2$ where $[K_i, Y_P] = V_i$, $[K_{3-i}, V_i] = 1$, and V_i is the orthogonal K_i–module, $i = 1, 2$. Further K_1 is not normal in P/C_P.

In all cases P is even minimal with respect not to be in M.

Proof: By 3.16 we have that K is alternating. Then $[Y_H, K]$ is quasi irreducible. Let first $[Y_H, K]/C_{[Y_H,K]}(K)$ be the permutation module. Then by 1.2 we have $n \leq 11$. If $n \geq 8$, we have that K is normal. In particular $C_{Y_H}(K)$ is S-invariant and so by 5.14 we have $C_{Y_H}(K) = 1$. If $n = 11$, then we have that $\langle C_K(v) \mid v \in C_{[Y_H,K]}(S)^\sharp\rangle = K$, contradicting 8.13. Suppose $n = 10$, then by 8.13 we have $K \cap M \cong \Sigma_8$. Hence there is some parabolic $P \cong 2^4\Sigma_5$ in K containing S which is not in M. By 5.14 we get $|Y_M| = 2$, as otherwise $K = \langle C_K(v) \mid v \in Y_M^\sharp\rangle$. Now $M_0 = N_H(S)$ and Y_P is the permutation module, so we have (i).

Let $n = 9, 7$ and $O_2(M \cap K) = 1$. So in case A_7 we have the 4–dimensional module. Suppose first that K is normal. Then again $Y_H = [Y_H, K]$. Now $H = C_HK$. Then $Y_H \leq Z(O_2(M))$. Assume that K is not normal. Then there are exactly two conjugates. So we replace $[Y_H, K]$ by $[Y_H, K^S]$, which shows that $Y_H = [Y_H, K^S]$ and so again $Y_H \leq Z(O_2(M))$. Set $V_M = Y_H^M$. Let $C = C_M(V_M)$. Then we have that $O_2(M/C) \neq 1$. Let $T \leq S$ such that $S \cap C \leq T$ and $TC/C = O_2(M/C)$. We have that $K \cap M \not\leq C$ and so we may assume that $K \cap M$ is in $N_H(T)$. But then $[T, Y_H] = 1$, which shows that $[T, V_M] = 1$, a contradiction.

As in case of $K \cong A_9$ we have that $K \cap M \cong A_8$, we just have to handle $n = 7$, 5 and the permutation module is in Y_H or $K \cong 3A_6$. Suppose $n = 7$. Then as in the case of $n = 11$ we see that $K = \langle C_K(v) \mid v \in C_{[Y_H,K]}(S)^\sharp\rangle$, contradicting 5.14 and 8.13.

Let $3A_6$ on the 6-dimensional module. Let $g \in H$ with $K^g \neq K$. As $m_3(H) \leq 3$, we must have $Z(K) \leq K \cup K^g$. But $Z(K)$ acts nontrivially on $[Y_H, K]$ and as $[Y_H, K]$ is an irreducible module K^g has to act trivially. So we have that K is normal. Let $x \in Y_M^\sharp \cap Z(S)$. Then $C_K(x) \cong \Sigma_4$. Hence $M \cap K \cong \Sigma_4$ or $3\Sigma_4$. Let $M \cap K \cong \Sigma_4$. Then obviously $|Y_M| = 2$ as $|C_{Y_H}(M \cap K)| = 4$ and normalized by $Z(K)$. Now $M_0 = N_H(S)$ and there is some minimal parabolic P not in $M \cap K$ with $|\langle Y_M^P\rangle| = 4$. So we are in case (ii). If $M \cap K \cong 3\Sigma_4$, the $|Y_M| = 4$. Now choose again some minimal parabolic P of K which is not in $M \cap K$. Then we get that $|\langle Y_M^P\rangle| = 16$ and we have two Σ_3–modules in Y_P. Again this is (ii).

Let $K \cong A_5$ and we have the permutation module. Suppose that K is normal in H. Then as before we see that $H = C_HK$ and we just may choose $H = P$ and get (i). So assume that K has exactly two conjugates under S. Now we have that $C_{Y_H}(\langle K^S \rangle) = 1$ by 5.14 and 8.13. This again shows that $H = C_H\langle K^S \rangle$ and again using $P = H$, we get (iii). $\square$

For the notations related to $2F$–modules compare 3.2.

compLieB

Lemma 11.5 *Let K be a component of H/C_H which is not covered by $M \cap H$ and which induces an F–module or $2F$–module with cubic but not quadratic offender A normalizing K on Y_H. If A is a $2F$–offender assume $|Y_H : C_{Y_H}(A)| < |A|^2$. Let K be of Lie type in characteristic two and assume that a Borel subgroup B is covered by $M \cap H$. Assume that all modules in $[Y_H, K]$ are of type $V(\lambda)$ for some weight λ. In case of $K \cong A_6$ we assume that $P\Gamma L_2(9)$ is not induced on K. Then there is a subgroup P of H containing M_0 such that one of the following holds*

- *(i) There is a normal subgroup P_1 containing C_PS such that $P_1/C_P \cong \Sigma_3$, or $E(P_1/C_P) \cong L_2(q)$ and Y_P involves natural modules and trivial modules, $P = P_1M_0$ and $Z(P_1) = 1$:*
- *(ii) There is a normal subgroup P_1 containing C_PS such that $P_1/C_P \cong \Sigma_3 \wr Z_2$, Y_{P_1} is an extension of up to 3 orthogonal modules and $|Y_M| \leq 8$, $P = P_1M_0$.*
- *(iii) $E(P/C_P) = K_1 \times K_2 \cong L_2(q) \times L_2(q)$, Y_P is the tensor product $V_1 \otimes V_2$ where V_i is the natural K_i–module, $i = 1, 2$. Further K_1 is not normal in P/C_P.*

In (i) - (iii) P is even minimal with respect not to be in M.

Proof: Let $K \cong G(q)$, $q = 2^n$, and assume first that K is normalized by S. Then $C_{Y_H}(K) = 1$ by 5.14 and 8.13. Let V be a irreducible KS–submodule of Y_H. Then $C_V(S) \neq 1$. As $C_H(C_V(S)) \leq M$, we get that $M \cap K$ is a maximal parabolic belonging to λ or we have $K \cong L_n(q)$ or $Sp_4(q)$ and $V = W_1 \oplus W_2$, where W_1 is the natural module and W_2 its dual. Assume first that we are not in this case. Let K_1 be the preimage of K and set $H_1 = K_1M_0$. Then $Y_M \leq Y_{H_1}$, so for what follows we may assume that $H = H_1$. Let P_1 be a minimal parabolic in K_1 not in $M \cap K$ and $P = P_1M_0$. Then we have that $E(P_1/C_P) \cong L_2(q)$ or $P_1/C_P \cong \Sigma_3$. Further we see that Y_P just involves natural modules or trivial ones. So we have (i).

Let next $K \cong L_n(q)$ or $Sp_4(q)$ and we have the natural and dual module involved, which are interchanged by S. As $B \leq M$, we get that

$C_{W_i}(S \cap K) \leq Y_M$ if $q > 2$. But as $K = \langle C_K(C_{W_1}(S \cap K)), C_K(C_{W_2}(S \cap K)) \rangle$, we get $q = 2$. As $\sigma(H) = \emptyset$, we get $n \leq 7$. For $L_n(2)$. By 3.16 there are only natural and dual modules involved. So we see that there are at most three such pairs. Let $n \neq 3$. Then there is a parabolic P in H such that $P/C_P \cong \Sigma_3 \wr Z_2$. By 3.4 we have that $C_H O_2(M_0)/C_H$ covers $O_2(M \cap K)$. Hence we have that in any nontrivial composition factor of Y_H that Y_M is contained in the centralizer $O_2(M \cap K)$. So in any such factor Y_M is of order at most two. Now P induces at most three orthogonal modules in Y_P. Further there are no trivial modules in Y_P. This is (ii).

So we have $K \cong L_3(2)$ or A_6 and then $[Y_H, K]$ is the direct sum of the natural and dual module. Let first $K \cong L_3(2)$. Then the offender A would act quadratically. Hence $Y_H/[Y_H, K] \neq 1$. Then we get $|Y_H : C_{Y_H}(A)| = |A|^2$, a contradiction. So we have that $K \cong A_6$ and H induces $P\Gamma L_2(9)$, a contradiction.

Let now K not be normalized by S. Then as $\sigma(H) = \emptyset$ we see with 1.1 and 1.2 that $K \cong L_2(q)$, $L_3(2)$, $SL_3(4)$, A_6. Let $K \cong SL_3(4)$. Then as $Z(K)$ acts nontrivially on $[Y_H, K]$ we see that $K^s \neq K$ has to act nontrivially on $[Y_H, K]$ too. In particular we have a tensor product module and so it is not an F–module. But now we see, as A normalizes K, and A does not act quadratically that $C_A(K) \neq 1$. We now see that $Y_H = [Y_H, K]$ and that $|Y_M| = 4$ as $Z(K)$ acts nontrivially on Y_M. So we get that $M \cap KK^s$ is one of the two minimal parabolic and so take as P the other one, than we get (iii).

If we have $K \cong A_6$, we have the $Sp(4,2)$–module. Now we easily see that $[Y_H, K, K^s] = 1$ for $K^s \neq K$. As we do not have quadratic action, we see that there is exactly one module in $[Y_H, K]$. And then we see that $|Y_M| = 2$, as $C_{Y_H}(\langle K^S \rangle) = 1$. So we get $\Sigma_3 \wr Z_2$ on the orthogonal module, which is (ii).

We now have $K \cong L_2(q)$ or $L_3(2)$. We see that $\langle K^S \rangle = K \times K_1$. Let $[Y_H, K, K_1] = 1$. This shows that $Y_M \cap [K, Y_H] = 1$. As the Borel subgroup is in M, this shows that $L_2(q)$ is not possible. So we have $L_3(2)$. If S induces $PGL(2,7)$ on K, we see as before that A acts quadratically. So we have that $M \cap K \cong \Sigma_4$. Now in any module involved in $[Y_H, K]$ we have that Y_M just induces a group of order 2. Now we have at most three modules involved and so as we may take the other minimal parabolic in K, we get again (ii).

So assume now that we have $[Y_H, K, K^s] \neq 1$ for $K \neq K^s$. If $K \cong L_2(q)$, we see that we have exactly two natural modules in $[Y_H, K]$, which shows that $Y_H = [Y_H, K]$ is the tensor product module for KK^s. As $B \leq M$, we have $|Y_M| = q$ Now with $\langle K, S \rangle$ we have (iii).

Let finally $K \cong L_3(2)$. Then we get exactly three natural modules in $[Y_H, K]$ and so again Y_H is the tensor product of the natural module for K with the one for K^s. Hence $M \cap K \cong \Sigma_4$. So there is some minimal parabolic P in $\langle K, s\rangle$, $P \cong \Sigma_4 \wr Z_2$, which induces at most three orthogonal modules in Y_P. This gives (ii). $\Box$

compLie

Lemma 11.6 *Let K be a component of H/C_H which is not covered by $M \cap H$ and which induces an F–module or $2F$–module with cubic but not quadratic offender A normalizing K on Y_H. If A is a $2F$–offender assume $|Y_H : C_{Y_H}(A)| < |A|^2$. Let K be of Lie type in characteristic two and assume that a Borel subgroup B containing S is not covered by $M \cap H$. Assume that all modules in $[Y_H, K]$ are of type $V(\lambda)$ for some weight λ, then there is a subgroup P of H containing M_0 such that $P = P_1M_0$, where P_1 is normal in P, $P_1/O_2(P_1)$ is cyclic of order $q-1$ and acts semiregularly on Y_P. Further $[S, P_1] \not\leq O_2(P_1)$. Either we have $q = 2^{2b}$ and P/P_1 induces a subgroup of $Gal(GF(q))$ on $P_1/O_2(P_1)$, the subgroup of order $2^b - 1$ is covered by M, or $q-1$ is prime. In both cases $Y_P = Y_M \times Y_M^g$ for some $g \in P$.*

Proof: Let first K be normalized by M_0. Replacing H by $H_1 = K_1M_0$, where K_1 is the preimage of K, we may assume that $H = K_1M_0$. As K is normalized by S, we see that $C_{Y_H}(K) = 1$. Let V be a irreducible KS–submodule of Y_H. Then $C_V(S) \neq 1$. As $C_H(C_V(S)) \leq M$, we get that $M \cap K$ is in a maximal parabolic belonging to λ or we have $K \cong L_n(q)$ or $Sp_4(q)$ and $V = W_1 \oplus W_2$, where W_1 is the natural module and W_2 its dual, or $\Omega^+(2n, q)$ and two half spin modules are involved. Suppose the former. As $B \not\leq M$, we see that $C_V(S \cap K) \not\leq Y_M$ and so there is some subgroup P_1 in that maximal parabolic of K such that P_1 is 2-closed and $P_1/O_2(P_1)$ cyclic of order $q-1$, acting semi regularly on $\Omega_1(Z(O_2(P_1)))$. As $N_G(S) \leq M$ by 7.3 there are elements in S acting nontrivially on $P_1/O_2(P_1)$. Next we show that P_1 is normalized by M_0. As $(M \cap K)P_1$ is a maximal parabolic L in K, we see that M_0 normalizes this parabolic. Let V_1 some irreducible K–module in V. As $O_2(M_0)$ is normal in S and $C_{\langle V_1^{M_0}\rangle}(O_2(M_0)) \leq Y_M$, we see that $O_2(L) \leq O_2(M_0) \cap K$. But then P_1 normalizes $O_2(M_0 \cap K)$. As P_1 does not normalize $O_2(M_0)$, otherwise it normalizes $\Omega_1(Z(O_2(M_0))) = Y_M$ by 3.4 there are elements in $O_2(M_0)$ which induce outer automorphisms on K and do not centralizes $P_1/O_2(P_1)$. We will assume that $M_0 \cap K \not\leq B$. Let the weight λ for V_1 correspond to a connected diagram. Then we have that $M \cap K = (C_M \cap K)(S \cap K)$. But as $M_0 \cap K)$ is a Sylow 2–subgroup of $C_M \cap K$, we now get that $M \cap K/O_2(M \cap K) \cong \Sigma_3$. As the Borel subgroup is not in M, we get $K \cong U_4(2)$, and V_1 is the natural module. Now $|Y_M \cap V_1| = 2$ and so there is a second module in Y_H. On the unitary module we have for A that $|V_1 : C_{V_1}(A)| \geq |A|$. In particular we have the situation of a $2F$–module and so as $|Y_H : C_{Y_H}(A)| < |A|^2$, the second one has to be an F–module with an over offender. In particular this is the orthogonal module. Now we see that $|A| \geq 8$. If $|A| = 8$, then A has to centralize in the orthogonal module a submodule of index 4, while in the unitary it centralizes one of index 16, which gives $|Y_H : C_{Y_H}(A)| = |A|^2$, a contradiction. So we have that $|A| = 16$. In particular $A \not\leq K$.But then $|V_1 : C_{V_1}(A)| \geq 2^6$, which would imply that A has to induce transvections on the orthogonal module, a contradiction. So we have that $M_0 \cap K \leq B$, if λ belongs to a connected diagram.

So assume by 3.16 or 3.29 that we have $V(\lambda_2)$ for $L_n(q)$ or $Sp(2n, q)$ or $V(\lambda_3)$ for $L_6(q)$. If $M \cap K = (C_M \cap K)(S \cap K)$, then we see as outer automorphisms of K act nontrivially on $M \cap K/O_2(M \cap K)$, that $M_0 \cap K \leq B$. So one of the two components of $M \cap K/O_2(M \cap K)$ is in C_M the other in M_0. Now the one in M_0 has to be centralized by an outer automorphism. In particular this is not a field autormorphism. So it is a diagram automorphism. But also this is impossible.

So in any case we have that $M_0 \cap K \leq B$. Hence also $M \cap K = (C_M \cap K)(S \cap K)$. In particular all modules involved in $\langle Y_M^K \rangle$ belong to the same weight. Let $U = C_{Y_H}(S \cap K)$ and $U_1 = C_U(P_1)$. Then $Z(S) \cap U_1 = 1$, so we get that $U_1 = 1$, i.e. P_1 acts semiregularly on Y_{P_1}. Set $P = P_1 M_0$, then P_1 is normal in P. Now P induces just field automorphisms on $P_1/O_2(P_1)$. Let $q = 2^{2^c r}$, r odd. Choose b minimal such that the subgroup of order $2^{2^b r} - 1$ in $P_1/O_2(P_1)$ is not in M. Then first $b \geq 1$, as the subgroup of order $2^r - 1$ normalizes a Sylow 2–subgroup and so is in M. Replace P_1 by that group and set $P = P_1 M_0$. Now the subgroup L_1 of order $2^{2^{b-1} r} - 1$ is in M and acts regularly on Y_M. Let V_1 be some irreducible K–submodule in V. Then $N_P(V_1)$ induces on $C_{V_1}(S \cap K)$ the group $P_1/O_2(P_1)$ extended by a group which induces field automorphisms. The same is true for all conjugates of V_1 under M_0. Hence we see that $[L_1, O_2(M_0)] \leq O_2(M_0)$ and so $|Y_P : C_{Y_P}(O_2(M_0))|^2 = |Y_P|$. As $C_{Y_P}(O_2(M_0)) = Y_M$, we have the assertion.

Suppose now that we have $L_n(q) \cong K$ and natural and dual modules are involved or $\Omega^+(2n, q)$ and both half spin modules are involved. Let $K \not\cong L_3(q)$. Then we get a subgroup P_2 with $P_2/O_2(P_2) \cong Z_{q-1} \wr Z_2$ and $C_V(O_2(P_2))$ is a direct sum of modules V_1 and V_2 , where $O^2(P_2)$ induces a semi regular group of order $q - 1$ on V_i, $i = 1, 2$ and V_1, V_2 are both not normal in P_2. Further we see that $O_2(M_0)$ cannot normalize V_1 as otherwise $V_1 \cap Y_M \neq 1$, and so the same is true for V_2, but then $P_2 \leq M$ and then even $K \leq M$. If now $M_0 \cap K \not\leq B$, we get as before that $O_2(M_0)$ centralizes $(M \cap K)/O_2(M \cap K)$. This is only possible if $K \cong \Omega^+(2n, q)$. By 3.16 and the assumption that $|Y_H : C_{Y_H}(A)| < |A|^2$, we get now $n = 3, 4$. In both cases we have that $[Y_H, K] = V$. But then $Y_M \cap [Y_H, K]$ is centralized by $K \cap S$. Hence $(M \cap K)' = C_M \cap K$. This shows that $S \cap K$ centralizes Y_M, a contradiction. So in any case we have that $M_0 \cap K \leq B$. Now we have as above that all the modules for KS involved in Y_H are of the same type.

Now let $x \in O_2(M_0)$ inducing the diagram automorphism on K, then set $P = ([P_2, x]M_0)C_{P_2}(x)$. If $C_{P_2}(x) \not\leq M \cap K$, then we may argue as before. So assume that $C_{P_2}(x) \leq M \cap K$. This contains a cyclic group of order $q-1$ in P_2. On this group S induces field automorphism. As now $O_2(M_0)$ must act trivially, we see that $|O_2(M_0) : O_2(M_0) \cap K| = 2$. Now set $P_1 = [P_2, x]S$. Then again $|Y_{P_1} : C_{Y_{P_1}}(O_2(M_0))|^2 = |Y_{P-1}|$. Now choose P_1 minimal. This gives the assertion.

If $K \cong L_3(q)$ or $Sp_4(q)$, then we have $[Y_H, K] = V$. Now we have $M_0 \cap K \leq B$ and so we may argue as before.

Suppose that K is not normalized by M_0. As the Borel subgroup is nontrivial we see that $K \cong L_2(q)$ or $SL_3(4)$.

Let $K \cong SL_3(4)$. Then as $Z(K)$ acts nontrivially on $[Y_H, K]$ we see that $K^s \neq K$ has to act nontrivially on $[Y_H, K]$ too. In particular we have a tensor product module and so it is not an F–module. But now we see, as A normalizes K, and A does not act quadratically that $C_A(K) \neq 1$. We now see that $Y_H = [Y_H, K]$. Further $|Y_M| \leq 4$. As $Z(K)$ is not in M, we get $|Y_M|$. Now $M \cap K$ centralizes Y_M and so $M_0 \cap K \leq B$.Let P_1 be the preimage of $Z(K)S$ and $P = P_1M_0$, then P_1 is normal in P and $|Y_P| = 4$. So we have the assertion.

Let K, K_1 be two conjugates. Suppose first that $[[Y_H, K], K_1] \neq 1$. Then $K^{M_0} = KK_1$, $[Y_H, K] = [Y_H, K_1]$, which is the tensor product of the natural module for K with the one for K_1. Now again $M_0 \cap K \leq B$ and so a Sylow 2-subgroup of KK_1 is in $O_2(M_0)$, which shows that $Y_M \leq [Y_H, K]$ as $C_{Y_H}(B) = 1$. Again $O_2(M_0) \not\leq KK_1$. Now B acts on $C_{[Y_H,K]}(S \cap K)$, a group of order q. As this group contains Y_M, we see that there is a group of order $q-1$ in $B \cap M_0$. This group is neither in K nor in K_1. Hence no element in $O_2(M_0)$ can induce a nontrivial field automorphism on K or K_1. This gives that $|O_2(M_0) : O_2(M_0) \cap K| = 2$. Hence now as above we get the situation of the lemma.

Let now $[Y_H, K, K_1] = 1$. Then we have that $[Y_H, K]$ is the natural module. Hence $[Y_H, K^{M_0}]$ is a direct sum of at most three natural modules. Again $M_0 \cap K \leq B$. Further we have that no component centralizes some $1 \neq x \in C_V(S)$. This gives that $K^{M_0} = KK_1$. But now some element in $O_2(M_0)$ interchanges K and K_1. Let $s \in O_2(M_0)$ with $K^s = K_1$. Set $P_2 = \langle B, B^s \rangle$. Then $W = [W, s]C_W(s)(S \cap KK^s)$. Now either $[W, s]$ or $C_W(s)$ is not in M. So with the same procedure as in the case of $L_n(q)$ with diagram automorphism we get the assertion. □

niceP

Definition 11.7 *Let P be one of the groups in 11.4, 11.5 or 11.6. Then we* *call P nice.*

component1

Lemma 11.8 *Let K be a component of H/C_H which is not covered by $M \cap H$ and which induces an F–module on $[Y_H, K]$ with offender A normalizing K. Then there is a nice P.*

Proof: By 3.16 we have that all modules for groups of Lie type are of type $V(\lambda)$ for some weight λ. Then the assertion follows from 11.4, 11.5 and 11.6. Recall that $P\Gamma L_2(9)$ does not admit an F–module.

fit

Lemma 11.9 *Let $F = F(H/C_H)$ and assume that there is some 2-group $A = \langle a \rangle$ which induces some transvection on Y_H such that $[A, F]$ is not covered by M. Then there is a subgroup P of H containing M_0 such that one of the following holds*

- *(i) There is a normal subgroup P_1 containing $C_P S$ such that $P_1/C_P \cong \Sigma_3$, Y_P is the natural modules and $P = P_1 M_0$.*
- *(ii) There is a normal subgroup P_1 containing $C_P S$ such that $P_1/C_P \cong \Sigma_3 \wr Z_2$, Y_{P_1} is the orthogonal module and $|Y_M| = 2$, $P = P_1 M_0$.*

In particular we have a nice P.

Proof: We may assume that F is a 3–group. As $F \not\leq M$, we have $C_{Y_H}(F) = 1$. There is some $b \in F$ such that $B = \langle a, a^b \rangle \neq A$ and $B \not\leq M$. We have that $|Y_M : C_{Y_M}(a^b)| \leq 2$. This gives with 5.14 that $|Y_M| = 2$. This shows $M_0 = N_G(S)$. Further $B/B \cap C_H \cong \Sigma_3$. Without loss of generality we may assume $a \in S$. Suppose there is another $h \in F$ with $a^h \notin M$. Then we have that $|Y_H : C_{Y_H}(\langle a, a^b, a^h \rangle)| \leq 8$. This shows that $\langle a, a^b \rangle C_H = \langle a, a^h \rangle C_H$. This shows that $\langle b \rangle C_H/C_H = [F, A]C_H/C_H$. Now set $F_1 = \langle [F, a]^{M_0} \rangle C_H/C_H$. Let $|F_1| = 3^n$, then F_1 is generated by n conjugates of $[F, a]$. But for any of them there was some element in S inverting exactly one and centralizing all the others. Let C be a critical subgroup in the preimage of F_1. Then 2.2 shows that $n \leq 4$. If $n = 4$, then C is extraspecial. But a induces a transvection on $C/Z(C)$, a contradiction. So $n \leq 3$. Let $n = 3$. Then M_0 induces on F_1 a subgroup of $GL(3, 3)$. As $M_0 = N_G(S)$, we see that M_0 induces $Z_2 \wr Z_3$. But then $|C_{Y_H}(S)| \geq 8$, contradicting $|Y_M| = 2$. So we have that $n \leq 2$.

Let $n = 1$. Now set $P_2 = BM_0C_H$. Then we have that $|Y_{P_2}| = 4$. Hnece we find some $P_1 \leq P$ with $S \leq P_1$ and $P_1/O_2(P_1) \cong \Sigma_3$. Set $P = P_1 M_0$, then we have (i).

Let $n = 2$. Set $P_2 = F_1 M_0 C_H$, then $|Y_{P_2}| = 16$. Further P_2 induces $O^+(4, 2)$ on Y_{P_2}. So we get P_1 with $P_1/C_{P_2} \cap P_1 \cong \Sigma_3 \wr Z_2$. With $P = M_0 P_1$ we have (ii). □

b21

Hypothesis 11.10 *There is some* $g \in G$ *such that* $1 \neq [Y_H, Y_H^g] \leq Y_H \cap Y_H^g$, *further* $Y_H \leq O_2(M)$.

Set $A = Y_H^g C_H/C_H$. Then A induces an F–module offender on Y_H. By 3.24 A fixes some component K on which it induces an F–module offender, or it induces an F–module offender on $F(H/C_H)$.

KM

Lemma 11.11 *Assume 11.10. Let* K *be a component of* H/C_H *on which* A *induces an* F*–module offender. Then there is also some component* $K \not\leq (M \cap H)/C_H$ *on which* A *induces some* F*–module offender.*

Proof: Suppose false. Let first K be not normal. Then we get with 1.1, 1.2 and 3.16 that $K \cong L_2(q)$, $L_3(2)$, $SL(3,4)$, $3A_6$, $3A_7$, $Sp_4(2)'$ or A_7. If $Z(K) \neq 1$ we have that $Z(K) = Z(K^h)$, $h \in H$. This with 3.16 shows that we must have $3A_6$ on the $Sp(4,2)$–module or $3A_7$ on the 4–dimensional module or on the permutation module. Hence in any case $[K, Y_H]$ is quasi irreducible and so $[K, Y_H, K^h] = 1$ for $[K, K^h] = 1$. If KK^h is normal, both components are conjugate under S and so both are in M. If $[Y_M, K] \neq 1$, then we have $[Y_H, K] \leq Y_M$ and we get a contradiction with 11.3. So we have $[Y_M, K] = 1$. Let $T_1 = N_S(K)$. If $C_{[Y_H,K]}(T_1) \not\leq C(K)$ then $C_{[Y_H,KK^h]}(S) \not\leq C(K)$ contradicting $C_{Y_H}(S) \leq Y_M$. Now with 3.38 we get $K \cong L_3(2)$. But then $C_{[Y_H,KK^h]}(KK^h)$ is of order at most 4, which shows that this group is centralized by H again contradicting 11.3. So we must have a third conjugate and then $K \cong L_2(q)$ or $L_3(2)$ and we have exactly three conjugates. Let $\langle K^H \rangle = K \times K_1 \times K_2$, where S normalizes KK_1 and K_2. So K_2 centralizes $[KK_1, Y_H]$. We see that K_2 centralizes some element in $Z(S)^\sharp$ and so by 8.11 K_2 is in M. If $K \cong L_2(q)$ we see again that $[Y_H, \langle K^H \rangle] \leq Y_M$, which contradicts 11.3. Hence $K \cong L_3(2)$. Now H induces Σ_3 on $\langle K^H \rangle$. By 5.16 we may assume that $N_H(K)$ and $N_H(K^h)$ for some h with $K \neq K^h$ both are in M. But H is generated by these normalizers, a contradiction.

Let next $[K, Y_M] = 1$. Now let T be a Sylow 2-subgroup of $C_{H/C_H}(K)$ and set $V_H = C_{U_H}(T)$. Then we get that also this group is an F- module for K with offender A. Let W be a quasi irreducible submodule W, then $C_W(S \cap N(W)) \leq C_W(K)$ as K is normal in H and $\Omega_1(Z(S)) \leq Y_M$. By 3.38 we get that $K \cong L_3(2)$ or A_n and we just have $W/C_W(K)$ is the natural module. This now shows that $W = [Y_H, K]$ and so $C_{[Y_H,K]}(K) \leq Y_M$, contradicting 11.3. So we have

(∗) $[K, Y_M] \neq 1$ for all M which share a Sylow 2–subgroup with H

Let U_H be the sum of all quasi irreducible submodules of $[K, Y_H]$. Let $[Y_M \cap U_H, K] = 1$. Let U be some submodule in Y/U_H for K, which is covered by Y_M. Let U_1 be the preimage. Then in $U_1/C_{U_H}(K)$ we have $U_H(Y_M \cap U_H)/C_{U_H}(K)$. But this gives $U_1 = U_H$, a contradiction. So we have $Y_M \leq U_H$ and then the contradiction $[Y_M, K] = 1$. Sü we have $Y_M \cap U_H \not\leq C_{U_H}(K)$. As $K \leq M$ then $\langle (U_H \cap Y_M)^K \rangle \leq Y_M$, we get that some of these submodules are in Y_M. Let $A = A_1 \times A_2$ with $A_1 = C_A(K)$. Let V be some submodule, which is not normalized by A_2, so assume $V^a \neq V$ for some $a \in A_2$. Now $(V \times V^a)^b$ contains $[V, a]$ for all $b \in A_2$. As $[V, a]$ is not a K–submodule, we get that A_2 normalizes $V \times V^a$. In particular $[V, a] = [V, A_2]$ and $C_V(a) = 1$ for all $1 \neq a \in A_2$. In particular $C_{C_V(S \cap K)}(a) = 1$. This shows $|A_2| \leq |C_V(S \cap K)|$. But as A_2 is an F–module offender, we have $[V, S \cap K] = 1$, a contradiction. So we have that A_2 normalizes each submodule, we get that A_2 normalizes $Y_M \cap U_H$. The same does $M \cap H$. Hence by 11.3 we now get $A_2 \leq M$ and then $A \leq M$. Let now $a \in A_1$. If $Y_M \cap Y_M^a \neq 1$, as A_2 normalizes and does not centralizes $Y_M \cap U_H$ and A acts quadratically. Hence in any case $A \leq M$. As $K \leq M$ and K is normal we even have that $A \leq M^g$ for all $g \in H$.

Suppose first that $U_H/C_{U_H}(K)$ is irreducible. In particular $U_H = (U_H \cap Y_M)C_{U_H}(K)$. Then for $h \in H$ we get $Y_M \cap Y_M^h \neq 1$. By 5.13 we have that $M = M^h$. But this yields $H \leq M$, a contradiction. So we have that U_H involves at least two nontrivial irreducible K–modules. Assume first that U_H and so Y_H induces an F–module offender on Y_{H^g} as well. Suppose that Y_H induces an F–module offender on $F(H^g/C_{H^g})$. Then A induces transvections on U_H by 4.5, which contradicts the fact that U_H involves at least two nontrivial modules. So we have that $[U_H, F(H^g/C_{H^g})] = 1$. Hence it induces an F–module offender on some component of H^g. We will show that $Y_H \leq M^x$ for all M^x which share a Sylow 2–subgroup with H^g. As seen above this is true if $[Y_H, K^g] \neq 1$. So assume that $[Y_H, K^g] = 1$. Then there is a second component L in H/C_H such that $[Y_H, L^g] \neq 1$ and L induces an F–module in Y_H. Suppose $m_3(K) = 1$. Then by 1.1 $K \cong L_2(q)$ or $L_3(2)$. As U_H is not irreducible we get $K \cong L_3(2)$ and we have exactly two modules in U_H. But now $[L, U_H] = 1$ and so $[Y_H, Y_{M^x}] = 1$ for all M^x which share a Sylow 2–subgroup with H^g. So assume now thet $m_3(L) = 1$ and $[U_H, L] \neq 1$. Then again $L \cong L_3(2)$ and there are exactly two L–modules. But then $|V| = 4$ for V an irreducible K–submodule of U_H, a contradiction. So we are left with $K, L \in \{SL_3(4), Sp(4,2)', 3A_6, A_7\}$. As U_H involves two irreducible modules, we get that $K \cong SL_3(4)$ and in U_H there are exactly two natural modules. As we may assume $[U_H, L] = 1$ and $Z(K)$ acts fixed point freely on U_H, we get that $Z(L) = Z(K)$. In particular we see that L induces at least three irreducible m odules, which contradicts the fact that L induces an F–module on Y_H. So in all cases we have that $Y_H \leq M^x$ for all M^x which share a Sylow 2–subgroup with H^g.

Let Y be the subgroup of Y_H generated by all the Y_M^x where M^x

and H^g share a Sylow 2–subgroup. Assume $[Y, Y_H] = 1$. Then also $[[F^*(H^g/C_{H^g}), Y_H], Y] = 1$. This shows that some component which induces an F–module has to centralize Y, which contradicts $(*)$.

Now we have some $Y_M^x \leq Y_H^g$, such that H^g and M^x share a Sylow 2-subgroup and $[Y_M^x, Y_H] \neq 1$. By 7.1 we have that $[Y_M, Y_M^x] = 1$ and so $[Y_M^x, K] \leq C_H$ by $(*)$. By quadratic action we see that $1 \neq L = [Y_M^x, F^*(H/C_H)]$ centralizes $[K, Y_H]$. Hence $[L, Y_H]$ is centralized by K. But as $Y_H \leq M^x$, we get that $1 \neq [[C_{Y_H}(K), L], Y_M^x] \leq Y_M^x$. By 5.14 we get that $H \cap M^x$ covers K. By assumption we have that $Y_{H^g} \leq O_2(M^x)$, which now contradicts $[K, Y_{H^g}] \neq 1$.

So we have that U_H does not induce an F–module offender on Y_{H^g}. By 3.16 and the fact that $\sigma(H) = \emptyset$, we get $K \cong L_n(2)$, $5 \leq n \leq 7$, or $K = L_4(q)$, q even. Now in any case K is some component of M_0/C_{M_0} as one of the modules in U_H is in Y_M. Then by 5.17 we get some ρ in K, $o(\rho) = 3$ or in case of $L_4(q)$ we might have $o(\rho)$ divides $q - 1$ with $N_G(\langle\rho\rangle) \leq M$. Let $H_0 = N_H(S \cap C_H)$. As $\sigma(H) = \emptyset$, we see that $[K, H_0 \cap C_H] \leq S \cap C_H$. So we have that $C_{H_0/S\cap C_H}(K)$ is covered by M. If $K \not\cong L_4(q)$, we have that $H_0/S \cap C_H = SKC_{H_0/S\cap K}(K)/S \cap K$. As $H \not\leq M$, this shows $K \cong L_4(q)$ and we have fieldautomorphisms involved. If $o(\rho)$ divides $q - 1$, then field automorphisms do not induce new conjugacy on the groups of order $o(\rho)$. If $o(\rho) = 3$ and 3 divides $q + 1$, then any conjugacy class of elements of roder 3 in $L_4(q)$ intersects nontrivially $L_4(2)$ which is cntralized by the field automorphisms. Hence also no new fusion happens. This shows that in any case $H_0/S \cap K = KN_{H_0/S\cap K}(\langle\rho\rangle)$, a contradiction. □

fitM

Lemma 11.12 *Assume 11.10. Then either there is some component K of H/C_H on which A induces an F–module offender and $K \not\leq (M \cap H)/C_H$, or A induces some F–module offender on a Sylow 3–subgroup F of $F(H/C_H)$ and we have that $[A, F] \not\leq M$.*

Proof: By 11.11 we just have to treat the case that A induces an F–module offender on $F(H/C_H)$ and it does not induce one on any component. Now as $m_3(H) \leq 3$, we get with 2.1 that $|A : C_A(F(H/C_H))| \leq 8$ and there is a group $D = D_1 \times D_r$ of r dihedral groups of order 6, $2^r = |A : C_A(F)|$ induced on Y_H, where A is a Sylow 2–subgroup of D. Now $|Y_H : C_{Y_H}(D)| \leq 2^{2r}$. Let $A = A_1 \times A_2$, $A_2 = C_A(F)$. Then A_2 does not induce an F–module offender by assumption. As A_1 induces a sharp offender and A is an F–module offender, we see $A_2 = 1$. This shows that A is generated by transvection. Suppose $[F, A] \leq M$. Let W be an irreducible $\langle [F, A], M_0 \rangle$–submodule of $[F, Y_H]$. Then we have that $W \cap Y_M \neq 1$ and so $W \leq Y_M$. Let $a \in A \setminus M$, then we get $C_W(a) = 1$. As A is generated by transvections we get $|W| = 2$, a contradiction. So we have that $A \leq M$. We now also have that $|Y_M| \geq 4$. By 4.5 Y_H is generated by elements inducing transvections on Y_H^g hence centralizing some nontrivial element in Y_{M^x} for any M_x which shares a Sylow 2–subgroup with H^g. This gives that $Y_H \leq M^x$. Suppose that Y_H centralizes all these Y_{M^x}. Then Y_H is in the normal subgroup of H^g centralizing all M^x. If there is some component in H^g/C_{H^g} on which Y_H acts nontrivially then this component is in M^x. This contradicts 11.11. So Y_H acts on F^g. But F^g does not centralize Y_M^x. So there is some M_x such that $[Y_H, Y_{M^x}] \neq 1$. But $[Y_M, Y_M^x] = 1$ by 7.1, so we have that $[Y_M^x, W] = 1$, i.e. $[[F, A], Y_M^x] = 1$. But $C_A([F, A]) = 1$. So we have that $[F, A] \not\leq M$. $\square$

12 The group H, the amalgam case

We are going to set up some amalgam. We fix the notation of the previous chapter. In particular H has the same properties as before.

$O2 = 1$

Lemma 12.1 *Let $W_M \leq M$, such that $H \cap M$ is maximal in W_M. Then $O_2(\langle W_M, H\rangle) = 1$.*

Proof: Set $H_1 = \langle W_M, H\rangle)$. Let $O_2(H_1) \neq 1$. We have that $H_1 \not\leq M$ and by 8.14 $C_{H_1}(O_2(H_1)) \leq O_2(H_1)$. By 3.4(iii) we have that $Y_H \leq Y_{H_1}$. The maximal choice of Y_H now gives that $Y_H = Y_{H_1}$. But then with 11.1 we get $H_1 \leq H$, a contradiction. □

Our aim is to choose some appropriate L_M in such a way that $Y_H \not\leq Z(O_2(L_M))$. Suppose there is some W_M as in 12.1 with $Y_H \leq Z(O_2(W_M))$. Then $Y_H \leq Z(O_2(M))$. Set $V_M = \langle Y_H^M\rangle$. Let $S \cap C_M(V_M) \leq T \leq S$ with $TC_M(V_M)/C_M(V_M) = O_2(M/C_M(V_M))$. Then consider $U_M = N_M(T)$. As $Y_M < V_M$, we have that $T \not\leq C_M(V_M)$ by 3.4. So we have that $Y_H \not\leq Z(O_2(U_M))$ since $V_M = \langle Y_H^{U_M}\rangle$.

$O2neu$

Lemma 12.2 *We have $O_2(\langle U_M, H\rangle = 1$.*

Proof: Let P be a Sylow p–subgroup of C_H, $p \in \sigma(M)$. By 5.7 we have that M is not exceptional with respect to p. Now $C_H \leq M$. If $N_G(P) \leq M$ then by Frattini we have the contradiction $H = C_H N_H(P) \leq M$. So we may assume that either P is cyclic or $p = 3$ and a Sylow 3–subgroup of G is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. So assume first that P is cyclic. Then the same is true for a Sylow p–subgroup P_1 of $C_M(V_M)$. Suppose that $P_1 \neq 1$, then $\Omega_1(P_1)$ is normal in a Sylow p-subgroup of M and so $N_G(\Omega_1(P_1)) \leq M$, recall that by 5.2 M is not exceptional. But as $C_M(V_M) \leq C_H$, we get that $\Omega_1(P_1) = \Omega_1(P)$ and again $H = C_H N_H(\Omega_1(P)) \leq M$. So we have seen that $C_M(V_M)$ is a p'–group. As $M = C_M(V_M)N_M(T) = C_M(V_M)U_M$, we have that U_M contains a good E. In particular $O_2(\langle U_M, H\rangle) = 1$.

Let now a Sylow 3-subgroup of M isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. Now P is elementary abelian of order 9 and contains some element ρ with $N_G(\langle\rho\rangle) \leq M$. In particular $N_H(P) \not\leq N_H(\langle\rho\rangle)$. This shows that in $N_G(P)$ all subgroups of order three in P are conjugate. In particular $N_G(P)/C_G(P) \cong GL_2(3)$ or $SL_2(3)$. As $C_H C_H(P)S \leq M$, we see that $N_H(P)/C_H(P)$ contains $SL_2(3)$. But now in H some E is good in M^g for certain $g \in G$, which contradicts 5.4. □

If $Y_H \not\leq Z(O_2(M))$, then set $L_M = W_M$ as in 12.1 otherwise $L_M = U_M$ as in 12.2. Hence in both cases we found a subgroup L_M in M such that

$O_2(\langle L_M, H\rangle) = 1$ and $Y_H \not\leq Z(O_2(L_M))$. Further in the first case obviously $M_0 \leq L_M$. In the second case let $C_M/C_M(V_M) \cap O_2(M/C_M(V_M) \neq O_2(M/C_M(V_M))$. Then $O_2(M/C_M) \neq 1$, a contradiction. So we have that $T \leq S \cap C_M$. Then we get that $M_0 \leq N_M(T)$ and so again $M_0 \leq L_M$. Hence in any case $M_0 \leq L_M$.

Now we set $R_H = H_0 = N_H(S \cap C_H(Y_H))$. We have that $C_H(Y_H) \leq C_M$ and so $S \cap C_H(Y_H) = S \cap C_M \cap C_H(Y_H)$, which gives $M_0 \leq R_H$. As $H = R_H C_H$, we have $R_H \not\leq M$. Further by 3.4 we have $Y_{R_H} = Y_H$. Hence by the maximality of Y_H we have that $O_2(\langle L_M, R_H\rangle) = 1$. Now choose R_M minimal in L_M containing $R_H \cap L_M$ such that $O_2(\langle R_M, R_H\rangle) = 1$. Let U be a maximal subgroup of R_M which contains $R_H \cap R_M$. Set $X = \langle U, R_H\rangle$. Then we have $O_2(X) \neq 1$ and by 3.4 we have $Y_H \leq Y_X$. As $X \not\leq M$, the maximality of Y_H gives us that $Y_X = Y_H$. But then by 11.1 we have $X \leq H$. This gives that $H \cap R_M$ is the only maximal subgroup of R_M containing $R_H \cap R_M$.

As $M_0 \leq L_M \cap R_H$, we get that $M_0 \leq R_M$.

As $R_M \leq L_M$, we get $O_2(L_M) \leq O_2(R_M)$ and then $Z(O_2(R_M)) \leq Z(O_2(L_M))$. So $Y_H \not\leq Z(O_2(R_M))$.

We now consider the amalgam $\Gamma(R_M, R_H)$. This has the following properties

(i) $Y_{R_M} \leq Y_M \leq Y_H = Y_{R_H}$.

(ii) $Y_H \not\leq \Omega_1(Z(O_2(R_M)))$.

(iii) Any 2–element in R_H centralizing Y_H is contained in $O_2(R_H)$

(iv) $H \cap R_M$ is the unique maximal subgroup in R_M containing $R_M \cap R_H$.

(v) $H = C_H R_H$

(vi) $M_0 \leq R_M \cap R_H$.

Let $b = b_\Gamma$. We will assume that $Y_H \leq O_2(M)$.

bH

Lemma 12.3 *If b is even, then $b = b_{R_H}$.*

Proof: This follows from $Y_M \leq Y_{R_H}$ by (i) □

nontrivial

Lemma 12.4 *Let b be even and let (R_H, R_{H_α}) be a critical pair, then $1 \neq [Y_H, Y_{H_\alpha}]$.*

Proof: This follows from the property (iii) of the amalgam. $\square$

Pb2

Lemma 12.5 *Assume that* $Y_H \leq O_2(M)$. *If* b *is even, then there is a nice* P.

Proof: By 12.4 we have that 11.10 is satisfied. By 11.11, 11.12 and 11.8 we just have to treat the case that A induces an F–module offender on $F(H/C_H)$. We may assume that it does not induce one on any component. Otherwise we have one of the cases before. Now as $m_3(H) \leq 3$, we get with 2.1 that $|A : C_A(F(H/C_H))| \leq 8$. Hence we may assume that $F = F(H/C_H)$ is a 3–group. Further by 4.5 we have that Y_H induces transvections on Y_H^g and so we may assume that A contains some transvections on Y_H. Then the assertion follows with 11.9. $\square$

From now on we assume that b_Γ is odd and further we assume that $Y_H \leq O_2(M)$. Under this assumptions we will show that there is also some nice P.

The first aim of this chapter is to show that we have a nice P or $Y_H \leq O_2(C_G(x))$ for all $x \in Y_H^\sharp$.

2cent

Lemma 12.6 *We have* $b_\Gamma > 1$ *and* $Y_H \leq O_2(C_G(x))$ *for all* $1 \neq x \in Z(S) \cap Y_H$.

Proof: This follows from $Y_H \leq O_2(M)$ and 8.11. $\square$

Htriv

Lemma 12.7 *If* $1 \neq [Y_H, Y_{H^g}] \leq Y_H \cap Y_{H^g}$, $g \in G$, *then there is a nice* P.

Proof: This follows from 11.8, 11.9, 11.11 and 11.12. $\square$

So for the remainder of this chapter we assume that $[Y_H, Y_{H^g}] = 1$ for $[Y_H, Y_{H^g}] \leq Y_H \cap Y_{H^g}$.

Sylow

Lemma 12.8 *Let* U *be some subgroup containing* $C_S(Y_H)$. *Let* T *be a Sylow 2–subgroup of* U *which contains* $C_S(Y_H)$. *Then* $N_U(T) \leq H$.

Proof: Let $N_U(T) \not\leq H$. Set $W = \langle R_H, N_U(T)\rangle$. Let $O_2(W) \neq 1$. As $S \leq W$, we have with 8.14 that $F^*(W) = O_2(W)$. Then we have by 3.4 that $Y_W \geq Y_H$. But the maximal choice gives $Y_H = Y_W$ and by 11.1 we have $W \leq H$, a contradiction. So we have an amalgam $(R_H, N_U(T))$. As $O_2(R_H) \leq C_S(Y_H) \leq T$ we have that for this amalgam the parameter b is even, so we have some g with $1 \neq [Y_H, Y_H^g] \leq Y_H \cap Y_H^g$, a contradiction. $\square$

Hconstrain

Hypothesis 12.9 There is a subgroup L of G such that

(i) $C_G(O_2(L)) \leq O_2(L)$

(ii) $Y_H \not\leq O_2(L)$

(iii) $C_S(Y_H) \leq L$

Until further notice we will work under 12.9. In fact, if there is some $1 \neq x \in Y_H$ with $F^*(C_G(x)) = O_2(C_G(x))$ and $Y_H \not\leq O_2(C_G(x))$, then there is such a group L. We will show that the existence of such a group L yields the existence of a nice P.

Assume 12.9. Then there is even some L with

(i) $|L \cap H|_2$ is maximal with respect to 12.9(i)-(iii)

(II) L is minimal with respect to 12.9(iv)

We denote by T a Sylow 2-subgroup of $L \cap H$. Without loss $T = S \cap L$.

*Sylow*1

Lemma 12.10 *We have that T is a Sylow 2–subgroup of L.*

Proof: This follows from 12.8. □

parT

Lemma 12.11 *We have that T acts transitively on the components of $L/O_2(L)$, Y_H normalizes any component of $L/O_2(L)$ and acts quadratically on $O_2(L)$.*

Proof: The first assertion is related to the minimal choice of L. As Y_H is normal in T, we have that it acts quadratically on $O_2(L)$ and as $Y_HO_2(L)/O_2(L)$ is an abelian normal subgroup of $T/O_2(L)$ we get with 3.24 that it has to normalize any component or $\langle K^{Y_H}\rangle \cong L_2(q) \times L_2(q)$, q a power of 2, where K is a component. Let B be a Borel subgroup of the preimage of that group. Then $Y_H \not\leq O_2(B)$. But this contradicts the minimal choice of L. □

squad

Lemma 12.12 *Let $T \leq P < L$ be a proper parabolic of L. Set $\langle Y_H^P\rangle = A$. Then A is elementary abelian and acts quadratically on $O_2(L)$.*

Proof: By the minmal choice we have that $Y_H \leq O_2(P)$. Let $g \in P$ with $1 \neq [Y_H, Y_H^g]$. As Y_H is normal in $O_2(P)$, we have a contradiction. So A is elementary abelian. As $[O_2(L), A] \leq A$, we have quadratic action. □

noncentral

Lemma 12.13 *We have that* $Y_L \neq \Omega_1(Z(T))$.

Proof: Suppose that $L \leq N_G(\Omega_1(Z(T)))$. If $S = T$, then by 8.11 we have that $L \leq M$. But $Y_H \leq O_2(M)$ and so $Y_H \leq O_2(L)$, a contradiction. So we have that T is not a Sylow 2-subgroup of G. Let $T_1 = N_S(\Omega_1(Z(T)))$. Then $T_1 > T$. As $Y_H \leq T$, we have that $\Omega_1(Z(S)) \leq \Omega_1(Z(T))$. In particular $C_G(\Omega_1(Z(T))) \leq M$ by 8.14. Now the $A \times B$–lemma shows that $E(N_G(\Omega_1(Z(T)))) = 1$. As $Y_H \not\leq O_2(N_G(\Omega_1(Z(T))))$ we get a contradiction to the choice of L as $|N_G(\Omega_1(Z(T))) \cap M|_2 > |L \cap M|_2$. □

transvec

Lemma 12.14 *If there are elements in* Y_L *which do induce transvections on* Y_H*, then we have a nice* P.

Proof: Let $x \in Y_L$ inducing a transvection on Y_H and assume that K is some component of H/C_H with $[x, K] \neq 1$. Then by 3.16 we see that $K/Z(K) \cong A_n$, $L_n(2)$, $Sp(2n, 2)$ or $\Omega^{\pm}(2n, 2)$. Further we see that $[Y_H, K]$ is quasi irreducible. Let first K be covered by $M \cap H$. If $Y_M \cap [K, Y_H] \not\leq C_{Y_H}(K)$, we have that $[Y_H, K] \leq Y_M$. Further we see that $E(H/C_H)$ normalizes $[K, Y_H]$, so by 11.3 $E(H/C_H)$ is covered by M. Set $\tilde{K} = \langle K^{M_0} \rangle$. If $\tilde{K}$ is normalized by H/C_H, we get with 11.3 that $H \leq M$, as $[\tilde{K}, Y_H] \leq Y_M$. So we have that $\tilde{K}$ is not normalized by H/C_H. In particular $\langle K^H \rangle$ has at least three components. This now shows that $K \cong A_5$ or $L_3(2)$ and $[Y_H, K]$ is the natural module. But then we see that $C_{[Y_H, \langle K^H \rangle]}(S)^{\langle K^H \rangle} = [Y_H, \langle K^H \rangle]$ and so by 8.13 we have that $[Y_H, \langle K^H \rangle] \leq Y_M$. But then again with 11.3 we would get $H \leq M$.

We may assume that $Y_M \leq C_{Y_H}(K)$. Then by 1.1 we have that $K \cong L_3(2)$ or A_n, $n \geq 8$. But the former does not admit transvections and in the latter K is normal and so also $1 \neq C_{[Y_H, K]}(K) \leq Y_M$ is normal in H contradicting 11.3 again. So K is not in M and we can quote 11.8.

So suppose that x acts on a Sylow p–subgroup P of $F(H/C_H)$. Then $p = 3$. By 5.15 we get that $[x, P] \not\leq M$. Then the assertion follows with 11.9. □

From now on we will assume that no element from Y_L induces transvections on Y_H.

trans

Lemma 12.15 *We have that* L *is nonsolvable. Further let* R *be a component of* $L/O_2(L)$ *and* $\tilde{Y}$ *be a complement in* Y_H *to* $C_{Y_H}(R)$*, then* $|\tilde{Y}| \geq 4$.

Proof: If L is solvable, then by by 4.5 we have that Y_L induces transvections on Y_H a contradiction. The same would be true if $|\tilde{Y}| = 2$. □

notM

Lemma 12.16 *Let A be an elementary abelian subgroup of H with $[y, A] = [Y_H, A]$ for all $y \in Y_H \setminus C_{Y_H}(A)$ and $|Y_H : C_{Y_H}(A)| \le |A|$. Suppose that K is a component of H/C_H with $[A, K] \neq 1$. Then we have a nice P.*

Proof: By 3.20 $[K, Y_H]$ is quasi irreducible if $[A, K] \le K$. In that case we get the assertion with 11.7 if K is not covered by M. So assume it is covered by M. Assume further that K is normal in H. Then with 11.3 we get that $Y_M \le C_{Y_H}(K)$. As $C_{Y_H}(S) \le Y_M$, we see with 3.38 that $K \cong A_n$ or $L_3(2)$. In both cases we get $Y_M = C_{[Y_H,K]}(K)$ and so by 11.3 $H \le M$. So we may assume that K is not normal in H. If KK^h is normal, both components are conjugate under S and so both are in M. If $[Y_M, K] \neq 1$, then we have $[Y_H, KK^h] \le Y_M$ and we get a contradiction with 11.3. So we have $[Y_M, K] = 1$. Let $T_1 = N_S(K)$. If $C_{[Y_H,K]}(T_1) \not\le C(K)$ then $C_{[Y_H,KK^h]}(S) \not\le C(K)$ contradicting $C_{Y_H}(S) \le Y_M$. Now with 3.38 we get $K \cong L_3(2)$. But then $C_{[Y_H,KK^h]}(KK^h)$ is of order at most 4, which shows that this group is centralized by H again contradicting 11.3. So we must have a third conjugate and then $K \cong L_2(q)$ or $L_3(2)$ and we have exactly three conjugates. Let $\langle K^H \rangle = K \times K_1 \times K_2$, where S normalizes KK_1 and K_2. So K_2 centralizes $[KK_1, Y_H]$. We see that K_2 centralizes some element in $Z(S)^\sharp$ and so by 8.11 K_2 is in M. If $K \cong L_2(q)$ we see again that $[Y_H,{}^{\langle} K^H\rangle] \le Y_M$, which contradicts 11.3. Hence $K \cong L_3(2)$. Now H induces Σ_3 on $\langle K^H \rangle$. By 5.16 we may assume that $N_H(K)$ and $N_H(K^h)$ for some h with $K \neq K^h$ both are in M. But H is generated by these normalizers, a contradiction.

So we are left with $[K, A] \not\le K$. By 3.24 we then have that $K \cong L_2(2^n)$. But then also by 3.24 A cannot induce an F–module offender. □

Wdual

Lemma 12.17 *Let $A \le Y_L$ with $|AC_H/C_H| > 2$ acting on Y_H such that $[y, A] = [Y_H, A]$ for all $y \in Y_H \setminus C_{Y_H}(A)$ and $|Y_H : C_{Y_H}(A)| \le |A|$. Then we have a nice P.*

Proof: By 3.21 we get that $[A, F(H/C_H)] = 1$. Let K be a component of H/C_H with $[A, K] \neq 1$. Then the assertion follows with 12.16. □

Wlemma

Lemma 12.18 *Let R be a component of $L/O_2(L)$. Let $W \le Y_L$ with W being minimal such that $[R, W] \neq 1$. If $C_W(R) = 1$ then we have a nice P.*

Proof: Let $\tilde{Y}$ be as in 12.15. As $\tilde{Y}$ acts quadratically we get that R is a group of Lie type in characteristic two or by 3.26 that R is alternating, $U_4(3)$ or some sporadic group.

If $R/Z(R) \cong U_4(3)$. Let X be the centralizer of an involution. Then by

12.12 we have that $\langle Y_H^X \rangle$ is abelian, contradicting that it has to contain $O_2(X)$, an extraspecial group.

If $R/Z(R)$ is sporadic, then since $\langle \tilde{Y}^P \rangle$ is quadratic for all proper parabolics P by 12.12, we get with 3.27 that $R \cong 3M_{22}$ on the 12–dimensional module and $|\tilde{Y}| = 4$. Then $\langle C_R(y) \mid 1 \neq y \in \tilde{Y} \rangle \cong 2^4 3A_6$. As $|C_W(y)| = 2^8$, this shows that $C_W(y) \neq C_W(\tilde{Y})$ for $1 \neq y \in \tilde{Y}$. Now choose $x \in C_W(y) \setminus C_W(\tilde{Y})$ for some $1 \neq y \in \tilde{Y}$. Then x induces a transvection on Y_H, a contradiction.

Let $R \cong A_n$. As $\tilde{Y} \leq O_2(P)$ for any proper parabolic P, we see that it is conjugate to $\langle (12)(34), (13)(24) \rangle$, which is not quadratic on the permutation module. So we have the spin module. Now we have $[y, W] = [\tilde{Y}, W]$ and $C_W(y) = C_W(\tilde{Y})$ for all $1 \neq y \in \tilde{Y}$. In particular we have a strong dual F–module.

Let next $R \cong 3A_6$ on the 6–dimensional module. Then by 12.12 we must have outer automorphisms on A_6 which do not induce Σ_6. Hence we have both 6–dimensional modules involved. In particular there is one W_1 such that $[W_1, y] = [W_1, \tilde{Y}]$ for all $1 \neq y \in \tilde{Y}$.

Let $R(Z(R)$ be a group of Lie type. With 3.28 we see that $\tilde{Y}$ is contained in some root group and so in some $L_2(q) = K_1$, or $R \cong Sz(q)$. Suppose the former. Then by 3.50 we have that K_1 induces a natural submodule W_1 in W as $C_W(R) = 1$. Hence again we have that W_1 is a strong dual offender on Y_H. If $R \cong Sz(q)$, then by 3.50 we have that W is the natural module and again W is a strong dual offender.

Now we get the assertion with 12.17

□

YLF

Lemma 12.19 *Let the notation be as in 12.18. Assume that $C_W(R) \neq 1$. Then we have that T is not a Sylow 2–subgroup of G and Y_L is an F–module.*

Proof: Let first T be a Sylow 2–subgroup of G. Then we may assume that $T = S$. But then by 12.18 we get some $x \in Z(S)^\sharp$ with $\langle x \rangle$ normal in L, contradicting 8.11.

Now let $\Omega_1(Z(T))$ or $J(T)$ be normal in L. Let X be one of these groups with X normal in L. As $T < S$, we have that $N_S(X) > T$. As X contains $\Omega_1(Z(S))$ the $A \times B$ – lemma and 8.14 imply that $E(N_G(X)) = 1$. But as $|N_G(X) \cap H|_2 > |L \cap H|_2$ so $Y_H \leq O_2(N_G(X)) \cap L = O_2(L)$, a contradiction. Hence Y_L is an F–module. □

boddcomp

Lemma 12.20 *If 12.9 holds we get a nice P.*

Proof: Let R and $\tilde{Y}$ be as in 12.15. By 12.19 we have that R induces an F–module. If R is alternating, then as above we see that $\tilde{Y}$ corresponds to $\langle(12)(34),(13)(24)\rangle$, which is not quadratic on the permutation module. So we have A_7 on the four dimensional module, $3A_6$ on the 6-dimensional one, or some Lie group in characteristic two. By 3.28, 12.18 and 3.36 we get $R \cong G_2(q)$ or $Sp(2n,q)$ and W is the natural module. Let $R \not\cong L_2(q$. If $\tilde{Y}$ is not a transvection group, then it is contained in some $L_2(q) \times L_2(q)$ which induces an orthogonal module in $W/C_W(R)$. By 3.36 this module splits and so we get a strong dual F–module, which with 12.17 shows that we have a nice P. So we have that $R \cong Sp(2n,q)$ and we get a strong F–module with an offender A of order q, or $K \cong G_2(q)$. In the latter $\tilde{Y} \leq K_1 \cong L_2(q)$, which induces the natural module in the $G_2(q)$–module. Now this module also does not split, otherwise we get the assertion again with 12.17. This now again shows that we have have a strong offender A of order q.

As $q > 2$, we see with 3.21 that A does act trivially on $F(H/C_H)$. So let K be a component on which A induces a strong F–module on $[K,Y_H]$. If K is not covered by $H \cap M$, we can quote 11.7. So we may assume that K is covered by M.

Assume first that K is normal in H. Suppose further that $[Y_H,K]/C_{[Y_H,K]}(K)$ is irreducible. Then we have $Y_M \cap [Y_H,K] \leq C_{[Y_H,K]}(K)$ as otherwise $[Y_H,K] \leq Y_M$ and so we get the contradiction with 11.3 that $H \leq M$. But now we have by 3.38 that $K \cong L_3(2)$ or A_n. As we have a strong offender of order at least 4, we just can have $L_3(2)$ by 3.17. As A acts quadratically and we have $|[Y_H,A]| \leq 4$, so $|[\tilde{Y},A]| \leq 4$. But we do know that $|[\tilde{Y},A]| > q$, as we had a nonsplit extension.

So we have that at least two nontrivial irreducible modules are in $[Y_H,K]$. Let $U \leq \tilde{Y}$, $|U| = 4$. As $\tilde{Y}$ projects into some $L_2(q)$ in R, which induces a nonsplit extension of the natural module, we see with 3.52 that $[A,U] = [A,\tilde{Y}]$. Hence there is no fours group in $\tilde{Y}$ which is contained in one of the nontrivial irreducible modules in $[Y_H,K]$. This shows that we have exactly two modules and $|\tilde{Y}| = 4$. Hence on each module we have that A induces transvections to a hyperplane and so $K \cong L_n(2)$ and we have two natural modules. Suppose again that $Y_M \cap [Y_H,K] \leq C_{[Y_H,K]}(K)$. Then by 3.38 we have $K \cong L_3(2)$. But there are no transvections on the 4-dimensional indecomposable module for $L_3(2)$. So we have that one of the two modules is in Y_M. Again $[Y_H,K] \not\leq Y_M$. As $H \not\leq M$, we get that $H/C_H([Y_H,K]) \cong L_n(2) \times \Sigma_3$. As $m_3(H) \leq 3$, so $n \leq 5$. Further we have $Y_H = [Y_H,K]$. We have that $C_H \leq C_M$. So M_0 normalizes $S \cap C_H$. Hence by replacing H by $N_H(S \cap C_H)$ we may assume that $O_2(H) = S \cap C_H$. Then we have that $|H : M_0| = 3$. Then there is some $P_1 \leq H$, $S \leq P_1$, $P_1/O_2(P_1) \cong \Sigma_3$ and a Sylow 3–subgroup of P_1 centralizes K. This shows that we have one of the cases of 11.8 and so we have a nice

P.

So we may assume that K is not normal. Then we get with 1.1 that $K \cong L_2(r)$ or $L_3(2)$. Assume that $\langle K^H \rangle \neq \langle K^S \rangle$. Then we get al least three conjugates of K on which H induces a Σ_3 or Z_3. As $N_G(S) \leq M$ by 7.3 we get that Σ_3 is induced. In particular one of the three conjugates, K_1 say, is normalized by S. As K_1 induces an F–module we see that $C_{Y_H}(K_1) \neq 1$. Then as $C_{C_{Y_H}(K_1)}(S) \neq 1$, we get that $K_1 \leq M$. As $[Y_H, K_1]$ induces at most two nontrivial irreducible modules, we see that $[\langle K_2^S \rangle, [Y_H, K_1]] = 1$. But as $C_{[Y_H,K_1]}(S) \neq 1$, we have that $\langle K^H \rangle \leq M$. The same is true if $\langle K^H \rangle = \langle K^S \rangle$. In particular in any case $\langle K^H \rangle$ is covered by $M \cap H$. In the case of $K \cong L_2(r)$ we get that $[K, Y_H]/C_{[Y_H,K]}(K)$ is irreducible and $C_{[Y_H,K]}(S \cap K) \nleq C_{Y_H}(K)$. This shows that $C_{Y_H}(S) \nleq C_{Y_H}(K)$ and then $[Y_H, \langle K^H \rangle] \leq Y_M$, contradicting 11.3. This gives $K \cong L_3(2)$ and $[Y_H, K]$ is a sum of two natural modules. This shows again that one of them is in Y_M and some Σ_3 is induced on them. In particular we have $(L_3(2) \wr Z_2) \times \Sigma_3$ on $[Y_H, K^H]$. But now 5.16 gives $H \leq M$, a contradiction. $\square$

ast

Proposition 12.21 *If $F^*(C_G(x)) = O_2(C_G(x))$ for all $1 \neq x \in Y_H$, then there is a nice P.*

Proof: By 12.20 we may assume that $Y_H \leq O_2(C_G(x))$ for all $1 \neq x \in Y_H$. As b is odd we now may apply 3.10 to the amalgam (R_H, R_M). By 12.7 the case 3.10(3) does not show up. Then we have that one of 3.12, 3.13 or 3.14 holds. Suppose that we are in 3.12 or 3.14. Recall that $Y_H \nleq Z(O_2(R_M))$. So we always have a strong dual F–module. By 12.16 we may assume that this is realized by $F(H/C_H)$. Then by 12.17 the offender $A = \langle x \rangle$ has to be of order two and so induces a transvection on Y_H. Now x acts on a Sylow p–subgroup P of $F(H/C_H)$. Then $p = 3$. By 5.15 we get that $[x, P] \nleq M$. Then with 11.9 we get a nice P.

Hence we may assume that we have the situation of 3.13. If A acts on $F(H/C_H)$ nontrivially we get again transvections and so the assertion follows with 5.15 and 11.9. Hence there is some component K on which the offender A acts nontrivially. If K is not in M we get the assertion with 11.8. So assume K is covered by $M \cap H$. Let first K be normal in H/C_H. If $Y_M \cap [Y_H, K] \leq C_{[Y_H,K]}(K)$. As $C_{Y_H}(S) \leq Y_M$, we see with 3.38 that $K \cong A_n$ or $L_3(2)$. In both cases we get $Y_M = C_{[Y_H,K]}(K)$ and so by 11.3 $H \leq M$. Hence $Y_M \cap [Y_H, K] \nleq C_{[Y_H,K]}(K)$. As $M \cap H = (M_0 \cap H)(C_M \cap H)$, we get that $KC_H/C_H \leq M_0C_H/C_H$. Then we may assume that $K \leq (M_\beta)_0$ where $d(\beta, \alpha') = b - 1$. As K is normal in H/C_H, we get that the same applies for R_H. Hence $K \leq (M_\beta)_0 \cap R_H \leq R_{M_\beta}$. By 3.13 we have that the offender A was in $O_2(R_{M_\beta})$, a contradiction.

So we have that K is not normal. Suppose first that we have exactly two conjugates of K, i.e. $\langle K^H\rangle = \langle K^S\rangle$. If $[Y_M \cap [Y_H, \langle K^H\rangle], \langle K^H\rangle] \neq 1$, then again we get $\langle K^H\rangle C_H/C_H \leq M_0C_H/C_H$. And then we get the same contradiction as before. So we have that $[Y_M, \langle K^H\rangle] = 1$. But then with 3.38 we get again that Y_M is normal in H, a contradiction. So we must have three components, which then are isomorphic to $K \cong L_2(r)$ or $L_3(2)$ by 1.1. If all components are covered by M_0 we may argue as before. So we have a component K_1, which is not covered by M_0. As $[Y_H, K]$ is an F–module it involves at most two nontrivial modules. Hence $[[Y_H, K], K_1]] = 1$. As K_1 is normalized by S, we then also get that $[K_1, [Y_H, \langle K^H\rangle]] = 1$. This shows with 11.3 that K_1 is covered by M. As K_1 is not covered by M_0 we get $[Y_M, K_1] = 1$. This now shows that $C_{[Y_H,K_1]}(S \cap K_1) \leq C(K_1)$. Then we see with 3.36 that $K \cong L_3(2)$ and $|[Y_H, K]| = 16$. Hence we have that $|C_{[Y_H,\langle K^H\rangle]}(\langle K^H\rangle)| = 8$. On $\langle K^H\rangle$ we have that H induces a group Σ_3. But then we have that $C_{Y_H}(H) \neq 1$. As $C_{Y_H}(H) \leq Z(S)$ we get $H \leq M$ with 8.11. □

From now on we work under the following hypothesis

Hnonconstrain

Hypothesis 12.22 There is some $1 \neq x \in Y_H$ such that $E(C_G(x)) \neq 1$. Further we have no nice P.

By 8.14 we have that the element x from 12.22 is not 2–central in G.

Now let x be as in 12.22 and set $U = C_G(x)$. Then obviously $C_S(Y_H) \leq U$. So by 12.8 we may choose x such that $S \cap U$ is a Sylow 2–subgroup of U.

stroncon

Lemma 12.23 *Let L_1 be a parabolic in U containing $S \cap U$ with $F^*(L_1) = O_2(L_1)$. Then $Y_H \leq O_2(L_1)$.*

Proof: Otherwise L_1 satisfies (i) - (iii) of 12.9 and so there is some L which satisfies 12.9, which contradicts 12.20. □

*stru*1

Lemma 12.24 *We have $Y_H \not\leq O_2(U)$ and $[N, Y_H] \leq N$ for any component N of U. Further $N \times \langle x\rangle$ is not contained in a uniqueness group.*

Proof: Let $Y_H \leq O_2(U)$. Then $[E(U), Y_H] = 1$. So $E(U) \leq C_H$. As $O_2(H) \leq U$, we have that $[E(U), O_2(H)] \leq O_2(E(U))$. This shows that $[O_2(H), E(U)] = 1$ contradicting 8.14.

Let $N^{Y_H} \neq N$. As Y_H is normal in $S \cap U$, we have that $N/Z(N)$ has abelian Sylow 2–subgroups. In particular we have $N/Z(N) \cong L_2(2^n)$, as $N \in \mathcal{C}_2$. Let $X = N_N(S \cap N)$. Then $L_1 = \langle X, S \rangle$ is a constraint parabolic with $Y_H \not\leq O_2(L_1)$, contradicting 12.23.

Suppose that $N\langle x \rangle \leq M_1$, M_1 be a uniqueness group. Then by 8.14 we have that $F^*(M_1) = O_2(M_1)$. As $[N, C_{O_2(M_1)}(x)] = 1$, the $A \times B$–lemma gives a contradiction. □

stru2

Lemma 12.25 *We have* $[N, Y_H] = N$ *for any component* N *of* U.

Proof: Suppose $[N, Y_H] = 1$. Then by 11.1 we have $N \leq H$. But then $[N, O_2(H)] \leq O_2(H)$ and so, as $O_2(H) \leq U$, we have $[N, O_2(H)] = 1$, contradicting 8.14. □

struM

Lemma 12.26 *Let* N *be a component of* U*, then* $N \not\leq M$*. In particular* $E(U) = N$ *and* $C_{Y_M}(N) = 1$.

Proof: We have $N \not\leq M$ by 12.24.

Suppose first that $Y_M \cap N \neq 1$. Then by 5.14 we get that $N = E(U)$. So we may assume that

$$Y_M \cap N = 1 \textit{ and then } [Y_M, S \cap N] = 1.$$

By 12.24 we have that $m_p(U) \leq 3$ for all odd primes p. Hence we may assume that $m_3(N) \leq 1$. By 1.1 and $N \in \mathcal{C}_2$ we now get that

$$N/Z(N) \cong L_2(q), U_3(q), L_3(q), Sz(q) \textit{ or } L_2(p).$$

Further we have that there are nontrivial elements in $Z(S \cap U) \cap Y_{M_0}$. As these elements cannot centralize N by 5.14 we have that $Z(N \cap S) \not\leq Z(N)$. So by 1.16

N is simple.

Let first N be a group in characteristic two i.e. $L_2(q)$, $L_3(q)$, $U_3(q)$ or $Sz(q)$. Let $q \geq 4$. If $N \not\cong Sz(q)$ or $L_2(q)$, then N is normal in U, as otherwise $m_p(U) \geq 4$ for some p.

Let R be a root group normal in $(S \cap N)$ and let K be some group of order $q - 1$ acting transitively on $R^\sharp$ and normalizing $S \cap N$. Suppose that $S \cap U$ normalizes $(S \cap N)K$ then set $B = (S \cap U)K$. If $S \cap U$ does not normalize $(S \cap N)K$, which is the case for $N \cong Sz(q)$ or $L_2(q)$ and N not normal in

U or $N \cong L_3(q)$ and some graph automorphism is induced by $S \cap U$). Then there is some $t \in S \cap U$ with $K^t = K_1$ and $(S \cap N)(S \cap N)^t(K \times K^t)$ is normalized by $S \cap U$). Then set $B = (S \cap U)(K \times K_1)$.

We first show that $B \leq H$. We consider the group

$$X = \langle R_H, B \rangle.$$

If $O_2(X) \neq 1$, then we have that $Y_X \geq Y_H$ and by maximality of Y_H we get $Y_X = Y_H$ and then $X \leq H$ and so $B \leq H$ the assertion.

So we may assume that (R_H, B) is an amalgam. Let first $N \not\cong L_2(q)$. Then we can use the property that there is no involution in U acting on B and acting nontrivially on K, further on $K \times K_1$ the only such involution in U acting nontrivially is one interchanging K and K_1. This all is true as $q = 2^n$, n odd.

Let the parameter b for the amalgam (R_H, B) be odd. Let (α, α') be a critical pair. We may assume that $Y_\alpha \sim Y_H$. So let us assume that $Y_\alpha = Y_H$. As $1 \neq [Y_H, Y_{\alpha'}] \leq Y_H \cap Y_{\alpha'}$, we get that Y_H acts nontrivially on $B_{\alpha'}/O_2(B_{\alpha'})$. Hence we have that $N \cong Sz(q)$ and N is not normal in U or $N \cong L_3(q)$ and $S \cap U$ induces a graph automorphism on N. In the latter, as $Y_{\alpha'}$ is abelian and normal in $B_{\alpha'}$, we see that $Y_{\alpha'}$ is centralized by a Sylow 2–subgroup of $B_{\alpha'}$, contradicting $[Y_H, Y_{\alpha'}] \neq 1$. So we have $N \cong Sz(q)$. Now Y_H interchanges two components of $U_{\alpha'}$. Let $r \in Y_{M_0}^\sharp$ with $r \in O_2(B_{\alpha'})$. Then r centralizes a Sylow 2–subgroup T of $\langle N_{\alpha'}^{Y_H} \rangle$. Hence by 5.14 $T \leq M$. Now also $\langle Y_H^T \rangle \leq O_2(M)$. But by 12.7 we have that this group is abelian, while form $U_{\alpha'}$ we see that it contains a Sylow 2–subgroup of $Sz(q)$, a contradiction. Hence we have that $|Y_{M_0})| = 2$ and interchanges two components in $U_{\alpha'}$. Then in $U_{\alpha'}$ we have that Y_{M_0} centralizes a group L isomorphic to $Sz(q)$, which by 5.14 is in M. But we have that $[Y_{\alpha'}, Y_H] \cap L \neq 1$ and so we get a contradiction to $Y_H \leq O_2(M)$ as before.

So we have shown that b is even. Now by 12.7 we get that $[Y_B, Y_{\alpha'}] \neq 1$, where $Y_{\alpha'}$ is conjugated to Y_B. But then Y_B must be an F–module, which is not the case.

So we have shown that

If N is a group of Lie type in characteristic two different from $L_2(q)$ then

$$B \leq H.$$

Let now $N \cong L_2(q)$. Further if $N \cong A_5$ we will assume that U does not induce a Σ_5. Let b be odd. So again we may assume $Y_H = Y_\alpha$ and $[Y_H, Y_{\alpha'}] \neq 1$.

Set $Y_1 = \langle Y_{M_0}^g \mid g \in H\rangle$. Asumme that Y_1 induces just inner automorphism on $N_{\alpha'}$. Let $T_{\alpha'}$ be a Sylow 2–subgroup of $N_{\alpha'}$. Then we have that $[Y_1, T_\alpha] = 1$. Hence $T_{\alpha'} \leq M^g$ for all $g \in R_H$.

Suppose there is some $t \in Y_H$ which induces a field automorphism on $N_{\alpha'}$. Let $W_{\alpha'} = C_{N_{\alpha'}}(t)$, then we have that $T_{\alpha'} \cap W_{\alpha'} = [T_{\alpha'}, t]$. Hence $N_{W_\alpha}(T_\alpha \cap W_\alpha) \leq N_G(Y_H) \leq H$ by 11.1. If $q > 4$, then we have that $Y_1 \cap W_{\alpha'} = T_{\alpha'} \cap W_{\alpha'}$, as there is no element in Y_{M_0} which centralizes $N_{\alpha'}$, and so $[T_{\alpha'}, Y_H] \leq Y_1$. But now elements of odd order in $\langle T_{\alpha'}^H\rangle$ centralize Y_H and so $T_{\alpha'}C_H/C_H \leq O_2(H/C_H) = 1$, a contradiction. So we have $N \cong A_5$. But then t induces Σ_5 on $N_{\alpha'}$, a contradiction.

So we may assume that Y_H does not normalize $N_{\alpha'}$. Let $t \in Y_H \setminus N_G(N_{\alpha'})$. Then $W_{\alpha'} = C_{N_{\alpha'} \times N_{\alpha'}^t}(t) \cong N_{\alpha'}$ and a Sylow 2-subgroup of this group is in Y_H. Hence again $N_{W_{\alpha'}}(C_{T_{\alpha'}}(t)) \leq H$ and so it normalizes Y_1. This shows that $Y_1 \cap W_{\alpha'} = [T_{\alpha'}, Y_H]$. But then as before elements of odd order in $\langle T_{\alpha'}^H\rangle$ centralize Y_H and so $T_{\alpha'}C_H/C_H \leq O_2(H/C_H) = 1$, a contradiction.

So we may assume that some $r \in Y_{M_0}$ does not induce an inner automorphism on $N_{\alpha'}$. Then $X_{\alpha'} = C_{\langle N_{\alpha'}, N_{\alpha'}^r\rangle}(r) \cong L_2(t)$, where $t = q$ if r does not normalize $N_{\alpha'}$ and $t^2 = q$ otherwise. In any case by 5.14 we have that $X_{\alpha'} \leq M$. Now we get that $[Y_{\alpha'}, Y_{M_0}]$ contains a Sylow 2–subgroup of $X_{\alpha'}$, which contradicts $O_2(X_{\alpha'}) = 1$.

So we have that b is even and then by 5.14 we have that $[Y_B, Y_\alpha] \neq 1$. So Y_B is an F–module, which gives $q = 4$ and $N \cong A_5$ and Σ_5 is induced.

So we have shown that

If N is a group of Lie type in characteristic two and $S \cap U$ does not

induce Σ_5 on N, if $N \cong A_5$, then $B \leq H$.

In case of $L_3(q)$ and no graph automorphism is involved we have that the full Borel subgroup is in H, as we have two choices for K. In all cases $[Y_{M_0}, B] \leq Y_H$. We have that $[Y_{M_0}, S \cap N] = 1$, so Y_{M_0} induces an inner automorphism from R. In particular there is some $r \in R$ and $r_1 \in C_U(N)$ with $1 \neq rr_1 \in Y_H$. Now $\langle (rr_1)^K\rangle \geq R$. This gives that $R \leq Y_H$.

Now set $W = \langle Y_H^g \mid g \in M\rangle$. By the assumption following 12.7 we have that W is abelian and so $W \leq U$. Then we have that $R \leq W$. Suppose that $W = C_W(N)R$, then we have that $B \leq N_G(W) = M$. But then with the same argument as above we have that $R \leq Y_{M_0}$ contradicting $Y_{M_0} \cap N = 1$. So we have that the projection of W onto N is greater than R and then

$N \cong L_3(q)$. As this projection is normal $W \cap N$ in $S \cap U$ and elementary abelian, we see that no graph automorphism can be induced. This then gives that the full Borel subgroup of N is in H. Now in B there is a subgroup K_1 of order $q-1$, which centralizes R and acts transitively on the projection of W onto N modulo R. Hence we see that $W \cap N$ is elementary abelian of order q^2. Let $L_1 = N_N(W \cap N)$. Then L_1 acts transitively on $(W \cap N)^\sharp$ and as L_1 normalizes W, we have that $L_1 \leq M$. But then we have that $Y_{M_0} \cap N \neq 1$, a contradiction.

So we have shown

$$E(U) = N \ \textit{or}\ N \cong L_2(p) \ \textit{where in case of}\ p = 5$$

$$\textit{the group}\ S \cap U\ \textit{induces}\ \Sigma_5\ \textit{on}\ N.$$

Let now $N \cong L_2(p)$, p odd. In case of $p = 5$ we assume that $S \cap U$ induces Σ_5 on N. We next show

$$(*) \qquad If\ \ E(U) \cong A_5 \times A_5\ \ then\ \ 3 \notin \sigma(M).$$

Assume $3 \in \sigma(M)$ We have that S induces on both components a group Σ_5. Let T be a Sylow 3–subgroup of $E(U)$ and M_1 be a conjugate of M with $T \leq M_1$. By 12.24 we have that $U \not\leq M_1$. Hence T contains some element $t \neq 1$ such that $C_G(t) \not\leq M_1$. Suppose first that M_1 is exceptional with respect to $p = 3$. Then there is some $t_1 \in T$ such that $C_G(t_1) \leq M_1$. Hence $x \in M_1$. This now shows with 5.6 that $C_{C_{O_2(M_1)}(x)}(t) = 1$. Now as $N \not\leq M_1$ by 12.24 we have that $t_1 \notin C_{E(U)}(N)$. In particular for $t \in C_{E(U)}(N)$, we have that $C_G(t) \not\leq M_1$. Hence $O_2(M_1) \cap U \leq C_{E(U)}(N)$. But the same applies for the other component N_1 of $E(U)$, a contradiction. So we have that M is not exceptional. In particular T is not centralized by an elementary abelian subgroup of order 27 in M_1. As T contains a 3-central element from M_1, we get that no $SL(2,3)$ is induced on T. The structure of $N_U(T)$ shows that $N_{M_1}(T)/C_{M_1}(T) \cong Z_2 \times \Sigma_3$. In particular there is $\tau \in T$, τ 3-central in M_1 such that $\langle \tau \rangle$ is normal in $N_{M_1}(T)$. This now shows that we may assume that $\tau \in N$ and so $\langle x \rangle N_1 \leq M_1$, contradicting 12.24. This proves $(*)$.

Set again $W = \langle Y_H^g \mid g \in M \rangle$. As $Y_H \leq O_2(M)$ we have with 12.7 that W is elementary abelian, so $W \leq U$.

Let first $[W, N] \not\leq N$. As W is elementary abelian and normal in $S \cap U$, we get that N has an abelian Sylow 2–subgroup and so $N \cong A_5$. Then there is some τ of order three in $N \times N^w$, $w \in W \setminus N_W(N)$ such that τ normalizes W. Hence $\tau \in M$. It τ centralizes Y_{M_0}, then we have that Y_{M_0} centralizes $E(U)$, contradicting 5.14 and 12.26. So we have that $1 \neq [Y_{M_0}, \tau] \leq N \times N^w$. By $(*)$ we have $3 \notin \sigma(M)$. Assume that $|W| \leq 8$. Then as Y_H normalizes N by 12.25, we have that $Y_H \leq N \times N^w$, which contradicts $x \in Y_H$. So we have that $|W| \geq 16$. So N centralizes a nontrivial subgroup of index 8 in W and then $\sigma(M) = \{7\}$. Further $e(G) = 3$. We have $L_1 = C_{N \times N^w}(w) \cong A_5$. Now L_1 centralizes a subgroup of index 4 in W and so $L_1 \leq M$. But W is normal in M and $[S \cap N, w]$ is a Sylow 2–subgroup of L_1, a contradiction.

Let $[W, N] \leq N$. Then we have that $|W : C_W(N)| \leq 4$. Assume that $|W| \leq 4$. As $Y_H \neq Y_{M_0}$, then $W = Y_H$ is normal in M, a contradiction. So we have that $|W| \geq 8$ and so $C_W(N) \neq 1$. As $N \not\leq M$, we have that no element in $C_W(N)$ is centralized by a good E. Thus $\sigma(M) = \{3\}$. Further $|W : C_W(N)| = 4$. As W is normal in a Sylow 2–subgroup of $S \cap U$ we have that $N \cong L_2(7)$ or A_5. Now $m_3(U) \leq 2$ and so $E(U) = N \times N_1$, where $N_1 \cong L_2(7)$ or A_5 too. By $(*)$ we have $E(U) \not\cong A_5 \times A_5$ before, we may assume $N \cong L_2(7)$. Then in $NC_U(N)$ there is a subgroup L_1 such that $L_1/C_{L_1}(N) \cong \Sigma_4$ and $W \leq O_2(L_1)$. But now L_1 is generated by involutions in $L_1 \setminus O_2(L_1)$, which centralize a subgroup of index two in W, which then gives that $L_1 \leq M$. Hence in N there is a subgroup $L_2 \cong \Sigma_4$, such that $L_2 \leq M$. As Y_{M_0} does not centralize N it projects nontrivially on $O_2(L_2)$ and so $Y_{M_0} \cap N = Y_{M_0} \cap L_2 \neq 1$, a contradiction.

So we have $N = E(U)$ and $C_{Y_M}(N) = 1$ follows with 5.14. □

We now choose U such that N is maximal.

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Lemma 12.27 *Let $V_0 = C_{Y_H}(E(U))$, and $g \in H$ with $V_0 \cap V_0^g \neq 1$, then $V_0 = V_0^g$.*

Proof: Let $v, w \in V_0$ with $v^g = w$. Then by maximality N and N^g both are components of $C_G(w)$. But then by 12.26 we get $N = N^g$ and so $V_0 = V_0^g$. □

Nocon

Lemma 12.28 *We have that $F^*(C_G(x)) = O_2(C_G(x))$ for all $x \in Y_H^\sharp$. In particular 12.22 is not satisfied, i.e. there is a nice P.*

Proof: Suppose false. By 12.26 and 5.13 we get that the centralizer $L_1 = C_U(y)$ for some involution $y \in Y_{M_0}$, which centralizes a Sylow 2–subgroup of N, is contained in M. As $Y_H \leq O_2(M)$, we see that $Y_H \leq O_2(L_1)$. By 12.7 we have that $\langle Y_H^{L_1} \rangle$ is abelian.

Let first

$N/Z(N) \not\cong Sp(2n, q)$ or $F_4(q)$.

Recall that $N \in \mathcal{C}_2$. Further by 12.24 we have that $m_3(N) \leq 3$. So with 1.1 we get that N is group of Lie type in characteristic two, including A_6, $L_2(p)$, $L_3(3)$, M_n, J_n, HS or Ru.

If $|Y_H : V_0| = 2$, and $|Y_H| > 4$ then by 12.27 that H normalizes V_0 and so H normalizes N by 12.26 but then $Y_H \leq C(U)$, a contradiction. So we have that $|Y_H| = 4$. But then H induces Σ_3 on Y_H and so by 11.9 we have a nice P. So we have that

$$|Y_H : V_0| > 2.$$

Assume $Y_H \leq O_2(U)Z(L_1 \cap N)$. This now shows that we have that N is a group of Lie type $G(q)$ and $|Y_H : V_0| \leq q > 2$. Further by 12.23 we have that Y_H does not induce outer automorphisms on L. Let R be the root subgroup corresponding to L_1. We have that $L_1 \cap N \leq C_H$ and so $O_2(L_1) \leq O_2(C_H) \leq O_2(H)$.

Suppose $O_2(H) \leq O_2(U)O_2(L_1)$, so $O_2(H) = O_2(L_1)(O_2(H) \cap O_2(U))$. Then the Cartan C subgroup corresponding to R of N normalizes $O_2(H)$ and so is in H. This shows that $|Y_H : V_0| = q$. Let $O_2(H) \not\leq O_2(U)O_2(L_1)$. Then $[O_2(H), L_1] \leq O_2(L_1)$, which shows that $N \cong L_n(q)$, $n \leq 4$. If $n = 4$, then $O_2(H)$ just induces graph automorphism. But then there is some element ρ of order $q-1$ acting transitively on R and centralizes $O_2(H)/O_2(L_1)$. This gives again $C \leq H$ and then $|Y_H : V_0| = q$. Let $n = 3$, then we have ρ with $o(\rho) = q - 1/gcd(3, q-1)$ and we get the same result, or $q = 4$. But as $|Y_H : V_0| > 2$, we get $|Y_H : V_0| = 4 = q$. So we are left with $N \cong L_2(q)$ and $O_2(H)$ induces field automorphisms. As $|Y_H : V_0| > 2$, and Y_H is centralized by $O_2(H)$, we have that $q > 4$. Hence we have that $J(O_2(C_H)) \leq O_2(U)N$ and $J(O_2(C_H)) \cap N$ is a Sylow 2–subgroup of N. Now $N_N(J(O_2(C_H))) \leq H$ and so again $C \leq H$ and so $Y_H \cap N$ is a Sylow 2–subgroup of N, contradicting $O_2(H) \not\leq NO_2(U)$. Hence in any case we have that $|Y_H : V_0| = q$.

We have that V_0 is not normal in H, then with O'Nan's Lemma [GoLyS2, (14.2)], we get that either $|Y_H| = 8$ and H/C_H is a Frobenius group of order 21, but then $H = C_H N_H(S) \leq M$, a contradiction, or $[Y_H, C] \cong V_0$ and $|H : N_H(V_0)| = 2 = |H : N_H([Y_H, C])|$, or even $[Y_H, C]$ is normal in H.

Assume first that $C \leq M$. We have that $N \geq [Y_H, C]$. But then as $Y_{M_0} \cap V_0 = 1$ by 5.14 and 12.26 and $Y_H = V_0 \times [Y_H, C]$ we have that $Y_{M_0} \leq [Y_H, C] \leq N$. As S normalizes Y_{M_0}, we have that $[Y_H, C]$ is normal in H. Then $H = (M \cap H)N_N([Y_H, C])$. As $C \leq M$ and $L_1 \leq M$, we have that $N_N([Y_H, C]) \leq M$. But then $H \leq M$. So we have shown that

$$C \not\leq M, \textit{ if } Y_H \leq O_2(U)Z(L_1 \cap N).$$

We consider $W = \langle Y_H^g \mid g \in M \rangle$. By 12.7 we may assume that W is abelian. Further $W \leq O_2(M)$. So the projection of W onto N is an elementary abelian normal subgroup in L_1. As N is a group of Lie type, we get either that $W \cap L_1 = Y_H \cap L_1$ and this is also the projection onto N or $N/Z(N) \cong L_n(q)$ or ${}^2F_4(q)$. Suppose the former. Let $W \neq (W \cap L_1)(W \cap C_U(N))$. Then there is some outer automorphism $1 \neq t \in W$ of N such that $[t, S \cap N] \leq R$, a contradiction. So we have that $N_N(Y_H \cap L_1) \leq M$. This now gives that $C \leq M$, a contradiction. So we have that

$$N/Z(N) \cong L_n(q), n \geq 3, \ \textit{or} \ \ {}^2F_4(q), \ \textit{if} \ \ Y_H \leq O_2(U)Z(L_1 \cap N).$$

If $N/Z(N) \cong {}^2F_4(q)$, then the projection is normalized by $Sz(q)$ and so the projection is equal to $W \cap N$. But then $W \cap N = O_2(L_1 \cap N)$ and so it is normalized by C, which then normalizes W too, and so is in M, a contradiction.

So let $N/Z(N) \cong L_n(q)$. If the projection is normalized by $SL_{n-1}(q)$, then we get the same contradiction as in the ${}^2F_4(q)$–case. Hence we have $N/Z(N) \cong L_3(q)$ or $L_4(q)$. In the latter some $t \in W$ invert some element ρ of order $q-1$, which centralizes $Y_H \cap N$ and so also Y_{M_0}, which then contradicts $W \leq O_2(M)$. So we have $N/Z(N) \cong L_3(q)$. If $q > 4$ we may argue as before with some ρ, $o(\rho) = q - 1/gcd(3, q-1)$.

So we have $N/Z(N) \cong L_3(4)$. Let first $3 \in \sigma(M)$. Let T be a Sylow 3–subgroup of N. Then there is some conjugate M^g of M with $T \leq M^g$ and some $1 \neq \rho \in T$ with $C_G(\rho) \leq M^g$. As T centralizes any 2–group in U, which is normalized by T, we see that $[C_{O_2(M^g)}(x), T] = 1$. With the $A \times B$–lemma we get the contradiction $[T, O_2(M^g)] = 1$. Hence $3 \notin \sigma(M)$. Let now P_1 be the parabolic in N such that W projects onto $O_2(P_1)$. We have that $|O_2(P_1)/Z(N)| = 16$. Then P_1 is generated by involutions i such that $|W : C_W(i)| \leq 4$. In particular any such involution centralizes some $j \in W$ with $C_G(j) \leq M$ by 5.8. Hence $P_1 \leq M$. As P_1 acts transitively on $O_2(P_1)/Z(N)$, we get that $Y_{M_0} \geq O_2(P_1)$. This gives $Z(N) = 1$ and $Y_{M_0} = O_2(P_1)$. But then we have that $Y_H \not\leq O_2(U)Z(L_1 \cap N)$. So we have that

$$Y_H \not\leq O_2(U)Z(L_1 \cap N).$$

In that case we must have a normal abelian subgroup in $L_1 \cap N$, which is not in $Z(L_1 \cap N)$. This shows that

$$N/Z(N) \cong L_n(q), \ {}^2F_4(q), M_n, \ \textit{or} \ Ru.$$

Further let Y be the projection of Y_H onto N. As $S \cap C_H$ centralizes Y_H it also centralizes Y. Hence by 3.4 we have that $Y = Y_H \cap N$.

We show

$$N/Z(N) \not\cong L_n(q).$$

If we have $N/Z(N) \cong L_n(q)$, there are two parabolics $L_i \cong q^{n-1}SL_{n-1}(q)$, $i = 2, 3$ in N such that $O_2(L_2)O_2(L_3) = O_2(L_1 \cap N)$ and $O_2(L_2) \cap O_2(L_3) = R$. Both are interchanged by the graph automorphism. As Y_H is normal in $S \cap U$, we have that there is no graph automorphism. But then $Y_H \not\leq O_2(L_i)$ for at least one $i = 2, 3$, contradicting 12.23.

Assume now that N is sporadic or ${}^2F_4(2)$. Let again $W = \langle Y_H^M \rangle$. Then by 5.8 there is a fours group V in W such that $C_G(v) \leq M$ for all $v \in V^\sharp$. Set $L = \langle L_1 \cap N, C_N(v) \mid v \in V^\sharp \rangle \leq M$. If $L_1 \cap N$ is maximal in N, then we have that $L = N$, a contradiction. Hence we have that $N \not\cong Ru$, M_{12} or ${}^2F_4(2)$. If $N/Z(N) \cong M_n$, $n = 22, 23$, or 24, then we have that $L/O_2(L) \cong A_6$, A_7 or A_8. In all cases we now have that $\langle Y_{M_0}^L \rangle$ contains $O_2(L)$. Hence $O_2(L) \leq Y_{M_0} \leq Y_H$, contradicting 12.23 with the 2-local $2^4\Sigma_5$,in M_{22} and M_{23} and $2^6 3\Sigma_6$ in M_{24}. We now have

$$N/Z(N) \cong {}^2F_4(q), q > 2.$$

Let $X = \langle Y_H^{C_N(R)} \rangle$. Then X is elementary abelian of order q^5. Let P be the other parabolic of N containing $S \cap N$. Then we have that $X \leq O_2(P)$ and $\langle X^P \rangle$ is nonabelian. We have that the intersection X_P of all conjugates of X in P is of order q^3. As $\langle Y_H^P \rangle$ is abelian by 12.7, we see that Y_H is contained in that intersection X_P. If Y_H projects into $Z(O_2(P))$, then $O_2(H) \cap N$ centralizes Y_H and so $O_2(P) \leq O_2(H_0)$ and then $Z(O_2(P)) = Y_H \cap N$. Hence $P \leq H$. If it does not project into $Z(O_2(P))$, then as $[Y_H, O_2(P)] = Z(O_2(P))$, we have $Z(O_2(P)) \leq Y_H$. Now we have that $C_P(X_P/Z(O_2(P)))/O_2(P) \cong L_2(q)$. So $N_P(Y_H)$ involves $L_2(q)$. Hence in both cases H induces an F–module on Y_H. As $q > 2$ this is affected by some component K which is not in M. So by 11.8 we have a nice P.

Hence we finally have to handle the cases

$$N/Z(N) \cong Sp(2n, q) \text{ or } F_4(q).$$

By 12.24 and 1.2 we get $N/Z(N) \cong Sp(4, q)'$ or $Sp(6, q)$.

Let first $q = 2$. Set as before $W = \langle Y_H^M \rangle$. As no Sylow 3–subgroup of N normalizes a nontrivial 2–group in N, we get as above that $3 \notin \sigma(M)$. Hence we have a subgroup V in W of order 8 such that $C_G(v) \leq M$ for all $v \in V^\sharp$. Let first $N \cong A_6$, Then W projects onto one of the two elementary abelian groups of order 8 in Σ_6 and so the same applies for V. But then the projection contains $(1, 2)$ and $(1, 2)(3, 4)(5, 6)$ as well and so

$N = \langle C_N(v) \mid v \in V^\sharp \rangle \leq M$, a contradiction. So we have that $N \cong Sp(6,2)$. As the projection of Y_{M_0} onto N contains some element in $Z(S \cap N)$, we see that in any case $C_N(Z(S \cap N)) \leq M$. If $M \cap N > C_N(S \cap N)$, then $M \cap N$ is the centralizer of some root element. As W is normalized by $M \cap N$, we see that W projects onto $Z(O_2(M \cap N))$. But then in both cases there is no elementary abelian subgroup of order 8 in $Z(O_2(M \cap N))$ all of whose centralizers are in $M \cap N$. This shows that $M \cap N = C_N(Z(S \cap N))$. Let $\rho \in M \cap N$, with $o(\rho) = 3$. Then we have that $|C_{O_2(M\cap N)}(\rho) : Z(O_2(M \cap N))| = 2$. As any involution in $Sp(6,2)$ is centralized by some element of order 3, we see that V must be contained in $C_N(\rho)$ and so $|V \cap Z(O_2(N \cap M))| \geq 4$. As V does not contain root elements, we see that $|V \cap Z(O_2(N \cap M))| = 4$. But then $C_{C_{O_2(M\cap N)}}(V \cap Z(O_2(N \cap M))) = Z(O_2(N \cap M))$, a contradiction. So we now have

$$q > 2.$$

Assume first that some element of $Y_{M_0}^\sharp$ projects nontrivially in some root group R. Then $C_N(R) \leq M$. Let W be as before. Suppose that W projects into R. Then we have that $C_N(R)$ normalizes Y_H and so $C_N(R) \leq H$. Hence we see that $O_2(H) \cap N \leq O_2(C_N(R))$. But then we see that the Cartan subgroup related to $S \cap N$ is contained in H, as it normalizes $O_2(H)$. Hence we get that $R = Y_H \cap N$, is the projection of Y_H onto N. Now also $W \cap N = R$ and so C is even contained in M. Now as above we get applying O'Nan's lemma that $|H : N_H(R)| \leq 2$. But $R = Y_{M_0}$, which is normalized by S and so Y_{M_0} would be normal in H, a contradiction.

So W does not project into R. As W is an elementary abelian normal subgroup in M, we get that $W \cap N = Z(O_2(C_N(R)))$. But now again $C \leq M$. Further again $Y_{M_0} = R$. Further we have that $Y_H \neq V_0R$, as we otherwise may argue as before.

Hence in any case we may assume that the projection of Y_H onto N is not contained in a root group. Assume first that the projection of Y_H onto N is contained in $Z(S \cap N)$. Then there is some $r \in Y_H$ which projects on neither of the two root groups in $Z(S \cap N)$. Now $Z(S \cap N) = Z(O_2(C_N(r))$ and so $Z(S \cap N) = Z(O_2(C_N(Y_H)))$. Now we have that $O_2(C_H) \cap N = O_2(C_N(Y_H))) = O_2(N_N(Z(S \cap N)))$. The latter group is normalized by the Cartan subgroup C and so $C \leq H$. Then $Y_H \cap N = Z(S \cap N)$.

Suppose that Y_H does not project into $Z(S \cap N)$. As Y_H is normal in $S \cap U$ and by 12.7 the projection of Y_H is in $O_2(L)$ for any parabolic L of N, we see that we must have $N \cong Sp(6,q)$. Let X be the intersection of $O_2(L)$ for the three maximal parabolics L of N. Then we see that $|X| = q^3$. Let $L_2 = N_N(X)$. Then $O_2(L_2) = C_{L_2}(X)$. As the projection of Y_H is in X, we get that $C_N(Y_H) = O_2(L_2)$. But now $N_N(X)$ normalizes $O_2(C_H)$ and so $N_N(X) \leq H$. This gives that a Cartan subgroup C of N is in H and so $Y_H \cap N = X$ is equal to the projection. Let E be the unique elementary abelian subgroup of order q^6 in $S \cap N$. Then $E \leq O_2(C_N)$. Hence we have that $E = Z(J(O_2(C_H))) \cap N$. This shows that $N_N(E)$ normalizes $Z(J(O_2(C_H)))$, so $N_N(E) \leq H$. But as X is not normal in $N_N(E)$, we get that $Y_H \cap N = E$, a contradiction. So we have shown

$$Y_H \cap N = Z(S \cap N).$$

Let now $N \cong Sp(6,q)$. We have that $S \cap C_H = (C_H \cap C_U(N) \cap S)(S \cap N)$. Let again E be the unique elementary abelian subgroup of order q^6 in $S \cap N$. Then we see that $E \leq Z(J(S \cap C_H))$. Hence $N_N(E)$ normalizes $Z(J(S \cap C_H))$. So $Z(J(S \cap C_H))$ is normal in $\langle H_0, N_N(E) \rangle = D$. By maximality of Y_H we have that $Y_D = Y_H$ and so $N_N(E) \leq H$. But then $N_N(E)$ must normalize $Z(S \cap N) = Y_H \cap N$, a contradiction. So we have shown that

$$N \cong Sp(4,q).$$

Let W be as before. If $W \cap N = Y_H \cap N$, then C normalizes W and so $C \leq M$. This now implies that $R = Y_{M_0}$ for some root group R. In particular $C_N(R) \leq M$. But $Z(S \cap N)$ is not normal in $C_N(R)$, a contradiction. So we have that $Y_H \cap N < W \cap N$. Let $E \leq S \cap N$ be elementary abelian of order q^3 such that $W \cap N \leq E$. Then we have that $[E, W] = 1$. As $C_{O_2(M)}(W) \leq U$, we see that $[E, C_{O_2(M)}(W)] = 1$. Hence the $A \times B$–lemma yields that $E \leq O_2(M)$. In particular $N_N(E)$ normalizes $C_{O_2(M)}(W)$. This shows that $N_N(E) \leq M$. Hence $E = W \cap N$, $C \leq M$ and $M = R$, where $N_N(E) = N_N(R)$.

Let $T = C_H \cap S$. We have that $T \cap N = EF$, where $E = T \cap N \cap O_2(M)$. Further we have that $T = (T \cap N)(T \cap C_U(N))$. Let first $T \cap C_U(N))$ be abelian. Then as $\Omega_1(Z(T)) = Y_H$ by 3.4 we have that $V_0 = \Omega_1(T \cap C_U(N))$ and so $J(T) = EFV_0$. As $F \not\leq O_2(M)$, we have that S normalizes V_0E. As there are exactly two elementary abelian subgroups of order $|EV_0|$ in $J(T)$, we get that H_0 normalizes EV_0. Hence V_0E is normalized by $D = \langle H_0, N_N(E) \rangle$. But then by maximality of Y_H we get that $Y_H = Y_D$ and so $N_N(E) \leq H$, which gives $E \leq Y_H$, a contradiction. So we have shown that $T \cap C_U(N)$ is nonabelian. As $(T \cap N)' = Y_H \cap N$ and $Z(S) \cap V_0 = 1$, we get that $(T \cap C_U(N))' \leq V_0$. In particular $[T, T \cap C_U(N)] \leq V_0$. Further we have that V_0 is not normal in H_0 and so there is $g \in H_0$ such that $V_0 \cap V_0^g = 1$. Then $[T, (T \cap C_U(N)) \cap (T \cap C_U(N))^g] \leq V_0 \cap V_0^g = 1$. In particular $(T \cap C_U(N)) \cap (T \cap C_U(N))^g \leq Z(T) \cap C_U(N) \cap C_U(N)^g$. By 3.4 we have that $\Omega_1(Z(T) \cap C_U(N)) = V_0$. Hence we have that $(T \cap C_U(N)) \cap (T \cap C_U(N))^g = 1$. In particular $T \cap C_U(N)$ is isomorphic to a nonabelian subgroup of $T \cap N$. This first shows that $V_0 = (T \cap C_U(N))'$ and $T \cap C_U(N)$ contains at most two elementary abelian subgroups of maximal order. Let J be an elementary abelian subgroup of maximal order in T, then we have that $J = EX$

or FX, where X is an elementary abelian subgroup of maximal order in $C_U(N)$. In particular there are either two or four such groups. Let $A = EX$ or $A = FX$ be normal in H_0 for some X, then we have that A is normalized by $D = \langle H_0, N_N(A)\rangle$, which again by maximality of Y_H is contained in H. But then $Y_H \cap N$ would be E or F, a contradiction.

Suppose next that S normalizes some EX, then EX has exactly 3 conjugates under H_0 and so some FX_1 is normal, a contradiction. So we have that EX either has two or 4 conjugates under S. We have that $E \leq W = \langle Y_H^M \rangle$, where W is normalized by S. Hence we have that EX must have exactly two conjugates under S and the same applies for FX. Finally FX is not conjugate to EX. Let X_1, X_2 be the two maximal elementary abelian subgroups in $C_U(N) \cap T$. Then we have that $1 \neq [EX_1, EX_2] \leq [X_1, X_2] \leq V_0$ is a normal subgroup in S. But then $V_0 \cap Z(S) \neq 1$. As $\Omega_1(Z(S)) \leq Y_{M_0}$ this shows $Y_{M_0} \cap V_0 \neq 1$, which with 5.14 gives $N \leq M$, contradicting 12.26.

This final contradiction proves that $F^*(C_G(x)) = O_2(C_G(x))$ for all $1 \neq x \in Y_H$. With 12.21 we now get the assertion. □

13 The group H for $Y_H \not\leq O_2(M)$

In this section we assume that $Y_H \not\leq O_2(M)$. Then we may apply 4.2. There are two cases to handle.

(1) There is $g \in M$ with $1 \neq [Y_H, Y_H^g] \leq Y_H \cap Y_H^g$

(2) There is the group L given by 4.2. We have that $A = Y_H^g \cap O_2(L)$ acts as 2F–module offender on Y_H with $[Y_H, A, A] = 1$. If A acts quadratically, then it induces a strong F–module on Y_H.

If we have case (1) we will denote by A the group Y_H^g.

nottrivial

Lemma 13.1 *Assume case (2) above. Then $A \cap O_2(M) \not\leq A \cap C_H$.*

Proof: We have that $[A \cap C_H, Y_H] = 1$. Hence we have that $A \cap C_H \leq Y_H \cap A$. So $A \cap O_2(M) \leq Y_H \cap Y_H^g$. We have that $[A, O_2(M)] \leq [Y_H^g, O_2(M)] = [Y_H, O_2(M)] \leq O_2(M) \cap A \cap Y_H \leq Y_H \cap Y_H^g$. Let $x \in L$, $o(x)$ be odd. Then $[O_2(M), x] \leq Y_H \cap Y_H^g$ and $[Y_H \cap Y_H^g, x] = 1$. Hence $x \in C_M(O_2(M))$, so $x = 1$. But then L is a 2–group, a contradiction. □

Again the aim of this section is to prove that we have a nice P.

notM1

Lemma 13.2 *Let K be a component or a Sylow subgroup of $F(H/C_H)$ which is in M, if A does induce an F–module offender on K, then there is a nice P. In case of a Sylow subgroup of $F(H/C_H)$ the same holds for a $2F$–module offender.*

Proof: Assume that we do not have a nice P. Let $W = [K, Y_H]$ and assume first that K is normal in H. If $Y_M \cap W \not\leq C_W(K)$ we get some submodule of W which is in Y_M. But as $Y_M \leq Y_H^g$ as $g \in M$, we have that $[A, Y_M] = 1$, so $[A, K] = 1$. So we have that $Y_M \cap W \leq C_W(K)$, in particular K is a component and not a Sylow group. Now by 3.16, 3.36 and 3.38 we get that $|Y_M \cap W| = 2$ and so $Y_M \cap W$ is normal in H, which contradicts 11.1.

So we have that K is not normal, and so again K is a component. If $\langle K^S \rangle$ is normal in H, we may argue as before and get a contradiction. So we have that K has three conjugates on which H induces Σ_3. By 1.1 we see that $K \cong L_2(r)$ or $L_3(2)$. In particular $[Y_H, K, K_1] = 1$ for a conjugate $K_1 \neq K$ of K. Again $Y_M \cap W \leq C_W(K)$. So assume that K is normalized by S, we get that $K \cong L_3(2)$ and $|Y_M \cap W| = 2$. If K is not normalized by S then $C_{W^S}(S) \leq Y_M$ and so again it has to be in $C(\langle K^S \rangle)$, which shows $K \cong L_3(2)$ and $|C_W(K)| = 2$.

As $\langle K^H\rangle$ centralizes $C_W(K)$ we get that $\langle K^H\rangle$ is covered by M. Now by 5.16 we get that $H/Core_H(M\cap H)\cong\Sigma_3$. As $H\not\leq M$, we see that for $\rho\in H\setminus M$ with $\rho^3\in M$, that $C_{[Y_H,\langle K^H\rangle]}(\langle K^H\rangle)\cap C(\rho)=1$. As $|C_{[Y_H,\langle K^H\rangle]}(\langle K^H\rangle)\leq 8$, we now get $|C_{[Y_H,\langle K^H\rangle]}(\langle K^H\rangle)=4$. Let H_1 be the preimage of $F^*(H/O_2/H)$. We have that $[H_1,C_{Y_H}(\langle K^H\rangle)]\leq Y_M$. If this commutator is nontrivial, we get with 5.14 that $H\leq M$, a contradiction. Hence we have that $[H_1,C_{Y_H}(\langle K^H\rangle)]=1$. Now $C_{Y_H}(\langle K^H\rangle)$ is a direct sum of Σ_3–modules. Now we he have that a Sylow 2–subgroup of $Core_H(M\cap H)$ centralizes Y_M In particular there is some $P_1\leq H$, $P_1/O_2(P_1)\cong\Sigma_3$ and P_1 commutes with M_0 such that Y_{P_1} is a direct sum of nontrivial Σ_3–modules. Hence we have one of the situations of 11.4, a contradiction. □

$Fmodule1$

Lemma 13.3 *Let case (1) above, then we have a nice P.*

Proof: Let K be some component of H/C_H such that $A=Y_H^g$ induces some F–module offender on K. By 13.2 we may assume that K is not covered by M. So we may apply 11.8.

So we may assume that A induces an F – module offender on a Sylow p–subgroup P of $F(H/C_H)$. By 13.2 we have that $P\not\leq M$. Then $C_{Y_H}(P)=1$. Set $\bar{P}=P/\Phi(P)$. As we have transvections on Y_H there is some $a\in A$ with $|[\bar{P},a]|=3$ and a induces a transvection on Y_H. So there is some preimage B of $[\bar{P},a]$ with $|[Y_H,B]|=4$. We may assume that $B\not\leq M$. Then the assertion follows with 11.9. □

From now on we assume without further notice that we are in case (2).

$Fquad$

Lemma 13.4 *If A does act quadratically we have a nice P.*

Proof: Now A induces an F–module offender. If this happens on some component, we get the assertion with 13.2 and 11.8. So it happens on $F(H/C_H)$. But Y_H is a strong F-module and so we get that $|A|=2$ and A induces transvections on Y_H. Then we get the assertion with 13.2 and 11.9. □

$2Ffaith$

Lemma 13.5 *Suppose A induces a cubic, not quadratic, $2F$–module offender, which acts faithfully on some component K of H/C_H. Then we have a nice P.*

Proof: Suppose first K be covered by M. As $AC_H/C_H\cap O_2(M)C_H/C_H\neq 1$ by 13.1, we see that $C_{AC_H/C_H}(K)\neq 1$, contradicting the faithful action. so we have that $K\not\leq(M\cap H/C_H)$.

The possible groups for K are given by 3.29 - 3.32. As $H \not\leq M$ we get by 9.1 that $m_3(K) \leq 3$. Hence $K \cong 3U_4(3)$ is not possible.

Assume first that K is normal in H. Then we have that $C_{Y_H}(K) = 1$ by 8.11.

Let K be alternating, $K \not\cong A_5, A_6$ or A_8. These groups will be handled as Lie type groups lateron. If we have the permutation module and $[Y_H, K]$ involves just one module, we have an F–module and so the result follows with 11.8. So we may assume that we have exactly two permutation modules. As in 11.4 we see that then $K \not\cong A_{11}$ or A_7 as in that cases K is generated by centralizers of elements in $C_{[Y_H,K]}(S)^\sharp$. If $K \cong A_9$, we have $K \cap M \cong A_8$. Then $[Y_H, K] \leq Z(O_2(M))$. Set $V_M = \langle [Y_H, K]^M \rangle$. Let $C = C_M(V_M)$. Then we have that $O_2(M/C) \neq 1$. Let $T \leq S$ such that $S \cap C \leq T$ and $TC/C = O_2(M/C)$. We have that $K \cap M \not\leq C$ and so we may assume that $K \cap M$ is in $N_H(T)$. But then $[T, [K, Y_H]] = 1$, which shows that $[T, V_M] = 1$, a contradiction.

Assume next that we have $K \cong A_9$ on the 8–dimensional spin module. Then we have that $M \cap K \cong 2^3 L_3(2)$, otherwise we may argue as before. Hence we have that $|Y_M| = 2$ and we have the subgroup $R \cong A_8$ which induces a 4-dimensional module on $\langle Y_M^R \rangle$. In particular we get some parabolic P, with $P/O_2(P) \cong \Sigma_3$, inducing the 2-dimensional module, which is a nice P.

Assume next that we have $K \cong A_7$ and $[Y_H, K]$ involves both 4-dimensional modules. Then $|Y_M| = 2$ and $M \cap K \cong \Sigma_4$. Now take any other parabolic, we get that we have Σ_3 on a natural module, which is a nice P.

Let $K \cong 3A_6$ on two 6-dimensional modules, then we have that offender act quadratically, a contradiction.

Assume next that we have a sporadic group. Then by 3.34 we just have one of the Mathieu groups M_{2i} on one of its natural modules.

If $K \cong M_{24}$, then $M \cap H/C_H \cap K = K_1 \cong 2^4 A_8$ or $2^6 3 \Sigma_6$ depending on the module. If $K \cong M_{23}$. Then $K_1 = M \cap H/C_H \cap K \cong 2^4 A_7$. If $Y_M \cap W > C_W(K_1)$, then $Y_M \cap W = C_{W/C_W(K_1)}(O_2(K_1))$. But then $C_{K_1}(Y_M) = 1$, which contradicts $[A, Y_M] = 1$. This gives $|Y_M| = 2$.

If $K \cong M_{22}$ then $K_1 = M \cap H/C_H \cap K \cong 2^4 A_6$ or $2^4 \Sigma_5$. In the first case we argue as before for $|Y_M| = 2$. In the latter we have that $C_W(O_2(K_1))$ is of order 4. But this group contains some x such that $C_K(x) \cong L_3(4)$, which shows that we have $x \notin Y_M$ by 5.14. So again we have $|Y_M| = 2$.

In any case we have that $|Y_M| = 2$. So in case of M_{24} we get Σ_3 on a 2-dimensional module, while in case of M_{22} or M_{23} we also might get Σ_5 on the orthogonal module. Hence we have groups as in 11.4, so a nice P.

Let next K be a group of Lie type. Let first be the Borel subgroup in M. Suppose that $[K, Y_H]$ involves just modules $V(\lambda)$. By 11.5 we get that $K \cong A_6$ and $P\Gamma L_2(9)$ is induced on K. In particular we have two modules V_1, V_2 in $[Y_H, K]$ and $|AC_H/C_H| = 8$. As A does not act quadratically we have that $[Y_H, K]$ cannot be in $O_2(L)$. Let $V_1 \not\leq O_2(L)$. Then by 4.2 we have that $|[V_1, A]| \geq |A|$. But this is not true for either of the two natural modules for Σ_6.

So assume that we have $L_n(q^2)$ or $Sp(4, q^2)$ on a tensor product module. Then in both cases we get $|Y_M| = q$ as it has to be in the center of the corresponding parabolic. Further there is some parabolic P with $E(P/C_P) \cong L_2(q^2)$ and Y_P is the orthogonal module.

So we have that the Borel subgroup is not in M. If all modules are of type $V(\lambda)$, we get the assertion with 11.6. So we have to handle the cases $L_n(q^2)$ and $Sp(4, q^2)$ on the tensor product module. Then we get a subgroup $U \cong L_{n-1}(q^2)Z_{q+1}$ in $M \cap H/C_H \cap K/O_2(H \cap H/C_H \cap K)$ centralizing Y_M, where $n = 3$ in case of $Sp(4, q^2)$. We have that $|C_{Y_H}(U)| = q$ and there is a subgroup of order $q - 1$ acting on $C_{Y_H}(U)$. Further we have that $C_{Y_H}(U) \geq Y_M$. As B is not covered by M, we get that $Y_M \neq C_{Y_H}(U)$. Hence there is some group P_1 containing S and the group of order $q-1$, which acts semi regularly on $C_{Y_H}(U)$. Obviously $\langle P_1, M_0 \rangle = P_1 M_0$.

Let $q = 2^{2^c r}$, r odd. Choose b minimal such that the subgroup of order $2^{2^b r} - 1$ in $P_1/O_2(P_1)$ is not in M. Then first $b \geq 1$, as the subgroup of order $2^r - 1$ normalizes a Sylow 2–subgroup and so is in M. Replace P_1 by that group and set $P = P_1 M_0$. Now the subgroup L_1 of order $2^{2^{b-1} r} - 1$ is in M and acts regularly on Y_M. As $[L_1, O_2(M_0)] \leq O_2(M_0)$ so $|Y_P : C_{Y_P}(O_2(M_0))|^2 = |Y_P|$. As $C_{Y_P}(O_2(M_0)) = Y_M$, we have a nice P.

So we may assume that K is not normal. Then we have that $K \cong Sz(q)$, $L_2(q)$, $SL_3(4)$, $L_3(2)$, $SU_3(8)$, $3A_6$, $3A_7$, $3M_{22}$. By 3.34 the last case cannot occur. If we have $SU_3(8)$ then just natural modules are involved and so an offender acts quadratically, a contradiction. The same applies for $SL(3, 4)$ and $Sz(q)$. If we have $L_3(2)$ we see as above that offender either act quadratically or exact, both is a contradiction. In the cases of $3A_6$ and $3A_7$ there are at most two modules in $[K, Y_H]$ so a conjugate has to centralize these modules, which shows that $Z(K)$ acts trivially. Now $[Y_H, K]$ involves at most two modules. Hence conjugates of K have to centralize $[K, Y_H]$. Now

$C_{Y_H}(\langle K^S \rangle) = 1$ and so we have exactly two conjugates. So we see that we get $\Sigma_3 \wr Z_2$ on the orthogonal module, which gives a nice P.

So what is left is $K \cong L_2(q)$. As A does not act quadratically, we have that $[Y_H, K]$ involves the orthogonal module just once and $q = r^2$ or $q = 4$ and $|A| = 2$, where we might have two such modules. In any case any component of H/C_H different from K has to centralize $[Y_H, K]$. Let the Borel subgroup be in M. Set $\langle K^S \rangle = K \times K_1$. Then $Y_H = [K, Y_H] \times [K_1, Y_H]$. Let $1 \neq x \in C_{Y_M}(S)$. Then $x \notin [K, Y_H] \cup [K_1, Y_H]$. Now $\langle x^B \rangle \cap [K_1, Y_H] \neq 1$. In particular $Y_M \cap [Y_H, K_1] \neq 1$, which with 5.14 gives K is covered by M, a contradiction. So we have that the Borel subgroup is not in M. Then we find a subgroup P_1 with $O^2(P_1/O_2(P_1)) \cong Z_{r-1} \times Z_{r-1}$ which is not in M. Now as in 11.6 looking for a minimal such group we get a nice P. □

KA

Lemma 13.6 *Suppose A is a cubic, not quadratic, $2F$–module offender and K is some component of H/C_H with $[K, A] \neq 1$, and $[K, A] \not\leq K$ then we have a nice P.*

Proof: Let $KK_1 = K^A$. By 4.3 we have that $|A| > 4$, $|Y_H^g : A| = 2$, and $K \cong L_n(2)$. Hence we have $K \cong L_3(2)$. Further as we have transvections we see with 3.38 that $[Y_H, KK_1]$ is the direct sum of two natural modules. We have $KK_1 = \langle K^S \rangle$. If $K \leq M$, then we have that $[Y_H, KK_1] \leq Y_M$, but his contradicts $[A, Y_M] = 1$. So $K \not\leq M$ and then $C_{Y_H}(KK_1) = 1$ and $|Y_M \cap [Y_H, KK_1]| = 2$. As there are no transvections on the nonsplit extension of the natural module by a trivial module, we have $Y_H = [Y_H, KK_1]$ and so $|Y_M| = 2$. Now $M_0 = N_G(S)$ and there is some P with $P/O_2(P) \cong \Sigma_3 \wr Z_2$ inducing an orthogonal module in Y_P, which the yields a nice P. □

KnonA

Lemma 13.7 *Suppose A is a cubic, not quadratic, $2F$–module offender and K is some component of H/C_H with $1 \neq [K, A]$, then we have a nice P.*

Proof: By 13.6 we may assume that $[K, A] \leq K$. By 13.5 we may assume that $B = C_A(K) \neq 1$. Then there is some further component (or a Sylow subgroup of $F(H/C_H)$), K_1 with $[B, K_1] \neq 1$. Choose K_1 with $|B : C_B(K_1)|$ maximal. Let $C = C_A(K_1)$. If $[C, K] = 1$, then $C \leq B$. Now choose K_2 with $[C, K_2] \neq 1$. Then we have $|B : C_B(K_2)| \leq |B : C_B(K_1)|$ by maximality. Hence there is some $b \in B$ with $[K_1, b] \neq 1$ but $[K_2, b] = 1$ as $C_B(K_1) \neq C_B(K_2)$. So we may choose two components K_1, K_2 (or Sylow subgroups of $F(H/C_H)$) with $A_i = C_A(K_i) \neq 1$ and $[A_i, K_{3-i}] \neq 1$, $i = 1, 2$.

Let $A = \tilde{A}_1 \times C_A(K_1)$ and V_1 be a quasi irreducible $K_1\tilde{A}_1$–submodule in Y_H. Suppose first that $V_1 \not\leq O_2(L)$. Let $V_1 \cap Y_H^g \not\leq C_{V_1}(K_1)$. Then for all $a \in A^\sharp$ we have $V_1 \cap V_1^a \not\leq C_{V_1}(K_1)$. In particular $V_1^A = V_1$. Then $[V_1, A_1] = 1$, which contradicts $V_1 \not\leq O_2(L)$. So we have that $V_1 \cap Y_H^g \leq C_{V_1}(K_1)$. Let

$v \in V_1 \setminus O_2(L)$, then we have that $[v, \tilde{A}_1] \cong \tilde{A}_1$. No element in $A_1^\sharp$ centralizes V_1. Let $v_1 \in V_1 \cap O_2(L_1) \setminus C_{V_1}(K_1)[v, \tilde{A}_1]$. Then there is some $a_1 \in A_1^\sharp$ and $v \in V_1 \setminus O_2(L)$ such that $v_1 = [a_1, v]$. In particular $[a_1, v_1] = 1$. As V_1 is quasi irreducible for $K_1\tilde{A}_1$ which is centralized by a_1, we see that $[a_1, V_1] = 1$, a contradiction. So we get that $[v, \tilde{A}_1]C_{V_1}(K_1) = V_1 \cap O_2(L)$ and $|V_1 \cap O_2(L)/C_{V_1}(K_1)| = |\tilde{A}_1|$. Now let $1 \neq a \in A_1$. Set $V_2 = V_1^a$. Then also V_2 is a quasi irreducible $K_1\tilde{A}_1$–module. Further $[V_1, a]$ is also such an module. As $[V_1, a] \leq O_2(L)$, we see that $[[V_1, a], \tilde{A}_1] \leq Y_H \cap Y_H^g$. So we have shown

$$\text{(1)} \qquad \text{For } a \in A_1 \text{ we have } [V_1, a, \tilde{A}_1] \leq Y_H \cap Y_H^g.$$

We collect some facts about the action on V_1. We have that $\tilde{A}_1$ acts quadratically on $V_1/C_{V_1}(K_1)$ and by (1) also on $[V_1, a]$. Further $V_1/C_{V_1}(K_1)$ is an F–module with offender $\tilde{A}_1$. We have that $C_{V_1/C_{V_1}(K_1)}(a_1) = C_{V_1/C_{V_1}(K_1)}(\tilde{A}_1)$ for all $1 \neq a_1 \in \tilde{A}_1$. Application of 3.17 now gives that K_1 is solvable or $K_1/Z(K_1) \cong L_n(r)$, $Sp(2n, r)$, r even, or A_7 or $3A_6$, or $|\tilde{A}_1| = 2$. Suppose the latter, then we have that $|V_1/C_{V_1}(K_1)| = 4$ as $|V_1 \cap O_2(L_1)/C_{V_1}(K_1)| = 2$. Then K_1 is solvable. If K_1 is solvable it is a 3–group as it induces an F–module. As we can look at $\langle V_1^{K_2} \rangle$, we see that there is also some module not in $O_2(L)$ and so K_2 also has the structure above. As one of both K_1, K_2 must be nonsolvable, we may assume that K_1 is nonsolvable.

Let first $K_1 \cong 3A_6$ acting faithfully. But then in the 6–dimensional module we see that there is no element v with $[v, \tilde{A}_1] = C_{V_1}(\tilde{A}_1)$.

Let next $K_1/Z(K_1) \cong A_7$. Then V_1 is the four dimensional module and $|\tilde{A}_1| = 4$. Now as $m_3(H) \leq 3$, we see that $K_2/Z(K_2) \cong L_2(r)$, $L_3(r)$, A_6 A_7, or K_2 is a 3–group of rank at most two. As $\langle V_1^{K_2} \rangle$ contains an irreducible module W for K_1K_2 with $W \cap Y_H^g \neq 1$ and $W \not\leq O_2(L)$, we see that A acts faithfully on K_1K_2. If we have $K_2/Z(K_2) \cong L_3(r)$, then W is a tensor product of the natural $SL_3(r)$– module with V_1 and so $|A|^2 > |W : C_W(A)| \geq r^8$, by 4.2 as A does not act quadratically. As $|A| \leq 4r^2$, we get a contradiction. If $K_2 \cong L_2(r)$, we see that $|W : C_W(A)| \geq r^6$, which shows $r = 2$, which is also the case for K_2 to be solvable. But now $|A| \leq 2^3$ which shows $|W : C_W(A)| = |A|^2$, contradicting 4.2 again. If $K_2/Z(K_2) \cong A_6$ or A_7, we get $|W : C_W(A)| \geq 2^8$ and again $|W : C_W(A)| = |A|^2$, a contradiction.

Let next $K_1/Z(K_1) \cong Sp(2n, r)$ and $V_1/C_{V_1}(K_1)$ be the natural module. Then $n \leq 3$. Let first $K_1 \cong Sp(6, r)$, then $m_3(K_2) = 0$, a contradiction. So we have $K_1/Z(K_1) \cong Sp(4, r)$. Further we have that $|\tilde{A}_1| = r^2$ as $|[V_1, \tilde{A}_1]C_{V_1}(K_1)/C_{V_1}(K_1)| = |\tilde{A}_1|$. We may assume that $K_2 \cong L_2(t)$, $L_3(t)$ or solvable, as the cases $K_2/Z(K_2) \cong A_6$ or A_7 have been handled before. In the case of $L_3(t)$, we have that $|W : C_W(A)| \geq s^8$, $s = \max(r, t)$.

Now $|A| \leq r^2t^2$. Let $K_2 \cong L_2(t)$, then $|W : C_W(A)| \geq s^6$, and $|A| \leq r^2t$. If K_2 is solvable we get that $|W : C_W(A)| \leq r^6$, and $|A| \leq 2r^2$. In all cases we get $|W : C_W(A)| = |A|^2$, a contradiction.

Let now $K_1/Z(K_1) \cong L_n(r)$. Then we have that $K_2/Z(K_2) \cong L_m(t)$ or K_2 is solvable, as all the other cases have been handled before. Suppose first $r > 2$, then we see that $n \leq 4$.

Let $K_1 \cong L_4(r)$. Then $V_1/C_{V_1}(K_1)$ is the natural module or the orthogonal module. In both cases by 3.36 we have $C_{V_1}(K_1) = 1$. Now $K_2 \cong L_2(t)$, $L_3(t)$ or solvable. Let $GF(\ell)$ be the largest common subfield of $GF(r)$ and $GF(t)$. Let $r = \ell^x$, $t = \ell^y$. Then $W = V_1 \otimes V_2$, V_2 be the natural K_2–module and $U = [V_1, N_A(V_1)] \otimes [V_2, N_A(V_2)] = C_{V_1}(N_A(V_1)) \otimes C_{V_2}(N_A(V_2))$ is contained in a complement of $Y_H \cap Y_H^g$ in $Y_H \cap O_2(L)$ and so of size at most $|A|$. We have that $|A| \leq \ell^{3x+2y}$, ℓ^{3x+y}, $2r^3$, respectively. Further $|U| \geq \ell^{5xy}$, ℓ^{4xy}, or K_2 is solvable. Let first K_2 be nonsolvable. Then we have that $t = \ell$ and $K_2 \cong L_2(t)$ and $K_1 \cong L_4(t^x)$. Now $t > 2$ and then for p dividing $t-1$, we get $m_p(H) \geq 4$, a contradiction to 9.1. So K_2 is solvable. Now $|U| = r^3$. We have that $|V_1 : C_{V_1}(\tilde{A}_1| = r$. So $|V_1 : V_1 \cap O_2(L)| = r$. Now we get $|W \cap O_2(L) : C_W(A)| = r^4 > 2r^3$, as $r > 2$, a contracition.

Let next $K_1/Z(K_1) \cong L_3(r)$. We have that $K_2/Z(K_2) \cong L_2(t)$, $L_3(t)$ or K_2 is solvable. Let $K_2 \cong L_2(t)$ and define ℓ as before. Then we have that $|A| \leq \ell^{2x+y}$ and again $y = 1$. Further we have that $|W \cap O_2(L) : C_W(A)| = r^3 \leq r^2t$, which shows $r = t$. As $Y_H^g \cap W \neq 1$ we have that A acts faithfully on K_1K_2. Now $W = [Y_H, K_1K_2]$. As S normalizes both groups and both are not in M, we see that $Y_M \leq W$, and $C_{Y_H}(K_1) = C_{Y_H}(K_2) = 1$. Now if the Borel subgroup is in M, we get a subgroup P with $E(P/C_P) \cong L_2(r)$, Y_P is the natural module and $M_0 \leq P$. This is 11.4(ii). If the Borel subgroup is not in M we get P_1 with $P_1/O_2(P_1) \cong Z_{r-1}$ acting on a group of order r containing Y_M. Then we choose P_1 minimal and get a nice P.

Let now $K_2 \cong SL_3(t)$. Then with the same notation as before we get $|A| \leq \ell^{2x+2y}$ and $|U| \geq \ell^{4xy}$. This shows $x = y$ and then $r = t$. Let now p be a prime divisor of $r - 1$, then we must have $m_p(H) \leq 3$ as otherwise by 9.1 we get $H \leq M$. This now implies $r = 4$, hence $K_2 \cong SL_3(4) \cong K_1$. If both components are normalized by S, we may argue as before. So assume that $\langle K_1^S \rangle = K_1K_2$. Again as $K_1 \not\leq M$, we see that $W = Y_H$ and Y_M is contained in a subgroup of order 4. If $Z(K_1) \not\leq M$, we just choose $P_1 = Z(K_1)S$ and $P = M_0P_1$, which gives us a nice P. So we may assume that $Z(K_1) \leq M$. In that case $|Y_M| = 4$. Now also a Borel subgroup of K_1K_2 is in M and we get a minimal parabolic P_1 in $\langle K_1^S \rangle$ with $E(P_1/O_2(P_1)) \cong L_2(4) \times L_2(4)$ inducing the orthogonal module on Y_P, a nice P again.

Let now K_2 be solvable. Then we get $|A| \leq 2r^2$. But as $|W : C_W(A)| \geq r^3$, we have a contradiction to $r > 2$.

Let next $K_1 \cong K_2 \cong L_2(r)$, then $W = [K_1K_2, Y_H]$ is the tensor product of two natural modules. Again A acts faithfully and $C_{Y_H}(K_1K_2) = 1$, as both components are not in M. Hence if the Borel subgroup is in M we just set $P = K_1K_2M_0$, if S does not normalize K_1, and $P = K_1M_0$ otherwise. If the Borel subgroup is not in M, then we have a subgroup P_1 with $P_1/O_2(P_1) \cong Z_{r-1} \times Z_{r-1}$ acting on a group of order r containing Y_M. But then again there is just one group Z_{r-1} which acts, and H just induces Galois automorphisms on this group, which then gives a nice P again.

Let now $r = 2$, then $n \leq 7$. As the order of K_2 is divisible by three, we even get $n \leq 5$. Let $n = 4$ or $n = 5$, then $K_2 \cong L_3(2)$ or a cyclic 3–group. Then both groups are normal in H and also both are not in M. This shows that $C_{Y_H}(K_1) = C_{Y_H}(K_2) = 1$. As $Y_H \cap W \neq 1$, we see that A acts faithfully on K_1K_2. Further there is no second tensor product involved. As A has to induce an F-module on Y_H/W, we get that $W = [Y_H, K_1K_2]$. In particular we have that $|Y_M| = 2$ and so we may take a minimal parabolic in K_1 not centralizing Y_M and so we have a nice P. Let finally $n = 3$, then either $K_2 \cong L_3(2)$ or K_2 is solvable. If S normalizes K_1, then we may argue as before. So we may assume that $K_1 \cong K_2 \cong L_3(2)$. Further $\langle K_1^S \rangle = K_1K_2$. Again $K_1 \not\leq M$ and so $C_{Y_H}(K_1K_2) = 1$, $W = [Y_H, K_1K_2]$ and $|Y_M| = 2$. Now there is some parabolic $\Sigma_4 \wr Z_2$ in $\langle K_1^S \rangle$, which induces an orthogonal module, so we get a nice P.

So we now may assume that any quasi irreducible submodule for K_1 is contained in $O_2(L)$ and the same applies for K_2. Let W_1 be the submodule generated by all these submodules for $K_1\tilde{A}_1$ and correspondingly W_2 the one for $K\tilde{A}_2$. As for any V_1, we have that $V_1 \cap Y_H^g \not\leq C_{V_1}(K_1)$, we see that $[V_1, A_1] = 1$, so we have that $[W_1, K_2] = 1$ and also $[W_2, K_1] = 1$. Let now $B = C_A(K_1K_2) \neq 1$. Then we have K_3 with $[K_1K_2, K_3] = 1$ and $[B, K_3] = K_3$. We have $[W_1, B] \leq Y_H^g$. Then we see that $[W_1, B] \leq C_{W_1}(K_1)$ and so we have that $[K_3, W_1] = 1$. By the same argument we have that $[K_3, W_2] = 1$. So we see that $[K_3, Y_H] \leq C_{Y_H}(K_1K_2)$. Assume $[K_3, Y_H] \not\leq O_2(L)$. Then there is some $v \in [K_3, Y_H]$ such that $[v, A](Y_H \cap Y_H^g) = Y_H^g \cap O_2(L)$. As $[v, A] \leq [K_3, Y_H]$, we see that $W_1W_2 \leq [K_3, Y_H](Y_H \cap Y_H^g)$. But A centralizes $[K_3, Y_H](Y_H \cap Y_H^g)/[K_3, Y_H]$ and so also $W_1W_2/[K_3, Y_H]$. Hence as $K_1K_2 = [K_1K_2, A]$, we have that $W_1W_2[K_3, Y_H]/[K_3, Y_H]$ is centralized by K_1K_2 and so $W_1, W_2 \leq [K_3, Y_H]$, which shows $[W_1W_2, K_1K_2] = 1$, a contradiction. So we have that $[K_3, Y_H] \leq O_2(L)$.

In particular there are $K_1 \cdot K_2 \cdots K_s$, such that A acts faithfully on $K_1K_2 \cdots K_s$ and there is a faithful module W for $K_1 \cdots K_sA$, which is in

$O_2(L)$. Further $W = W_1 \oplus W_2 \oplus \cdots \oplus W_s$ with $[W_i, K_i] = W_i$ and $[W_j, K_i] = 1$ for $i, j = 1, \ldots, s$ and $i \neq j$. Hence A acts quadratically and as an F–module offender on W.

We may assume that K_1 induces an F-module on W_1 with offender $\tilde{A}_1$. As $s > 1$ and F–module offender for solvable groups are exact by 3.15 we even may assume that K_1 is nonsolvable.

Let K_i be some Sylow subgroup of $F(H/C_H)$. Then $[S, W_i] \leq W_i$ and so $1 \neq Y_M \cap W_i$. By 5.14 we have that K_1 is covered by M, which now with 13.2 yields a nice P. So we may assume that all K_i are nonsolvable. Further we get that

$$W_2 \oplus \cdots \oplus W_s \cap Y_M = 1. \tag{2}$$

By (2) we have that $K_2^S \neq K_2$. Let $K_1^S = K_1$. Then we have that $\langle W_2^S \rangle$ is centralized by K_1. But $1 \neq Y_M \cap \langle W_2^S \rangle$ and so by 5.14 we have K_1 is covered by M, which with 13.2 gives a nice P. So we have that $K_1^S \neq K_1$. Further we see that K_1 cannot centralize $\langle K_2^S \rangle$, which gives that for $t \in S$ with $K_2^t \neq K_2$, we must have $K_1^t \in \{K_2, K_2^t\}$. So we may assume $K_2 = K_1^t$ for some $t \in S$. But the same applies for any K_i, $i = 2, \ldots, s$, which gives $s = 2$. As $K_1^S \neq K_1$ we have that all Sylow r–subgroups, r odd, of K_1 are cyclic or r divides $|Z(K_1)|$. As K_1 induces an F–module, we get with 3.16 that K_1 is $L_2(q)$, $L_3(2)$, $SL(3,4)$, $3A_6$ or $3A_7$. Further W_1 contains at most two nontrivial irreducible modules. Now $K_1 K_1^t = \langle K_1^S \rangle$, as we cannot have four conjugates of K_1.

If A is an exact offender on $W_1 W_2$, then by 4.2 we see that $|Y_H/W : C_{Y_H/W}(A)|^2 \leq |A|$. Hence if $K_1 \ncong L_3(2)$ or $SL(3,4)$, we see that $W_1 W_2 = [Y_H, K_1 K_2]$. But then A acts quadratically on Y_H, a contradiction. So we have $K_1 \cong L_3(2)$ or $SL(3,4)$. As the element of order 3 in $Z(K_1)$ is the same as in $Z(K_2)$, we get that $SL(3,4) \cong K_1$ is not possible.

Suppose first again $W_1 = [Y_H, K_1]$, then Y_H has to contain a nonsplit extension of the natural module, on which A does not act quadratically. As $|A| = 16$, we now get that W_1 is irreducible. We see that $|Y_H| \leq 2^8$ and so $|Y_M| = 2$ and again using the $\Sigma_4 \wr z_2$ in $\langle K_1^S \rangle$ we get a nice P.

So we may assume that $W_1 W_2 \neq [Y_H, K_1 K_2]$. We get that $Y_M \cap W_1 W_2 = C_{W_1 W_2}(S)$, which is of order at most 4. If $Y_M \leq W_1 W_2$, we can choose the minimal parabolic not in M and so we get one or two orthogonal modules, which is the situation of 11.5(ii), so we have a nice P.. So assume $Y_M \not\leq W_1 W_2$. If $Y_M W_1 W_2 / W_1 W_2$ is centralized by $K_1 \cap M$, we may proceed as before getting at most three orthogonal modules, which still is the situation of 11.5(ii). So we may assume that $M \cap K_1$ acts on $Y_M W_1 W_2/W_1 W_2$. Then W_1 is the natural module for K_1 and in Y_H/W_1 we have the dual module. Now we have that there is some $U \leq Y_M$, $|U : W_1W_2 \cap U| = 4$ and $[U, K_2] \leq W_1W_2$. As $|UW_1W_2 : C_{UW_1W_2}(K_2)| \leq 16$, as $UW_1W_2/C_{UW_1W_2}(K_2)$ involves exactly one nontrivial module. As $Y_M \cap C(K_2) = 1$ by 5.14 and 13.2, we see that $|UW_1W_2/C_{UW_1W_2}(K_2) : Y_M C_{UW_1W_2}(K_2)/C_{UW_1W_2}(K_2)| \leq 2$. As A centralizes Y_M, we see that A induces transvections on $UW_1W_2/C_{UW_1W_2}(K_2)$ and so this is the natural modules, i.e. $|UW_1W_2/C_{UW_1W_2}(K_2)| = 8$ and then $UW_1W_2 = Y_M C_{UW_1W_2}(K_2)$. But now A would even act trivially, a contradiction. □

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Lemma 13.8 *Let $Y_H \not\leq O_2(M)$, then there is a nice P.*

Proof: By 13.3, 13.4, 13.5, 13.6 and 13.7 we just have to handle the case that we have (2) and A acts faithfully on $F(H/C_H)$, induces a cubic, not quadratic, $2F$–module offender and centralizes any component of H/C_H. This first shows $|A| > 2$. By 2.1 we also have $|A| \leq 8$, as otherwise $m_p(H) \geq 4$ for some odd p, which with 9.1 contradicts $H \not\leq M$. Hence 4.2 now shows that $|Y_H : C_{Y_H}(A)| \leq 2|A|$. This in particular gives us that A acts faithfully on a Sylow 3–subgroup P of $F(H/C_H)$. If we have a transvection in A, then we get the assertion with 11.9 and 13.2.

So there is no transvection in A. Let $|A| = 8$ and $D_1 \times D_2 \times D_3$ the subgroup of H/C_H given by 2.1. Then we may assume that $D_1 \times D_2$ has to induce a 4-dimensional orthogonal module W. But then $|W : C_W(A)| \geq |A|$, which says that A has to induce transvections on $[O_3(D_3), Y_H]$, which gives that we have transvections on Y_H.

Thus we have that $|A| = 4$. Now $U = D_1 \times D_2 \cong \Sigma_3 \times \Sigma_3$ and $[U, Y_H]$ is the orthogonal module. Hence we have that $[P, A]A$ induces an orthogonal module on Y_H and so $|[P, A]| = 9$. Let $T = N_S([P, A])$ and $s \in N_S(T)$ with $s^2 \in T$ and $[P, A]^s \neq [P, A]$. We have that $[P, A]$ is normal in P. Set $R = [P, A][P, A]^s$, then $|R| \leq 3^4$. Let $|R| = 3^4$. Then $[P, A] \cap [P, A^s] = 1$. As $A^s \in T$, we see that $[P, A, A^s] = 1$. Hence we have $[R, [A, A^s]] = 1$. But then also $[P, [A, A^s]] = 1$. This now gives $[[A, A^s], F^*(H/C_H)] = 1$ and so $[A, A^s] \leq C_H$. Now we have that $R\langle A, A^s, s\rangle$ acts on $[[P, A], Y_H][[P, A^s], Y_H] = V$, which is of dimension 8. In $X = GL(8, 2)$ we have that R is uniquely determined and $N_X(R)/R \cong Z_2 \times Z_2 \times \Sigma_4$. As AA^s now is elementary abelian of order 16, there are just two possibilities. But in both cases we have that AA^s contains the center of a Sylow 2–subgroup of $N_X(R)$. Hence AA^s contains transvections on V. As $[P, A, A^s] = 1$, we see that these are in $A \cup A^s$, a contradiction.

So we have that $|R| = 3^3$. Let V be as before. Now $R\langle A, A^s\rangle \leq X \cong$

$GL(6,2)$, since now $[[P,A],Y_H] \cap [[P,A]^s,Y_H] \neq 1$. If R is not elementary abelian, then R is extraspecial and so $Z(R)$ acts fixed point freely on V. But $Z(R) \leq [P,A] \cap [P,A^s]$ and $|[Y,[P,A]]| = 16$. Hence R is elementary abelian and so again R is uniquely determined. Now $N_X(R)/R \cong Z_2 \times \Sigma_4$. Further also $s \in N_X(R)$ and so we see that AA^s is elementary abelian. Again AA^s contains a transvection i. We have $|C_R(A)| = |C_R(A^s)| = 3$. As $|[R,i]| = 3$, we may assume that $[C_R(A),i] = 1$. But A^s does not centralizes $C_R(A)$. Hence $|C_{A^s}(C_R(A))| = 2$ and so $C_{A^s}(C_R(A)) = A \cap A^s$. In particular $C_{AA^s}(C_R(A)) = A$, which gives the contradiction $i \in A$.

So we have $T = S$ and then $[P,A]$ is normalized by S. Let $C_{Y_H}([P,A]) \neq 1$, then $Y_M \cap C_{Y_H}([P,A]) \neq 1$. By 5.14 we get that $[P,A]$ is covered by M. This now contradicts 13.1 and $C_A([P,A]) = 1$. So we have that $C_{Y_H}([P,A]) = 1$ and then $|Y_H| = 16$ and then H/C_H is contained in $\Sigma_3 \wr Z_2$ inducing the orthogonal module, which is the situation of 11.5(ii), hence we have a nice P. □

14 The amalgam (M, P), $b \neq 2$

In this chapter we will study the amalgam set up in the last three chapters. We just collect

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Proposition 14.1 *There is a subgroup P containing M_0, such that one of the following holds*

(1) $E(P/C_P) \cong L_2(q^2)$ and Y_P is the orthogonal module.

(2) $E(P/C_P) \cong L_2(q) \times L_2(q)$ and Y_P is the $\Omega^+(4,q)$–module. Further the components of $E(P/C_P)$ are not normal in P.

(3) There is a normal subgroup P_1 such that $P = P_1M_0$ and $E(P_1/C_P) \cong L_2(q)$ or $P_1/C_P \cong \Sigma_3$ and Y_P is a sum of natural modules.

4) $E(P/C_P) = K_1 \times K_2$, $K_1 \cong K_2 \cong A_5$, $Y_P = V_1 \times V_2$, where $[K_i, Y_P] = V_i$ and $[K_{3-i}, V_i] = 1$. Further V_i is the orthogonal K_i–module and K_1 is not normal in P/C_P.

(5) There is a normal subgroup P_1 such that $P = P_1M_0$ and $P_1/O_2(P_1) \cong \Sigma_3 \wr Z_2$ or $\Sigma_3 \times \Sigma_3$ and Y_P involves just orthogonal modules and at most three of them. Further $|Y_M| \leq 8$.

(6) P/C_P is an extension of a cyclic group of order $q^2 - 1$ by Galois automorphisms and $Y_P = Y_M \times Y_M^t$ for some $t \in P$ and P semiregularly on Y_P, where a group of order $q - 1$ normalizes Y_M.

7) P/C_P is an extension of a cyclic group of prime order greater than three, which acts semiregularly on Y_P, Further $Y_P = Y_M \times Y_M^t$ for some $t \in P$.

In (1) - (5) the group P is minimal with respect not to be in M.

From 14.1 we get the following important lemma.

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Lemma 14.2 *Let $u \in M$ be a p–element with $p \in \sigma(M)$ and $C_G(u) \leq M$. Then $u \notin C_G(Y_P)$. If $u \in N_G(S)$ then $p = 3$ and Sylow 3–subgroups of M are isomorphic to $Z_3 \wr Z_3$ and $N_G(S)$ contains an elementary abelian subgroup of order 9.*

Proof: Let first $u \in C_G(Y_P)$. Set $W = C_G(Y_P)P$. As $Y_M \leq Y_P$, we have that $C_G(Y_P) \leq M$. Now let R be a Sylow p–subgroup of $C_G(Y_P)$ with $u \in R$. If R is noncyclic then $N_G(R) \leq M$, or Sylow 3–subgroups of M are $Z_3 \wr Z_3$ and R is elementary abelian of order 9. Suppose the latter. If

$W = (M \cap W)C(R)$, we get again $W \leq M$, as $u \in R$. Hence 3 divides $|N_W(R)/C_W(R)|$. Now a Sylow 3–subgroup $\tilde{P}$ of W is extraspecial of order 27 and so $\tilde{P} \leq M^g$ for some $g \in G$. But in M^g this group contains some good E and so $W \leq M^g$. As $S \leq W$, we get with 9.1 that $M = M^g \geq P$, a contradiction.

So we have that $N_G(R) \leq M$ if R is noncyclic. If R is cyclic then $\langle u \rangle = \Omega_1(R)$ and so again $N_G(R) \leq M$. Hence in any case $N_G(R) \leq M$. But then $W = C_G(Y_P)N_G(R) \leq M$, contradicting $P \not\leq M$.

Now let $u \in N_G(S)$ then in particular $u \in M_0$ and so $u \in P$. We have $u \notin C_P(Y_P)$. Let R be a Sylow p–subgroup of $C_P(Y_P)\langle u \rangle$. By assumption we have that $N_P(R) \leq M$. But in none of 14.1(4), (6) or (7) the normalizer of an element of odd order in P/C_P is in $M \cap P/C_P$.

So assume that we have 14.1(1) or (2). If u induce a field automorphism, this shows that u centralizes in P/C_P a group isomorphic to $L_2(r)$. Hence by Frattini we have that $N_P(R)$ involves $L_2(r)$ and so $P = \langle N_P(R), M_0 \rangle$, a contradiction. If u induces an inner automorphism, then u is inverted by some element, which is not in M, a contradiction again. So we are left with $[u, E(P/C_P)] = 1$ and then by Frattini $N_P(R) \not\leq M$.

So assume now that we have 14.1(3) or (5). If $L_2(q)$, $q > 2$ is involved, we can argue as above. So assume that P_1/C_P is solvable. If $[u, P_1] \leq C_P$, then again $N_P(R)$ covers P_1/C_P, a contradiction. So we must have $P_1/C_P \cong Z_3 \wr Z_3$ and $p = 3$. But $u \notin N_G(S)$ and so u has to centralize P_1/C_P again, as a Sylow 2–subgroup of P_1 is dihedral. □

goodp

Lemma 14.3 *Suppose that P is one of the groups in 14.1. There is no good p–element in $P \cap M$, or $p = 3$ and a Sylow 3–subgroup of M is isomorphic to $Z_3 \wr Z_3$.*

Proof: Let $u \in P \cap M$ be a good p–element. By 14.2 we have that $u \not\leq C_P$. Set $T = S \cap C_P$ and $P_1 = N_P(T)$. Then $P = C_P P_1$. We have that $M \cap P = C_P(M \cap P_1)$.

We first show that $u \not\leq M \cap P_1$. Suppose false. Let R be a Sylow p–subgroup of $M \cap P_1$ with $u \in R$. Then $N_G(R) \not\leq M$. This shows $p = 3$ and Sylow 3–subgroups of M are isomorphic to $Z_3 \wr Z_3$.

Now we have that p divides $|C_P|$ and so $m_p(P) \geq 2$. Let now again R be a Sylow p–subgroup of P, then we have that $N_G(R) \leq M$, a contradiction again. □

Lemma 14.4 *Let $3 \in \sigma(M)$. Assume that there is some component K of $M/O_{3'}(M)$, $K = SL_3^{\epsilon}(q)$, $SL_3^{-1}(q) = SU_3(q)$, $SL_3^{+1}(q) = SL_3(q)$. Assume that there is some M–module V on which K induces a 2F–module with quadratic offender. Then all 3–elements are good if one of the following holds*

(i) 3 divides the order of $|P \cap M|$, P as in 14.1.

(ii) V is irreducible.

Proof: By 3.29 and 3.56 we have that V just involves natural modules for K. If $e(G) \geq 4$, we are done, so we may assume that $e(G) = 3$. Let first $m_3(K) = 1$. Then we see that $C_M(K)$ cannot be a $3'$–group. But now any 3–element centralizes an elementary abelian group of order 27, and we are done again. So we may assume that $m_3(K) = 2$ and 3 divides $q - \epsilon$. Let U be a Sylow 3–subgroup of M. We first show that we may assume $U \not\cong Z_3 \wr Z_3$. By 5.11 all 3–elements in K now are good and so all other 3–elements in U are in the elementary abelian subgroup of order 27, so all are good. Hence we may assume that $U \not\cong Z_3 \wr Z_3$. In particular $N_G(H) = N_M(H)$ for any subgroup H of U with $m_3(H) \geq 2$.

Assume first that 3 divides $|P \cap M|$. We may assume that $U \cap P$ is a Sylow 3–subgroup of $P \cap M$. As we can see $N_P(U \cap P) \not\leq M$. So we have that $m_3(U \cap P) = 1$. Let ω be the element of order three in $U \cap M$, then we have that $N_G(\langle \omega \rangle) \not\leq M$. By [GoLy, (29.1)] we have that ω induces a non inner but inner diagonal automorphism on K. Let now M_1 be the normal subgroup of M generated by $KC_M(K)$ and the possible field automorphism of odd order. Then we have that M_1/M is isomorphic to a subgroup of the outer automorphism group of K, which is a $\{2,3\}$–group. Hence we have that $M = M_1(P \cap M)$. By 5.18 we have that ωM_1 is inverted in M/M_1. Hence there is some $x \in P \cap M$ with $(\omega M_1)^x = \omega^{-1} M_1$. But then $\omega^x = \omega^{-1}$.

Let $K \cong SL_3(q)$. Suppose there is some field automorphism, which inverts ω. As P contains a Sylow 2–subgroup S of M, we have that $P \cap K$ is contained in a parabolic subgroup. As $\omega \notin K$, this is a $3'$–group and so it is contained in a Borel subgroup. But ω and x act on a Cartan subgroup C. As 3 divides $q - 1$, we have that there is some elementary abelian subgroup of order 9 in C on which $\langle x, \omega \rangle$ acts. Now x either centralizes this group or inverts this group. In both cases ω centralizes this group, which gives that $N_G(\langle \omega \rangle) \leq M$, a contradiction. So we have that ω is inverted by a graph or graph-field automorphism. We have now have that $[V_M, K]$ is a direct sum of the natural and the dual module. If $K \cong SU_(q)$. Then we have that $[V_M, K]$ is the natural module by 3.29. Hence in both cases (i) and (ii) we have that $[V, K]$ is either the natural module or we have $K \cong SL_3(q)$ and

$[V,K]$ is a direct sum of the natural and the dual module. In any case by 5.12 we see that $C_M([V_M,K])$ is a 3-prime group. As $C_M(K)$ has to act on the natural module by field multiplication, we get that $C_M(K)$ has cyclic Sylow 3–subgroup Y, where $|Y|$ divides $q-\epsilon$. So we see that $L = \langle K, Y, \omega\rangle$ is a subgroup of $\Gamma GL_3^\epsilon(q)$. Suppose that 9 divides $q-\epsilon$. Then $GL_3^\epsilon(q)/Z(SL_3^\epsilon(q))$ is a direct product of a cyclic group of order greater than three with a simple group. Hence only field automorphisms correspond to elements of order three. As there are elements of order three in L which are not in $SL_3^\epsilon(q)$ and which do not induce field automorphisms, we therefore get that 9 does not divide $q-\epsilon$. So $Y = Z(K)$. But then $m_3(K\langle\omega\rangle) = 3$ and so any 3-element in M is good, a contradiction. □

We will change our group P a little bit. For what follows it is not important that $M_0 \leq P$ it is just important that $Y_M \leq Y_P$. Hence we may replace P by $N_P(S \cap C_P)$. By 3.4 Y_P does not change. So we get

$niceamal1$

Proposition 14.5 *There is a subgroup P containing S but $P \not\leq M$, such that one of the following holds*

(1) $E(P/C_P) \cong L_2(q^2)$ and Y_P is the orthogonal module.

(2) $E(P/C_P) \cong L_2(q) \times L_2(q)$ and Y_P is the $\Omega^+(4,q)$–module.

(3) $E(P/C_P) \cong L_2(q)$ or $P/C_P \cong \Sigma_3$ and Y_P is a sum of natural modules.

(4) $E(P/C_P) = K_1 \times K_2$, $K_1 \cong K_2 \cong A_5$, $Y_P = V_1 \times V_2$, where $[K_i, Y_P] = V_i$ and $[K_{3-i}, V_i] = 1$. Further V_i is the orthogonal K_i–module and K_1 is not normal in P/C_P.

(5) $P/O_2(P) \cong \Sigma_3 \wr Z_2$ or $\Sigma_3 \times \Sigma_3$ and Y_P involves just orthogonal modules and at most three of them.

(6) P/C_P is an extension of a cyclic group of order q^2-1 by Galois automorphisms and P acts semiregularly on Y_P, with an element of order $q-1$ in M.

(7) P/C_P is an extension of a cyclic group of prime order greater than three, which acts semiregularly on Y_P, Further $Y_P = Y_M \times Y_M^t$ for some $t \in P$.

In (1) - (5),(7) the group P is minimal with respect not to be in M.

cen

Lemma 14.6 *If x is a 2-element of P with $[x, Y_P] = 1$, then $x \in O_2(P)$.*

Proof: This follows as by construction $S \cap C_P$ is normal in P. □

We will define a group $\tilde{Y_P}$. In the cases 14.5(3),(6) and (7) we just set $\tilde{Y_P} = Y_P$. If we are in (1) or (2) then let $\tilde{Y_P}$ be the preimage of $C_{Y_P/Y_M}(S \cap E(P/C_P))$. In case (5) let $\tilde{Y_P}$ the group generated by the commutators of the transvections in S. In case (4) let $\tilde{Y_P} = C_{Y_P}(S \cap E(P/C_P))$.

Now set

$$V_M = \langle \tilde{Y_P}^M \rangle.$$

Suppose that $C_M(V_M)$ contains a good E. As $N_G(Y_P) \not\leq M$, we get that P is not as in (3), (6) or (7). Set $\tilde{P} = \langle C_P(x) \mid 1 \neq x \in \tilde{Y_P} \rangle$. In case of (4) or (5) we have $P = \tilde{P}S$, a contradiction. In case (1) and (2) we have always some element $y \in \tilde{Y_P} \setminus Y_M$ whose centralizer in P/C_P involves $L_2(q)$. Hence $\langle \tilde{P}, S \rangle = P$ by minimality. So by 5.11 we have $m_p(C_M(V_M)) \leq 1$. Let $T \leq S$ such that $S \cap C_M(V_M) \leq T$ and $TC_M(V_M)/C_M(V_M) = O_2(M/C_M(V_M))$. Set $\hat{M} = N_M(T)$. Then we have with 2.5 that $\hat{M}$ contains some good E. So

$$O_2(\langle \hat{M}, P \rangle) = 1.$$

We have $C_M(V_M)T \leq C_M$. Hence we get that $Y_M = Y_{\hat{M}}$.

In this chapter we study the amalgam $\Gamma(\hat{M}, P)$. Let $b = b_\Gamma$.

In fact there might be several groups P satisfying 14.5 in one of its cases. So we will assume that we choose P as in 14.5(3) whenever this is possible.

Yalpha

Lemma 14.7 *Let $x \in O_2(\hat{M}) \cap C(Y_M)$ with $[V_M, x] \neq 1$, then $[V_M, x] = Y_M$.*

Proof: First of all we have that $[V_M, x] \leq [V_M, O_2(\hat{M})] \leq Y_M$. All we have to show is equality. Hence we may assume that $[\tilde{Y_P}, x] \neq 1$. If we are in 14.5(3)(6) or (7), we have that $Y_P = \tilde{Y_P}$. In (6) or (7) we have $Y_P = Y_M \times Y_M^t$ and so we have the assertion. In case (3) for any natural module V we have that $[V, x] = C_V(x)$, again the assertion. In 14.5(1) and (2) we have that $\tilde{Y_P} = C_{Y_P/Y_M}(S \cap E(P/C_P))$. As $xC_P \in S \cap E(P/C_P)$ and Y_P is a module over $GF(q)$, we have $[\tilde{Y_P}, x] = Y_M$. In case 14.5(4) we have that $\tilde{Y_P} = C_{Y_P}(S \cap E(P/C_P))$. Now this group is of order 4 and $|Y_M| = 2$, the assertion. Assume 14.5(5). Then in each module W we have that $\tilde{Y_P} \cap W$ is of order 4 and $W \cap Y_M$ is of order 2. As x acts on each module nontrivially, we get the assertion again. □

nogoodE

Lemma 14.8 *There is no good E in M centralizing V_M/Y_M.*

Proof: Let E be a good E with $[V_M, E] \leq Y_M$. Then E normalizes $\tilde{Y_P}$. We have that some $x \in \tilde{Y_P} \setminus Y_M$ is centralized by some good E. Hence $C_P(x) \leq M \cap P$. This shows that we do not have 14.5(1)-(3),(5)-(7). So we have (4) with more than one module.Further we have that E normalizes $O_2(M/C_M(V_M))$. But if we take $x \in \tilde{Y}_P \backslash Y_M$, then $|[x, O_2(M/C_M(V_M))]| = 2$, so $[E, Y_M] = 1$. Hence $[E, V_M] = 1$, a contradiction. □

For the remainder of this chapter set $R_M = C_M(V_M/Y_M)$. With 5.11 we have that $m_p(R_M) \leq 1$ for any $p \in \sigma(M)$.

RM

Lemma 14.9 *Let $\rho \in R_M$ be a p–element with $p \in \sigma(M)$, then $\rho \in C_M$. and Sylow p–subgroups of R_M are cyclic.*

Proof: Assume that $[Y_M, \rho] \neq 1$. First of all we have that $\langle \rho \rangle$ is the only subgroup of order p in a Sylow p–subgroup of R_M, as otherwise by 5.11 some good E centralizes V_M/Y_M a contradiction to 14.8. Let T be a Sylow 2–subgroup of C_M. Then we may assume that ρ normalizes $T \cap R_M$. Set $N = N_{R_M}(T \cap R_M)T$. Then by Frattini we get that $N = N_{R_M}(T \cap R_M)N_N(\langle \rho \rangle)$. As $T \leq C_M$, but $\rho \notin C_M$, we get that $N = N_{R_M}(T \cap R_M)C_N(\rho)$. This shows that $\rho \in N_M(T) = M_0$. Now by 14.3 we get that $p = 3$ and $U \cong Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. In particular we have that $Z(U) = R_M \cap U$. Hence we get that $C_M \cap U = 1$. But then we may assume that $U \leq M_0$, contradicting $P \not\leq M$. □

3insigma

Lemma 14.10 *Suppose that we have 14.5(4) or (5). If $3 \in \sigma(M)$, then a Sylow 3–subgroup of M is isomorphic to $Z_3 \wr Z_3$ and not all 3–elements are good. In particular $M/O_2(M)$ does not involve $U_4(2)$, $Sp_6(2)$, $\Omega^-(8,2)$ or A_9.*

Proof: Let R be a Sylow 3–subgroup of P. If all 3–elements are good then $P \leq M^g$ for some $g \in G$. By 9.1 then $M = M^g$ and so $P \leq M$, a contradiction. So we have that not all 3–elements are good. Let now M^g with $R \leq M^g$. Let R_1 be a Sylow 3–subgroup of M^g containing R. Then we have that $R = \Omega_1(C_{R_1}(R))$. We first have that $N_{R_1}(R) \not\leq C_{R_1}(R)$. Assume that $N_G(R) \leq M^g$. In P we see that there is some D_8 induced on R. But then $N_{M^g}(R)/C_{M^g}(R) \cong GL_2(3)$ and so all 3–elements in R are conjugate and then they all are good, a contradiction.

So we have that $N_G(R) \not\leq M^g$ and then $R_1 \cong Z_3 \wr Z_3$, the assertion. The second assertion follows from the fact that in these groups any element of order 3 is centralized by an elementary abelian group of order 27. □

bsmall2

Lemma 14.11 *If b is even, then one of the following holds*

i) $b = 2$ and $E(P/C_P) \cong L_2(q)$ or $P/C_P \cong \Sigma_3$ and Y_P is the natural module.

ii) $P/C_P \cong \Sigma_3 \wr Z_2$ *and* Y_P *is the natural module.*

Proof: As $Y_{\hat{M}} \leq Y_M \leq Y_P$, we have that $b = b_P$. Let (γ, α) be a critical pair. We may choose notation such that $P_\gamma = P$. By 14.6 we have that $1 \neq [Y_P, Y_{P_\alpha}] \leq Y_P \cap Y_{P_\alpha}$. Hence we may assume that Y_P is an F–module with Y_{P_α} as offender. This now shows that we have $E(P/C_P) \cong L_2(q)$ or $P/C_P \cong \Sigma_3$ and Y_P is the natural module, or $P/C_P \cong \Sigma_5$ or $\Sigma_3 \wr Z_2$ and Y_P is the orthogonal module, or $E(P/C_P) \cong A_5 \times A_5$ and Y_P is the sum of two orthogonal modules.

Choose β such that $\beta \in \Delta(\gamma)$ and $d(\beta, \alpha) = d(\gamma, \alpha) - 1$.

Let first P/C_P an automorpism group of $L_2(q)$ and Y_P be the natural module. Then we have that $[Y_P, Y_{P_\alpha}] = Y_{M_\beta} = Y_{M_{\alpha-1}}$. Now we have by 9.1 that $M_\beta = M_{\alpha-1}$. But then $d(P, P_\alpha) = 2$ and so $b = 2$. This is (i).

If $P/C_P \cong \Sigma_5$ or $E(P/C_P) \cong A_5 \times A_5$. We have that $(C_M(V_M) \cap P) \leq C_P$. Let T be the preimage of $O_2(M/C_M(V_M))$, then we see that TC_P/C_P is contained in a Sylow 2–subgroup of $E(P/C_P)$. Hence we see that $\langle T^{M \cap P} \rangle$ is a 2–group by 14.6 and the minimal choice of P. Hence T is normal in $M \cap P$, i.e. $M \cap P = \hat{M} \cap P$. Now we have that Y_{P_α} induces a transvection on Y_P. Hence $Y_{P_\alpha} \not\leq O_2(G_{\Delta_\beta})$. Let now $\delta \in \Delta(\beta)$ with $d(\delta, \alpha) = d(\beta, \alpha) - 1$. Then $[Y_{P_\alpha}, Y_{P_\delta}] = 1$. Let $\rho \in G_{\Delta\beta}$ be of order three inverted by some element in Y_{P_α}. Then $[\rho, Y_{P_\delta}] = 1$. But by 8.12 we have that $C_{P_\delta}(Y_{P_\delta})$ is a 2–closed, a contradiction.

So we are left with $P/C_P \cong \Sigma_3 \wr Z_2$, which is (ii). □

In what follows we have always b even and $P/C_P \cong \Sigma_3 \wr Z_2$ until we reach the contradiction. We fix the following notation as in the proof of 14.11. Let (γ, α) be a critical pair. We choose notation such that $P_\gamma = P$. Choose β such that $\beta \in \Delta(\gamma)$ and $d(\beta, \alpha) = d(\gamma, \alpha) - 1$. Further let $\delta_1 \in \Delta(\gamma)$ with $d(\delta_1, \alpha) = d(\gamma, \alpha)$. We may assume that $M = M_{\delta_1}$.

bg2

Lemma 14.12 *We have* $[V_M, Y_P] = 1$.

Proof: Assume that $[V_M, Y_P] \neq 1$. Then in particular $b = 2$. Hence and there $[\tilde{Y}_P, Y_{P_\alpha}] \neq 1$. But then Y_{P_α} cannot act quadratically on Y_P. □

noVM

Lemma 14.13 *Let* $1 \neq x \in V_{M_{\alpha-1}}$, *then* x *is not centralized by a good* E *in* M.

Proof: By 14.12 we have $[x, Y_{P_\alpha}] = 1$. Then we get that $Y_{P_\alpha} \leq M$. But then $P = \langle P \cap M, Y_{P_\alpha} \rangle \leq M$, a contradiction. □

noYM

Lemma 14.14 *We have that* $Y_M \not\leq V_{M_{\alpha-1}}$.

Proof: This follows from that fact that $|Y_M| = 2$. $\square$

FmoduleVM

Lemma 14.15 *Either* $V_{M_{\alpha-1}}/Y_{M_{\alpha-1}}$ *is an* F*–module with offender* V_M *or* V_M *is a proper* $2F$*–module with offender* $V_{M_{\alpha-1}}$.

Proof: As $[Y_P, V_{M_{\alpha-1}}] = 1$ by 14.12, we see that $[V_M, V_{M_{\alpha-1}}] \leq V_M \cap V_{M_{\alpha-1}}$. Let $x \in Y_{P_\alpha} \setminus M$. Then by 5.8 $|V_M : C_{V_M}(x)| \geq 4$. If $V_M \leq P_\alpha$ then by quadratic action we would get that Y_{P_α} is generated by elements, which centralize a hyperplane in V_M, a contradiction. So we have that $V_M \not\leq P_\alpha$. This shows that $V_M \not\leq O_2(\hat{M}_{\alpha-1})$. We have that $[V_M \cap O_2(\hat{M}_{\alpha-1}), V_{M_{\alpha-1}} \cap O_2(\hat{M})] \leq Y_M \cap Y_{M_{\alpha-1}}$. By 14.14 this shows that $[V_M \cap O_2(\hat{M}_{\alpha-1}), V_{M_{\alpha-1}} \cap O_2(\hat{M})] = 1$. Moreover as $[V_{M_{\alpha-1}} \cap O_2(\hat{M}), V_M] \leq Y_M$ we get with 14.14 that $[V_{M_{\alpha-1}} \cap O_2(\hat{M}), V_M] = 1$. We now have that $|V_{M_{\alpha-1}} : C_{V_{M_{\alpha-1}}}(V_M \cap O_2(\hat{M}_{\alpha-1}))| = |V_M \cap O_2(\hat{M}_{\alpha-1}) : C_{V_M}(V_{M_{\alpha-1}})|$. So if $V_{M_{\alpha-1}}/Y_{M_{\alpha-1}}$ is not an F–module with offender V_M, we get that $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M})| < |V_M : C_{V_M}(V_{M_{\alpha-1}})|^2$, the assertion. $\square$

noF

Lemma 14.16 *There is no* $A \leq V_M$ *such that* A *induces an* F*–module offender on* $V_{M_{\alpha-1}}/Y_{M_{\alpha-1}}$.

Proof: Let first K be some component of M/R_M such that A normalizes K and induces an F–module offender on $V = [V_{M_{\alpha-1}}/Y_{M_{\alpha-1}}, K]$. So we have that the assumptions of 3.42 are satisfied. By 14.13 we have that V is not centralized by a good E in $M_{\alpha-1}$. Suppose that we are in 3.42(2) or (3). Let W be the submodule in V and W_1 be the corresponding submodule in V_M/Y_M. Then by 14.13 we have that $[W, W_1] = 1$. Hence $[K, W_1] \leq R_{M_{\alpha-1}}$. If $[W_1, V_{M_{\alpha-1}}] \neq 1$, we get by quadratic action that $[K, [W_1, V_{M_{\alpha-1}}]] = 1$. Hence $M \cap M_{\alpha-1}$ covers K, which contradicts $[K, A] \neq 1$. So we have that $[W_1, V_{M_{\alpha-1}}] = 1$. Then $W_1 \leq P_\alpha$. Now we get that $Y_{P_\alpha} = \langle x \mid |W_1 : C_{W_1}(x)| \leq 4\rangle$. Hence any such x is in M, which shows that $Y_{P_\alpha} \leq M$, a contradiction.

So we have one of the cases in 3.42(4). By 14.10 we do not have (v), (vi), (vii). Let V_1 be the module corresponding to V in V_M and K_1 be the component corresponding to K. Suppose that we do not have 3.42(i). Then $m_p(K) \geq 2$ for some $p \in \sigma(M)$, recall that $m_p(R_M) \leq 1$ and so K centralizes a Sylow p–subgroup of $R_{M_{\alpha-1}}$. Let $[V_1, V] = 1$. Then $[V, V_{M_{\alpha-1}}]$ is centralized by K_1 and so by a good E in M, contradicting 14.13. So we have that $[V, V_1] \neq 1$. As now all p–elements are good, we see that no p–element from M can be in

$M_{\alpha-1}$ by 5.5. As any element in V_1 is centralized by a p–element in M, we see that $Y_{M_{\alpha-1}}$ is not in V_1 and so we get that V or V_1 induces an F–module offender on V_1, V, respectively. Further no element in $[V, V_1]^\sharp$ is centralized by a good E. By 3.52, 3.53 and 3.54 $[V, V_1]$ always contains the centralizer K and so we do not have (iii) or (iv). Hence $K \cong \Omega^-(6, q)$ and V is the orthogonal or unitary module. In the unitary case the commutator always contains some element which is centralized by some $L_2(q) \times Z_{q+1}$. As there is some $p \in \sigma(M)$ with p divides $q+1$, we are done. So we have the orthogonal module and all elements in $[V, V_1]$ are singular. But then $|[V, V_1] = q^2$ and all elements in V_1 have the same commutator. But this shows that V_1 induces a group of order at most q, which cannot be an offender.

So we are left with $K \cong L_2(q)$. Now V is a nonsplit extension of the trivial module by the natural module. Suppose first that as before $[V, V_1] \neq 1$. We see that both V_1 and V induces F–offenders on each other. This in fact shows that $[V_1, V]$ is normalized by an elementary abelian group of order p^3 in M. Hence V contains elements which are centralized by a good E in M contradicting 14.13. So we have that $[V, V_1] = 1$. Again in $[V, V_M]$ we have elements which are centralized by a good E in M_α. This shows that we have that the corresponding component to K in M is covered by $M \cap M_{\alpha-1}$. In particular K is centralized by some component $\tilde{K} \cong L_2(q)$. But now any element in V_1 is centralized by a good p–element in M. Hence $[V_{M_{\alpha-1}}, V_1]$ is centralized by a good p–element in M. As K is not covered by $M \cap M_{\alpha-1}$, we get that $[V_1, V_{M_{\alpha-1}}] = 1$. But then as above we get that $Y_{P_\alpha} \leq M$, a contradiction.

So we may assume that A induces an F–module offender on $F(M_{\alpha-1}/R_{M_{\alpha-1}})$. Then we get that A acts faithfully on the Sylow 3–subgroup of $F(M_{\alpha-1}/R_{M_{\alpha-1}})$. Let U be a Sylow 3–subgroup of the preimage of this group, which is normalized by A. If $3 \notin \sigma(M)$, then by 2.3 there is a good E such that $[E, U] \leq R_{M_{\alpha-1}}$. Set $S_1 = [U, A]$ and $V = [V_{M_{\alpha-1}}, S_1]$. Then we see that V is centralized by a good E in $M_{\alpha-1}$. Let V_1 be the corresponding group in V_M. By quadratic action we have that $[V_1, V] \leq V$. If $[V_1, V] = 1$, we get that $[V_{M_{\alpha-1}}, V_1]$ is centralized by S_1 and as $S_1 \not\leq M$, we have that $[V_1, V_{M_{\alpha-1}}] = 1$. As $|V_1| \geq 4$, we now get that $Y_{P_\alpha} \leq M$, a contradiction. So we have that $[V, V_1] \neq 1$. By 14.13 we see that $[V, V_1] \not\leq V_1$. Let $x \in V$ with $|V_1 : C_{V_1}(x)| \leq 4$. There is $\rho \in F(M/R_M)$, such that $|[V_M, \rho]| = 4$ and $[V_M, \rho] \leq V_1$. Now $|[V_M, \langle \rho^x \rangle]| \leq 16$. But also this group is centralized by a good E, as the same good E centralizes $\langle \rho, \rho^x \rangle$, As V_1 is generated by subgroups $[V_1, \rho]$ for elements ρ of that type, there is one such that $[x, [V_1, \rho]] \neq 1$, Hence there is some element in $V_{M_{\alpha-1}}$ which is centralized by a good E in M, contradicting 14.13.

Hence $3 \in \sigma(M)$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Again there is some $\rho \in U$ with $|[V_{M_{\alpha-1}}, \rho]| = 4$ Set $V = [V_{M_{\alpha-1}}, \rho]$ and $X = \langle \rho^{M_{\alpha-1}} \rangle \cap U$. Then we have that $|[V_{M_{\alpha-1}}, X]| \leq 64$. Let ρ_1, V_1 and X_1 be the corresponding elements in M.

Suppose first that $|X_1| > 3$. Then any element in $C_{V_M}(X_1)$ is centralized by a good E. In particular $[C_{V_M}(X_1), V_{M_{\alpha-1}}] = 1$ by 14.13. This now shows that $C_{V_M}(X_1) = Y_M$, as $Y_{P_\alpha} \not\leq M$. Hence we have that $|V_M| \leq 2^7$ further $[V, V_1] \neq 1$. If V_1 induces a group of order 8 on V, the same applies for V on V_1. But then there is some x in V such that $[V_1, x]$ is centralized by a good E in M and $M_{\alpha-1}$ as well. Assume now that V_1 induces a fours group. Then by quadratic action still there is some $u \in V$ such that $[V_1, u]$ is of order two. So we may choose x again such that $[V, x]$ is centralized by a good E in M and $|[V, x]| = 2$. But then also $[V, x]$ is centralized by a good E in M. Hence we have that $|V_1 : V_1 \cap R_{M_{\alpha-1}}| = 2$. Then $[V_1, V]$ is centralized by a good E in $M_{\alpha-1}$. If there is some $x \in V$ with $|[V_1, x]| = 2$, we get that $[V_1, x]$ is centralized by a good E in M, a contradiction. So we have that $[V_1, x] > Y_{M_{\alpha-1}}$. In particular $|V : V \cap R_M| = 2$. Now in V_1 we have a fours group W_1 such that $|V : C_V(W_1)| = 2$ and $C_V(W_1) = C_V(w)$ for $w \in W_1^\sharp$. Further we see that $[V_1, V]$ is centralized by an 3–element μ with $C_G(\mu) \leq M$. This first gives $\mu \in M_{\alpha-1}$ and further that μ acts on $[U, V_1]$. Hence $[U, V_1]$ is not of order 3. As $|[V, V_1]| = 2$, this gives that $[U, V_1]$ is not abelian and we see that $[U, V_1] \cap C(V) \neq 1$. Hence some 3-central element $\tau \in M_{\alpha-1}$ centralizes V. In fact this element τ also centralizes μ. Then $\tau \in M$ and so V_1 centralizes τ. As $[U, V_1]$ is not abelian, we have that $\mu \in U$. Further U is not a Sylow 3–subgroup of $M_{\alpha-1}$. Hence U is extraspecial and V_1 centralizes a group of order 9, which is not possible.

So we are left with $|X| = 3$. As $[V, X]$ is of order 4, we have that $[V, X]$ is centralized by a good E. Let X_1 the group corresponding to X in M. Again X_1 cannot centralize $V_{M_{\alpha-1}}$ but has to centralize $C_{V_{M_{\alpha-1}}}(X)$. So $1 \neq [X, X_1]$. But $X_1 R_M / R_M$ is normal in M/R_M, and then X contains some element centralized by a good E in M, contradicting 14.13. □

VMover

Lemma 14.17 *Let K be a component of M/R_M on which $V_{M_{\alpha-1}}$ induces a $2F$–module offender, then it does not induce an F–module with over offender.*

Proof: Assume false. Then we may apply 3.42. Let $V = [V_M, K]$. By 14.13 we have that we are in 3.42(4). By 14.10 we have (i), (ii), (iii), (iv) or (viii). As we have an over offender, we have that we are in (iii). Further $V_{M_{\alpha-1}}$ induces a group of order at least $2q^2$. By 3.53(ii), we get that $[V_{M_{\alpha-1}}, V]$ contains nontrivial elements centralized by K, contradicting 14.13. □

good5

Lemma 14.18 *Let K be a component of M/R_M such that $V_{M_{\alpha-1}}$ normalizes K and $V_{M_{\alpha-1}}$ induces a quadratic proper 2F–module offender on V_M/Y_M as K–module. Then we have one of the situations of 3.43(5).*

Proof: Let first M be exceptional. Then we may apply 3.41. By 14.13 we have that V_M is not centralized by a good E. So 3.41 implies that $3 \in \sigma(M)$ which contradicts 14.10. Hence M is not exceptional. So we can apply 3.43. Again by 14.13 we do not have 3.43(1). As $\tilde{Y_P} \leq C_{V_M/Y_M}(S)$, we see that 3.43(2) is not possible. Suppose that we are in 3.43(3)(4) and let W_M be the corresponding module. By 14.13 we have that not any element in W_M is centralized by a good E. We still have that in W any element is centralized by some good p–element and K (K as in 3.43) contains a good E. This shows that $[V_{M_{\alpha-1}}, C_{V_M}(K)] = 1$. Hence $[V_{M_{\alpha-1}}, V_M \cap O_2(\hat{M}_{\alpha-1})] \leq W_M$. If $Y_{M_{\alpha-1}} \leq W_M$, then there is a good p–element $\rho \in M$ which is contained in $M_{\alpha-1}$. This shows that $p = 3$ and a Sylow 3–subgroup U of M is isomorphic to $Z_3 \wr Z_3$. But as K contains a good E this is not possible. So we have that $[V_{M_{\alpha-1}}, V_M \cap O_2(\hat{M}_{\alpha-1})] = 1$. No with 14.16 we get that K even induces an F–module. Then 3.42 applies.

So we have one of the cases in 3.42(4). By 14.10 we have not (v), (vi) or (vii). In (iii) and (iv) we get with 3.53 or 3.54 some element in $C_{[V_M,K]}(K) \cap V_{M_{\alpha-1}}$, which contradicts 14.13. So we have (ii) or (viii). Recall that (i) is not possible as K contains a good E. In (ii) $[V_M, V_{M_{\alpha-1}}]$ always contains a non singular vector, but such vectors are centralized by a good E. So we have (viii). Let $K_{\alpha-1}$ the corresponding component in $M_{\alpha-1}$ then we see that $[[V_{M_{\alpha-1}}, K_{\alpha-1}], C_{V_M}(K)] = 1$ by 14.13 and so $[V_{M_{\alpha-1}}, K_{\alpha-1}]$ induces an F–module offender on $[V_M, K]$. But then in $[V_{M_{\alpha-1}}, K_{\alpha-1}]$ there are elements which centralize in $[V_M, K]$ just $C_{[V_M,K]}([V_{M_{\alpha-1}}, K_{\alpha-1}])$. As $N_K(C_{[V_M,K]}([V_{M_{\alpha-1}}, K_{\alpha-1}]))$ acts transitively on $[V_M, K]/C_{[V_M,K]}([V_{M_{\alpha-1}}, K_{\alpha-1}])$, this would imply that K acts transitively on $[V_M, K]$, a contradiction. □

Knotnormal

Lemma 14.19 *Let $V_{M_{\alpha-1}}$ induces a quadratic proper 2F–module offender on V_M/Y_M as $E(M/R_M)$–module, then there is no component K which is normalized by $V_{M_{\alpha-1}}$ such that $V_{M_{\alpha-1}}$ induces a proper 2F–module offender as a K–module.*

Proof: Assume false and let K be the corresponding component. Let $x \in V_{M_{\alpha-1}}$ with $[K, x] \in R_M$. Then by quadratic action we have that $[[V_M, K], x] = 1$. So we have that $[[V_M, K], C_{V_{M_{\alpha-1}}}(K)] = 1$. Let $B_{M_{\alpha-1}}$ be a complement to $C_{V_{M_{\alpha-1}}}(K)$ in $V_{M_{\alpha-1}}$. Then $B_{M_{\alpha-1}}$ is a quadratic 2F–module offender on $[V_M, K]$. Let X_M be a complement to $C_{[V_M,K]}(B_{M_{\alpha-1}})$ in $[V_M, K]$. Set $Z_M = X_M \cap R_{M_{\alpha-1}}$. By 14.16 we have that $|X_M/Z_M| < |B_{M_{\alpha-1}}|$. As by 14.17 we have that $[V_M, K]$ is not an F–module with over offender, we get that $Z_M \neq 1$. Further for $y \in Z_M^\sharp$, we have that $|B_{M_{\alpha-1}} : C_{B_{M_{\alpha-1}}}(y)| = 2$. Further $Y_{M_{\alpha-1}} \leq [V_M, K]$.

By 14.18 we have one of the cases of 3.43(5). If we have 3.43(5)(i), then $|[y, B_{M_{\alpha-1}}]| = |B_{M_{\alpha-1}}|$, hence $|B_{M_{\alpha-1}}| = 2$ and so $q = 4$. But now $5 = p \in \sigma(M)$ and so there is a good E centralizing $[V_M, K]$, contradicting 14.13.

Let 3.43(5)(ii). Then p divides $q+1$ and any element in $[V_M, K]$ is centralized by a good p–element ρ. As $Y_{M_{\alpha-1}} \leq [V_M, K]$, we get $\rho \in M_{\alpha-1}$. But then with 5.5 and 1.17 we have a contradiction.

Let next 3.43(5)(iii) or (iv). Then p divides $q-1$. As all elements in $[V_M, K]/C_{[V_M,K]}(K)$ is centralized by $L_2(q)$, we again see that any element in $[V_M, K]$ is centralized by a good p–element ρ, a contradiction again.

By 14.10 we do not have 3.43(5)(v),(vi),(vii), (ix), (xii) or (xxiii).

Let now 3.43(5)(viii). Then any element in $[V_M, K]$ is centralized by $L_2(q)$ or $U_3(q)$. Hence any element is centralized by a good p–element ρ, a contradiction.

Let now 3.43(5)(x). Then $K/Z(K) \cong A_n$, $n \leq 7$. As we have at most two nontrivial modules in $[V, K]$ we see that $3 \in \sigma(M)$, which by 14.10 implies $n = 5$. But then $[V_M, K]$ involves just one nontrivial module and so is centralized by a good E, a contradiction.

Let 3.43(5)(xi). Then by 14.10 we have $K \cong J_2$ and $5 = p \in \sigma(M)$. Hence there is some p–element ρ with $[K, \rho] \in R_M$ and so $[V_M, K, \rho] = 1$, a contradiction.

Let 3.43(5)(xiii), then $K \cong SU_3(q)$ and $[V_M, K]$ is the natural module. Now $|B_M| = q$ and $|X_M| = q^2$. This further shows that $C_{X_M}(t) = 1$ for all $t \in B_{M_{\alpha-1}}^\sharp$. As $|B_{M_{\alpha-1}} : C_{B_{M_{\alpha-1}}}(y)| = 2$, we get $q = 2$, a contradiction.

In 3.43(5)(xv) or (xvi) we do not have quadratic offenders by 3.56.

From now on we have more than one nontrivial irreducible module in $[V_M, K]$. Now let $1 \leq V_1 \leq \cdots \leq V_r$ be such a series. Then we may assume that X_M contains a complement X_i to $C_{V_i/V_{i-1}}(B_{M_{\alpha-1}})$ for all i. Hence there is exactly one i such that $X_i \cap R_{M_{\alpha-1}} \neq 1$. Then for all other j we have that $|X_j| < |B_{M_{\alpha-1}}|$. This shows that up to one nontrivial module involved in $[V_M, K]$ all other nontrivial irreducible modules are F–modules with over offender $B_{M_{\alpha-1}}$.

Let 3.43(5)(xvii) or (xxii), then just natural modules are involved, but by 3.18 these do not have over offender.

Let 3.43(5)(xviii). Now $K \cong Sp(6,q)$ and we have natural modules and spin modules in $[V_M, K]$. Let W_M be a spin submodule. Then any element in W_M is centralized by a good p–element and so $Y_{M_{\alpha-1}} \cap W_M = 1$. Hence $[W_M \cap R_{M_{\alpha-1}}, V_{M_{\alpha-1}}] = 1$. Now $B_{M_{\alpha-1}}$ has to induce an over offender on W_M and so $|W_M : C_{W_M}(B_{M_{\alpha-1}})| = q^4$. Let now $K_{\alpha-1}$ the component in $M_{\alpha-1}/R_{M_{\alpha-1}}$. Let $1 \neq x \in W_M \cap C(K_{\alpha-1})$, then $[x, C_{V_{M_{\alpha-1}}}(K_{\alpha-1})] \neq 1$. But $K_{\alpha-1}$ contains a good E. So some $1 \neq t \in [x, C_{V_{M_{\alpha-1}}}(K_{\alpha-1})]$ is centralized by a good p–element in M and a good E in $M_{\alpha-1}$. This contradicts 5.5, recall that all 3–elements of $Sp(6,q)$ are good by 1.17. So we have that W_M induces a group of order q^4 on $K_{\alpha-1}$. But then we have that $|V_{M_{\alpha-1}} : C_{V_{M_{\alpha-1}}}(W_M)| \geq q^7$, as W_M has to act nontrivially on a spin module and a natural module in $V_{M_{\alpha-1}}$ as well. But this contradicts the fact that $Sp(6,q)$ has no elementary abelian subgroups of order q^7.

Let now 3.43(5)(xix). Then $K \cong Sp_4(q)$ and $[V_M, K]$ involves two natural modules. We have that p divides $q^2 - 1$ and so K contains a good E. Let W_M be a natural submodule. Then as before we have that $[W_M \cap R_{M_{\alpha-1}}, V_{M_{\alpha-1}}] = 1$ and the as before W_M induces a group of order q^2 on $K_{\alpha-1}$. This gives $|[V_{M_{\alpha-1}}, K_{\alpha-1}] : C_{[V_{M_{\alpha-1}}, K_{\alpha-1}]}(W_M)| \geq q^4$, a contradiction as before.

Let now 3.43(5)(xxi). Let first $K \cong L_4(q)$. Then we have two 4–dimensional modules and an orthogonal module involved. As we have quadratic offender, we have to induce transvections on the natural modules., otherwise $|[V, B_{M_{\alpha-1}}]| = q^2$ for the natural module V, so $B_{M_{\alpha-1}}$ is in the stabilizer U of a 2–space. The largest group here which also acts quadratically on the orthogonal module is of order q^2. But then $B_{M_{\alpha-1}}$ cannot be a 2F–module offender. So we have that $B_{M_{\alpha-1}}$ is a transvection group. Hence for the orthogonal module i is not an over offender. By 14.10 we have that $3 \notin sigma(M)$, so $q > 2$. As no p–element can centralize $[V_M, K]$, we see that p has to divided $q - 1$. In the orthogonal module now any element is centralized by a good p–element, which shows that there is no such submodule, as otherwise since there is no over offender on the orthogonal module we must have $Y_{M_{\alpha-1}}$ in this submodule. So we have a submodule W_M which is the natural module. Again any element is centralized by a good p–element and so $W_M/C_{W_M}(B_{M_{\alpha-1}}$ acts faithfully on $K_{\alpha-1}$. But then we have that $|V_{M_{\alpha-1}} : C_{V_{M_{\alpha-1}}}(W_M)| \geq q^4$, contradicting $|B_{M_{\alpha-1}}| \leq q^3$.

Let now $K \cong SL_3(q)$. Now we have two 3–dimensional modules in $[V_M, K]$. As on one of then $B_{M_{\alpha-1}}$ has to induce an over offender, we have that

$|B_{M_{\alpha-1}}| > q$. As the index of the centralizer of $B_{M_{\alpha-1}}$ in the natural module is at least q and $|B_{M_{\alpha-1}}| \leq q^2$ we see with 14.16 that we have exactly two natural modules involved. Let W_M be a natural submodule. then any element in W_M is centralized by a good p–element, so $Y_{M_\alpha} \not\leq W_M$. Suppose now that $B_{M_{\alpha-1}}$ induces transvections to a hyperplane on $[V_M, K]/W_M$. Then for any $y \in [V_M, K] \setminus W_M$ with $y \in R_{M_{\alpha-1}}$ we have that $C_{B_{M_{\alpha-1}}}(y) = 1$, contradicting $Z_M \neq 1$. So $B_{M_{\alpha-1}}$ induces transvections to a point on $[V_M, K]/W_M$. Let now R be the 1-space containing $y \in Z_M$ as before. Then $|B_{M_{\alpha-1}} : C_{B_{M_{\alpha-1}}}(R)| = 2$. This shows that $|B_{M_{\alpha-1}}| = 2q$. As $|V_M : C_{V_M}(B_{M_{\alpha-1}})| \geq q^3$, we get $q = 2$ or 4. In both cases we get that $3 \in \sigma(M)$. But then by 14.10 we get $K \cong L_3(2)$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. But then a good E centralizes $[V_M, K]$, a contradiction.

Let now 3.43(5)(xxiv). Then $K \cong L_6(2)$, $7 \in \sigma(M)$ and at least 6 natural modules are involved in $[V_M, K]$. Let W_M be a natural submodule. As K contains a good E and any element in W_M is centralized by a good p–element, we see that some element $w \in W_M$ acts nontrivially on $K_{\alpha-1}$. Hence $|V_{M_{\alpha-1}} : C_{V_{M_{\alpha-1}}}(w)| \geq 2^6$. So $|B_{M_{\alpha-1}}| \geq 2^6$. Hence we have that for any natural module V involved in $[V_M, K]$ we have that $|V : C_V(B_{M_{\alpha-1}})| \geq 2^2$. This shows that $|X_M/Z_M| \geq 2^{10}$. As $|B_{M_{\alpha-1}}| \leq 2^9$ this contradicts 14.16

Let now 3.43(5)(xxv). Then $m_p(K) = 1$. As $Y_{M_{\alpha-1}} \leq [V_M, K]$ no good p–element can centralize $[V_M, K]$, in particular p does not divide the order of R_M. As some $E \cong E_{p^2}$ centralizes K, we get that there are at least 6 modules involved. Hence $n \geq 4$. Now by 14.10 $3 \notin \sigma(M)$. Let $\rho \in C_{M/R_M}(K)$, $o(\rho) = p \in \sigma(M)$, such that $W_M = C_{[V_M,K]}(\rho) \neq 1$. Then we have $Y_{M_{\alpha-1}} \not\leq W_M$. So $B_{M_{\alpha-1}}$ has to induce an over offender on W_M. As W_M is a sum of at least three natural modules, we get $n = 5$ and $p = 7$. Further W_M is the sum of exactly three natural modules and $|B_{M_{\alpha-1}}| = 16$. Now we choose a natural submodule L_M of W_M. Then all elements in L_M are centralized by a good E. Further $C_{V_{M_{\alpha-1}}}(K_{\alpha-1})$ is centralized by a good p–element. So with 5.5 we get that $x \in L_M \setminus R_{M_{\alpha-1}}$ acts faithfully on $K_{\alpha-1}$. This gives that $|V_{M_{\alpha-1}} : C_{V_{M_{\alpha-1}}}(x)| \geq 2^6$, contradicting $|B_{M_{\alpha-1}}| = 2^4$.

So we are left with $K \cong Sz(q)$, i.e. 3.43(5)(xxvi). But there are no proper $2F$–module offender on the natural module. □

EM

Lemma 14.20 *We have that $V_{M_{\alpha-1}}$ does not induce a proper $2F$–module offender on $[E(M/R_M), V_M]$.*

Proof: By 14.19 we have that there is some component K such that $L = \langle K, V_{M_{\alpha-1}} \rangle \neq K$ and $V_{M_{\alpha-1}}$ induces a $2F$–module offender on $[E(L), V_M]$. By quadratic action and 3.24 we either have that $V_{M_{\alpha-1}}$ induces a group of order 2 or $E(L) \cong \Omega^+(4,q)$ and $[V_M, E(L)]$ is the natural module. Assume that $V_{M_{\alpha-1}}$ induces at least a fours group. So we have the latter. Then we have that L contains a good E. Hence as in 14.19 we see that $[[V_M, L] \cap R_{M_{\alpha-1}}, V_{M_{\alpha-1}}] = 1$. In particular $Y_{M_{\alpha-1}} \leq [V_M, E(L)]$. Now no good p–element centralizes $Y_{M_{\alpha-1}}$, which shows that p divides $q-1$ if $p \in \sigma(M)$. Now there is some $x \in V_{M_{\alpha-1}}$ such that $C_{E(L)}(x) \cong L_2(q)$. As we have that $V_{M_{\alpha-1}}$ induces at least a fours group, we see that $[x, [V_M, E(L)]]$ is centralized by $C_{E(L)}(x)$, in particular $Y_{M_{\alpha-1}} \not\leq [x, [E(L), V_M]]$. This shows that $|[E(L), V_M] : [E(L), V_M] \cap R_{M_{\alpha-1}}| \geq q$. By 14.16 we even get equality and so $V_{M_{\alpha-1}}$ induces a group of order $2q$. Now $[V_M \cap R_{M_{\alpha-1}}, V_{M_{\alpha-1}} \cap E(L)] = Y_{M_{\alpha-1}}$. Hence for $t \in V_{M_{\alpha-1}} \cap E(L)$ we have that $|[V_M, E(L)] : C_{[V_M, E(L)]}(t)| \leq 2q$. As $q > 2$ and $|[V_M, E(L)] : C_{[E(L), V_M]}(t)| = q^2$, we have a contradiction.

So we have that $V_{M_{\alpha-1}}$ induces a group $\langle t \rangle$ of order 2. But as $V_{M_{\alpha-1}}$ has to induce a proper $2F$–module offender, we have that it has to induce an F–module offender, and so it induces a transvection. But it does not normalize the component K, so t inverts an element of odd order $r > 3$ in $E(L)$, a contradiction. □

$b = 2$

Lemma 14.21 *If b is even, then $b = 2$ and 14.11(i) holds.*

Proof: By 14.11 we may assume that $P/C_P \cong Z_3 \wr Z_2$ and Y_P is the orthogonal module. Then by 14.15 and 14.16 we have that $V_{M_{\alpha-1}}$ induces a proper 2F–module offender on V_M. By 14.20 it even induces a proper $2F$–module offender on $[V_M, U_M]$ for some Sylow r–subgroup U_M of $F(M/R_M)$. Let $B_{M_{\alpha-1}}$ be a complement to $C_{V_{M_{\alpha-1}}}(U_M)$ in $V_{M_{\alpha-1}}$. Then $B_{M_{\alpha-1}}$ induces a proper $2F$–ofender on $[V_M, U_M]$. By 2.1 we have a subgroup $D_M \cong D_1 \times \cdots \times D_t$ of M, D_i dihedral groups of order $2r$, with $B_{M_{\alpha-1}}$ as a Sylow 2–subgroup. By quadratic action we now get with 4.5 that $[V_M, O_r(D_M)]$ is generated by elements which centralize a subgroup of index two in $B_{M_{\alpha-1}}$ modulo Y_M. As $Y_M \not\leq V_{M_{\alpha-1}}$ by 14.14, we get that these elements even centralize a subgroup of index two in $V_{M_{\alpha-1}}$. By 14.16 we get that all these elements are in R_{M_α} and so we get that $[[V_M, O_r(D_M)], B_{M_{\alpha-1}}] = Y_{M_{\alpha-1}}$. This shows $t = 1$ and $r = 3$. In particular $B_{M_{\alpha-1}} = \langle x \rangle$ and x induces a transvection on $[V_M, U_M]$. Let first $3 \notin \sigma(M)$. We have that there is no good E in M such that $[U_M, E] \leq R_M$ as otherwise E acts on $[D, V_M]$ and would centralize this group and then also $Y_{M_{\alpha-1}}$, a contradiction. Let $p \in \sigma(M)$ and X_M be a Sylow p–subgroup of M. Then X_M normalizes U_M. Let U be a Sylow 3–subgroup of the preimage of U_M. Then by 2.3 U just admits cyclic p–groups. Hence we have that $X_M/C_{X_M}(U_M)$ is cyclic. Hence $C_{X_M}(U_M)$ contains a good E, a contradiction.

So we have $3 \in \sigma(M)$ and so by 14.10 a Sylow 3–subgroup of M is isomorphic to $Z_3 \wr Z_3$. Let $\rho \in U \setminus R_M$ with $\rho^x = \rho^{-1}$. Then we have that

$[V_M, \rho]$ is of order 4 and contains $Y_{M_{\alpha-1}}$ In particular ρ is not centralized by an elementary abelian group of order 27. Now as $xR_M \in U_M$, we get that $U \cong 3^{1+2}$ or $Z_3 \wr Z_3$. Now in both case we have that $Z(U) \leq R_M$. But now we get a good E which centralizes xR_M and so centralizes also $[x, V_M]$, a contradiction. □

Until further notice we will assume that $b > 1$ is odd. Let $(\hat{M}, P_\alpha)$ be a critical pair. Then in particular $M \neq M_{\alpha-1}$. We see that $[V_M, V_{M_{\alpha-1}}] \leq V_M \cap V_{M_{\alpha-1}}$.

$2FVM$

Lemma 14.22 *We have that* $[V_M, V_{M_{\alpha-1}}] \neq 1$.

Proof: Suppose that $[V_M, V_{M_{\alpha-1}}] = 1$. Then as Y_{P_α} does not centralize V_M, we get that $Y_P \neq \tilde{Y}_P$. Hence we have 14.5(1)(2),(4) or (5). In cases (1) and (2) we have that $C_P(\tilde{Y}_P) = C_P(Y_P)$. So we have (4) or (5). As V_M acts quadratically on Y_{P_α}, we have that Y_{P_α} contains elements, which induce transvections on V_M. By 3.41 M is not exceptional. Now we may apply 3.42. Let $x \in Y_{P_\alpha}$ inducing a transvection on V_M. Let first K be a component of M/R_M on which x acts nontrivially. Then by 3.33 we have that $K \cong L_n(2)$, $Sp(2n, 2)$, $\Omega^\pm(2n, 2)$ or A_n. Then we get with 3.42 that in all case $[x, V_M]$ is centralized by a good E in M or we have 3.42(4). With 14.10 we now see that $3 \notin \sigma(M)$, but then (4) is not possible. Recall that $|Y_M| = 2$, so $M = C_M$.

Let now x act nontrivially on $F(M/R_M)$, so it acts on a Sylow 3–subgroup U of $F(M/R_M)$ and $|[U, x]| = 3$. Let X_1 be a Sylow p–subgroup with $p \in \sigma(M)$ and assume that $3 \notin \sigma(M)$. Then we see with 2.3 that there is a good E in X_1 which centralizes U and so it centralizes $[[U, x], V_M]$ and then $[V_M, x]$. So we have that $3 \in \sigma(M)$. By 14.10 a Sylow 3–subgroup U_1 of M is isomorphic to $Z_3 \wr Z_3$. Let $U_2 \leq U_1$ such that U_2 is a Sylow 3–subgroup of UR_M. We also may assume that x normalizes U_2. Let $\rho \in U_2 \setminus R_M$ with $\rho^x = \rho^{-1}$. Suppose that $|C_{U_1}(\rho)| = 9$. Then U_2 contains an extraspecial group U_3 of order 27. Further there is $g \in U_1$ with $Z(U_3) \leq \langle \rho, \rho^g \rangle$. Hence $|[U_3, V_M]| \leq 16$ and U_1 acts on $[V_M, U_3]$. As $L_4(2)$ does not contain 3–groups of order 27, we see that $|C_{U_1}([U_3, V_M])| \geq 9$ and so there is a good E centralizing $[U_3, V_M]$ and then also $[V_M, x]$.
So in any case we have $[x, V_M]$ is centralized by a good E in C_M. We have that $[V_M, x]$ is centralized by a 2–group T in P_α such that $|T| = |S|/2$. Now $T \leq M \cap P_\alpha$. We may assume that $T \leq S$. Now we have that $N_G(T) \not\leq M$, as P_α and M cannot share a Sylow 2–subgroup. Further $N_G(T)$ induces Σ_3 on T. But this contradicts the choice of P, as in that case we could have chosen P of type (3). □

As $[O_2(\hat{M}), V_M] \leq Y_M$ and $Y_M \cap Y_{M_{\alpha-1}} = 1$, we may by symmetry assume that $V_{M_{\alpha-1}} \not\leq O_2(\hat{M})$.

$2FVM1$

Lemma 14.23 *We have that $V_{M_{\alpha-1}}$ induces a quadratic $2F$–module offender on V_M/Y_M.*

Proof: Let U be a hyperplane in Y_M, then we see that $C_{O_2(\hat{M})}(V_M/U) = C_{O_2(\hat{M})}(V_M)$. Let $|V_M : V_M \cap O_2(\hat{M}_{\alpha-1})| = 2^u$, $|V_M \cap O_2(\hat{M}_{\alpha-1}) : C_{V_M}(V_{M_{\alpha-1}})| = 2^t$. Let $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M})| = 2^v$ and $|V_{M_{\alpha-1}} \cap O_2(\hat{M}) : C_{V_{M_{\alpha-1}}}(V_M)| = 2^s$. Suppose that $v \geq u$. Let $t > v$. We have that $|V_{M_{\alpha-1}} : C_{M_{\alpha-1}}(V_M)| \geq 2^t$. So there is $x \in V_{M_{\alpha-1}} \cap O_2(\hat{M})$ and $y \in V_M \cap O_2(\hat{M}_{\alpha-1})$ with $[x, y] \neq 1$. But $[x, y] \in Y_M \cap Y_{M_{\alpha-1}} = 1$, so we see that $v \geq t$. Hence $2v \geq u + t$. In particular $V_{M_{\alpha-1}}$ induces a $2F$–module offender on V_M/Y_M, which is quadratic. □

ecM

Lemma 14.24 *Let K be a component of M/R_M such that $K \leq C_M R_M/R_M$ and K induces an $2F$–module in V_M/Y_M with quadratic offender. Let $\mathcal{P}$ be the set of primes $p \in \sigma(M)$ with p divides $|K|$. If $\mathcal{P} \neq \emptyset$, then there is some $p \in \mathcal{P}$ with $m_p(C_M) \geq 3$.*

Proof: Suppose false. Then we have that $m_p(C_M) \leq 2$ for all $p \in \mathcal{P}$. We have that $M = C_M(P \cap M)$, where we assume for the moment that we have P as in 14.1. Hence $|P \cap M|$ is divisible by p. Suppose first that all p-elements are good. Then for P_1 a Sylow p–subgroup of $P \cap M$ we have that $N_P(P_1) \leq M$, but $P = (P \cap M)N_P(P_1)$. So we have with 5.12 that $m_p(Aut_M(K)) \geq 3$ or p divides $|Z(K_1)|$, where K_1 is an image of K in $C_M/O_{p'}(R_M)$. Suppose that M/C_M has a noncyclic Sylow p–subgroup, then we have that also M_0 has such a Sylow p–subgroup. Now we may argue as before with 14.3, besides that M has a Sylow 3–subgroup R isomorphic to $Z_3 \wr Z_3$. But in that case we must have that M_0 has an elementary abelian subgroup of order 9, which complements a Sylow 3–subgroup of C_M. But this is not possible, as $Z(R)$ has to be in $M_0 \cap C_M$.

So we have that M/C_M has cyclic Sylow p–subgroups. This now shows that we have $e(G) = 3$. We have $m_3(K) \leq 2$. Assume that $m_p(K) = 1$. Now application of 1.2 and $m_p(Aut_M(K)) \geq 3$, shows that this is not possible. So we have $m_p(K) = 2$. By 1.1 and 3.29, 3.30, 3.31, 3.32 and either K has to admit an outer automorphism of order p or p divides $|Z(K_1)|$, we get that $K/Z(K)L_3(q)$, $U_3(q)$, $PSp(4, q)$, $G_2(q)$, $L_4(q)$, $U_4(q)$, $L_5(q)$ or $U_5(q)$, q a power of 2. If p divides $|Z(K_1)|$ we see that $K_1 \cong SL_3(q)$ or $SU_3(q)$ and $p = 3$, as in the cases of $L_5(q)$ and $U_5(q)$, $p = 5$, we would have $m_p(K) > 2$.

Suppose first that we have an field automorphism of order p. Let R be a Sylow p–subgroup of K. Let $p > 3$, then we see that R is abelian and so all p–elements are good, but p divides $P \cap M$. With 14.3 we have that $p = 3$. Let $K_2 \leq K$, the corresponding Lie group over $GF(2)$, then all 3–elements are good, a contradiction again.

Hence we are left with $K/Z(K) \cong L_3(q)$ or $U_3(q)$, $p = 3$. But then with 14.4 we get a contradiction. □

epC

Lemma 14.25 *Let K be a component of M/R_M such that $K \leq C_M R_M/R_M$ and K induces an $2F$–module on some composition factor in $O_2(M)$ with quadratic offender. Let $\mathcal{P}$ be the set of primes $p \in \sigma(M)$ with p divides $|K|$. Let $\mathcal{P} \neq \emptyset$, and $e(G) \geq 4$, then we have that there is $p \in \mathcal{P}$ with $m_p(C_M) \geq 4$.*

Proof: By 14.24 there is some p with $m_p(C_M) \geq 3$. Assume that $m_p(C_M) = 3$. Then p divides $|M \cap P|$, where we assume P to be as in 14.1. Hence by 14.3 not all p–elements can be good, which shows that $p > 3$, as $e(G) \geq 4$. Further by 5.12 either $m_p(C_M(K)) = 0$ or p divides $|Z(K_1)|$, where K_1 again is the preimage of K in $C_M/O_{p'}(C_M)$. Finally $m_t(K) \leq 3$ for all odd primes t. By 1.2 and 3.29, 3.30, 3.31, 3.32 we get that $K/Z(K) \cong L_2(q)$, $Sz(q)$, $L_3(q)$, $U_3(q)$. $PSp_4(q)$, $G_2(q)$, $L_4(q)$, $U_4(q)$, $Sp_6(q)$ or $\Omega^-(8, q)$. As $p > 3$, we see that $Z(K_1) = 1$ and just field automorphism are possible. Hence $m_p(K) = 3$. This shows that $K/Z(K) \cong L_4(q)$, $U_4(q)$, $Sp_6(q)$ or $\Omega^-(8, q)$, $q = r^p$. In all cases a Sylow p–subgroup R of K is abelian. But we have that p divides $r - 1$ or in the case of $U_4(q)$, p divides $r + 1$. This shows that R is abelian. Hence all p–elements are good, a contradiction. □

pnotK

Lemma 14.26 *Let $t = min(e(G), 4)$. Suppose that for some component K in $C_M R_M/R_M$ we have 3.43(1) with V is not centralized by a good E. Then there is a good E in C_M which centralizes $[V, K]$ or $e(C_M) \geq t$.*

Proof: Suppose first that a Sylow p–subgroup of M, $p \in \sigma(M)$ normalizes K. Let $m_p(C(K)) \geq 3$. Let F be elementary abelian of order p centralizing K. Suppose that $[F, [V, K]] \neq 1$. If $[V, K]$ is irreducible, then some element in F has to induce field multiplication. But then with 3.29, 3.30, 3.31, 3.32 we see that p divides $|K|$, a contradiction. So we have that some $\rho \in F$ acts nontrivially on the set of irreducible submodules in $[V, K]$, which then have to be F–modules. But then we see that p divides $|L_n(q)|$, where $n \leq m$ for $K \cong SL_m(q)$ and $n = 2$, else. Recall that $m_3(K) \leq 3$ and so we have at most two such modules for $K \not\cong SL_m(q)$. But in all cases we get p divides $|K|$. Hence we have that $[F, [K, V]] = 1$. As M/C_M has cyclic Sylow p–subgroups we have that F contains a good E in C_M.

So we have that $m_p(C(K)) \leq 2$. Assume that there is no good E in C_M, which centralizes $[V, K]$. As $m_p(Aut(K)) \leq 1$, we get that $e(G) = 3$ and there are field automorphisms of order p of K in M. In particular K is of Lie type in characteristic two and $m_3(K) \leq 2$. Let F be an elementary abelian subgroup of order p^2 in $C(K)$. As before we see that $[F, [V, K]] = 1$. Let R be a Sylow p–subgroup of M and $R_1 = \Omega_1(R)$. If $Z(R_1)$ is not cyclic, all p–elements are good. But as we may assume that p divides the order on M_0, this contradicts 14.3. So we have that $Z(R_1)$ is cyclic. Now we have that $\Phi(R_1) \leq C(K) \cap C_M$. Hence we may assume that $\Phi(R_1)$ is cyclic. then we get that R_1 is extraspecial. As $m_p(R_1) = 3$, we have that $|R_1| = p^5$. Now we have that $|R_1 : R_1 \cap C(K) \cap C_M| \leq p^2$. But then $R_1 \cap C(K) \cap C_M$ contains a good E which centralize $[V, K]$, a contradiction.

So we have that a Sylow p–subgroup does not normalize K. Then we get $p = 3$ and we have just three conjugates of K. In particular $e(G) > 3$. Now all 3–elements are good and so by 14.3 we have that M_0 is a $3'$–group. Hence any good E is contained in C_M, the assertion. □

Assume first that $V_{M_{\alpha-1}}$ induces some $2F$–module offender on some component K of M/R_M with $[K, V_{M_{\alpha-1}}] \leq K$. By 3.41 we have that M is not exceptional. We will now apply 3.43 .

If p divides $|K|$ for some $p \in \sigma(M)$, then by 14.25 and 14.24 we may even apply 3.43 to C_M.

*nogood*11

Lemma 14.27 *We do not have K as in 3.43(1).*

Proof: Assume that we are in 3.43(1). By 14.8 and 14.26 we may apply 3.43 to C_M again or there is a good E in C_M centralizing $[V_M, K]$. Hence in any case we have a good E in C_M centralizing $[V_M, K]$. Now let $K_{\alpha-1}$ be the corresponding component in $M_{\alpha-1}/R_{M_{\alpha-1}}$. Then we have that $[[V_{M_{\alpha-1}}, K], [V_M, K]] = 1$. Hence we get that $[K, [V_{M_{\alpha-1}}, K_{\alpha-1}]] = 1$. Assume that $[V_M, [V_{M_{\alpha-1}}, K_{\alpha-1}]] = 1$. Hence $[V_M, K]$ centralizes $K_{\alpha-1}$ and so K_α centralizes $[V_{M_{\alpha-1}}, [V_M, K]]$, so $K_{\alpha-1}$ is covered by M. Let $L \leq M$ such that L is minimal such that it covers $K_{\alpha-1}$. As $\sigma(M) \cap \pi(K) = \emptyset$, we see that K has at most three conjugates under L and so as L is perfect we get $[L, K] \leq R_M$. Let $\tilde{K}$ be the preimage of K. If $K \cong Sz(q)$ set $\pi = \pi(K)'$ otherwise set $\pi = 3'$. Let $U = O_\pi(R_M)$. If K is a component of $\tilde{K}/U$. Then we have that any π'–element x in K is centralized by a good E in M. This with 5.3 shows that $C_G(x) \leq M$, as $C_{O_2(M)}(x) \neq 1$. Hence the same is true for $K_{\alpha-1}$. But we have that L centralizes K modulo U. Hence there is a π'–element in $K_{\alpha-1}$ which centralizes K. So we have that K is covered by $M_{\alpha-1}$, which contradicts $V_{M_{\alpha-1}} \leq O_2(M_{\alpha-1})$ and $[K, V_{M_{\alpha-1}}]$ involves K. This implies that that K is not a component. Then it acts on a Sylow r–subgroup

U_1 of $F(\tilde{K}/U)$ nontrivially. As $m_r(U_1) \leq 3$, we see that $K \not\cong Sz(q)$. Hence we have $r = 3$. By 2.4 we get that $m_3(U_1) = 3$ and so $U_1 = F^*(\tilde{K}/U)$. Let U_2 be a critical subgroup of U_1, then with 2.4 we get that $\Omega_1(U_2)$ is elementary abelian of order 27 or extraspecial of order 3^5. In both cases we see that K is the only simple composition factor in $\tilde{K}/U$, which acts nontrivially on U_1. So in KL there is a subgroup L_1, which is nonsolvable and centralizes U_1. But then $m_3(U_1 C_L(U_1)) \geq 4$, a contradiction.

So we have that $[V_M, [V_{M_{\alpha-1}}, K_{\alpha-1}]] \neq 1$. Then we get that K is covered by $M_{\alpha-1}$, which again conradicts $V_{M_{\alpha-1}} \leq O_2(M_{\alpha-1})$. □

*comp*5

Lemma 14.28 *Let V_{M_α} normalize some component K of M/R_M and induces a $2F$–module offender on $[V_M, K]$, then K is as in 3.43(5).*

Proof: Suppose false then by 3.43 and 14.27 we have that K is as in 3.43(2),(3) or (4).

Suppose 3.43(2). Then $\tilde{Y_P}$ contains elements centralized by some good E which are not in Y_M. But then we easily see that $P \leq M$, a contradiction.

Suppose that we are in 3.43(3)(4). Suppose first that we have W_M that any element in W_M is centralized by a good E. Let $W_{M_{\alpha-1}}$ be the corresponding module in $M_{\alpha-1}$. Then $[W_M, W_{M_{\alpha-1}}] = 1$. Hence any element x in $[W_M, V_{M_{\alpha-1}}]^\sharp$ is centralized by some good p–element in $M_{\alpha-1}$. As $M \neq M_{\alpha-1}$, we get with 5.5 that $p = 3$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Further not all 3–elements are good. We have that K cannot contain a good E, so 3^2 does not divide $|K|$. But as the center of a Sylow 3–subgroup of M is of order three, we get that $K \cong Sz(q)$ and $[V_M, K]$ is the natural module. But this would be the situation of 3.43(1), a contradiction.

So we may assume that in 3.43(3) or (4) we always have the second possibility. Which means that K contains a good E and any element in W_M is centralized by a good p–element. If $M \cap M_{\alpha-1}$ contains a good p–element in M. Then by 5.5 we get $p = 3$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Further 3 does not divide $|R_M|$ as otherwise $M \cap M_{\alpha-1}$ would contain a good E.

Suppose first $m_3(K) = 3$. Then by 1.1, 3.29, 3.30, 3.31, 3.32 we get $K \cong A_9$, A_{10}, A_{11}, $Sp_6(q)$, $\Omega^-(8, q)$, $L_n(q)$, $4 \leq n \leq 7$, $U_n(q)$, $4 \leq n \leq 7$. But in A_9 any 3–element is good. Hence the same applies for A_{10} and A_{11}. By 1.17 all 3–elements in $U_4(q)$, $Sp_6(q)$ and $\Omega^-(8, q)$ are good. Hence we are left with $L_4(q)$, $L_6(2)$ and $L_7(2)$. Also in $L_6(2)$ all 3–elements are good, so we are left with $K \cong L_4(q)$, $3 \mid q - 1$. But then all 3–elements can be diagonalized and so they are good.

So we have that $m_3(K) = 2$, as K contains a good E. Then with 1.1, 3.31 and 3.32 we get that $K \cong 3A_6$, $3A_7$, $3M_{22}$, A_6, A_7 or a group of Lie type over a field of characteristic two. As $m_p(K) \leq 3$ for all odd p, we get in the latter that $K \cong SL_3(q)$, $SU_3(q)$, $PSp_4(q)$, $G_2(q)$, $L_4(q)$ or $U_4(q)$. As the center of a Sylow 3–subgroup is cyclic, we have that 3 does not divide the order of $C_{M/R_M}(K)/Z(K)$. So in all cases we have an outer automorphism of order 3. Hence K is a group of Lietype. Suppose that $K/Z(K)$ has a non abelian Sylow 3–subgroup. Then $K \cong G_2(q)$. But then a field automorphism centralizes $G_2(2)$ and so an extraspecial group of order 27, which contradicts the structure of a Sylow 3–subgroup. Now if $Z(K) = 1$, we have that K admits an outer automorphismgroup of order 9, so we have $K/Z(K) \cong L_3(q)$ or $U_3(q)$. If 3 divides the order of $Z(K)$, then $K \cong SU_3(q)$ or $SL_3(q)$. But now all 3–elements in $K \setminus Z(K)$ are conjugate and so we have that all 3–elements are good, a contradiction. So we have that $K \cong L_3(q)$ or $U_3(q)$. Now with 3.29 we have that $K \cong L_3(q)$ and $[V_M, K]$ is a tensor product module. But this module has not a quadratic offender by 3.56.

So we may assume that $M \cap M_{\alpha-1}$ does not contain some good p–element. Hence $[W_M, C_{V_{M_{\alpha-1}}}(K_{\alpha-1})] = 1$ and so $[W_M, W_{M_{\alpha-1}}] \neq 1$. Further $[W_M \cap O_2(\hat{M}_{\alpha-1}), V_{M_{\alpha-1}}] = 1$ and $[W_{M_{\alpha-1}} \cap O_2(\hat{M}), V_M] = 1$. Now by symmetry we may assume that $W_{M_{\alpha-1}}$ induces an F–module offender on W_M. So we may apply 3.42. In particular we have 3.42(4). Further as elements in $C_{V_{M/Y_M}}(S)$ are centralized by good p-elements, we get that we have 14.5(1)(2),(4) or (5). Suppose that $3 \in \sigma(M)$. Then we cannot have 14.5(5), as here $M \cap P$ contains a good 3–element. If all 3–elements are good, we must have 14.5(1) or (2). But as 3 divides $q^2 - 1$, we also get some 3–element in $M \cap P$, a contradiction. Hence in case of $3 \in \sigma(M)$, we have that not all 3–elements are good. So we cannot have cases 3.42(4)(v), (vi) or (vii). In (ii) and (iii) we have nonsplit extensions and so $[V_M, V_{M_{\alpha-1}}]$ contains elements centralized by a good E, a contradiction. So we are left with (ii) or (viii), recall that (i) is not possible, as K contains a good E. In (ii) $[V_M, V_{M_{\alpha-1}}]$ always contains a non singular vector, but such vectors are centralized by a good E. So we have (viii). But then in $[V_{M_{\alpha-1}}, K_{\alpha-1}]$ there are elements which centralize in $[V_M, K]$ just $C_{[V_M,K]}([V_{M_{\alpha-1}}, K_{\alpha-1}])$. As $N_K(C_{[V_M,K]}([V_{M_{\alpha-1}}, K_{\alpha-1}]))$ acts transitively on $[V_M, K]/C_{[V_M,K]}([V_{M_{\alpha-1}}, K_{\alpha-1}])$, this would imply that K acts transitively on $[V_M, K]$, a contradiction. □

noEM

Lemma 14.29 *There is no component K of $E(M/R_M)$ which is normalized by $V_{M_{\alpha-1}}$ such that $V_{M_{\alpha-1}}$ induces a $2F$–module offender on $[V_M, K]$.*

Proof: Assume false. Then by 14.28 we have to treat 3.43(5). If we have 3.43(5)(v), (vi), (vii), (ix), (xii), (xx), (xxiii), then $3 \in \sigma(M)$ and all 3–elements are good. So we get P is as in 14.5(3)(5) or (6). In particular $C_{[V_M,K]}(S \cap K)$ is not centralized by some 3–element. But this is always the case.

Let us assume that we do not have 3.43(5)(i). Suppose further that any element in $[V_M, K]$ is centralized by a good p-element from K and K contains a good E. Then we have the situation of (3) or (4). Hence we may argue as before. So we do not have 3.43(5)(ii), (iii), (iv), (viii),(xiv). As we have quadratic action we see with 3.26 that we do not have (xi). Further we do not have (xv) or (xvi).

Let now 3.43(5)(x). Then $K/Z(K) \cong A_n$, $n \leq 7$. As we have at most two nontrivial modules in $[V, K]$ we see that $3 \in \sigma(M)$ and all 3-elements are good. Hence we see that $Y_P = \tilde{Y}_P$. As $C_{[V_M,K]}(S \cap K)$ is centralized by a good 3–element, we see that the same is true for $C_{V_M}(S)$. Hence there is some $1 \neq x \in Y_{M^g} \cap Y_P$, which is centralized by a good 3–element in M, where $M^g \neq M$. But then M^g contains agood 3–element from M. As all 3–elements are good we get with 5.5 that $M = M^g$, a contradiction.

Assume now 3.43(5)(xiii) or (xxvi). In case of $U_3(q)$ any element in $[V_M, K]$ is centralized by some good p–element and we have some good E in K, a contradiction. So we have (xxvi). As no good p–element can be in $N_G(S)$, otherwise it would also be in M_0 and so by construction via H we would get $P \leq M$, we get that p does not divide $q - 1$. But then we see that there is a good p-element centralizing $[V_M, K] = W_M$ and it is not centralized by a good E. So we have an outer automorphism of order p. As p does not divide the order of $N_G(S)$, we have that K is not normalized by S. Hence there is a conjugate L of K. Let $x \in W_M$, then $|[V_{M_{\alpha-1}}, x] : [V_{M_{\alpha-1}}, x] \cap Y_M| = q$. Let $[K_{\alpha-1}, x] \neq 1$. Then as $[x, W_{M_{\alpha-1}}]$ is of order q^2, we have that $Y_M \cap W_{M_{\alpha-1}} \neq 1$. But now we get $M = M_{\alpha-1}$ by 5.5, as $p \neq 3$. So we have that $[W_M, K_{\alpha-1}] = 1$. But the same also applies for $L_{\alpha-1}$, the group corresponding L. Hence we have that $[W_M, V_{M_{\alpha-1}}]$ is centralized by a good E in $M_{\alpha-1}$. But then $M \cap M_{\alpha-1}$ contains a good p–element from M, which contradicts 5.5.

Suppose now that we have 3.43(5)(xvii), (xviii), (xix) or (xxi) with $K \cong L_4(q)$. We first assume the $m_p(K) \geq 2$. Let W_M be a submodule of $[V_M, K]$. We first show

(1) If $x \in W_M$, then $C_K(x)$ contains a good p–element

If $K \cong G_2(q)$, we have $p \mid q^2 - 1$ and any element in W_M is centralized by $L_2(q)$. If $K \cong U_4(q)$, then any element in W_M is centralized by $L_2(q)$ or

$Sp_4(q)$ and $p \mid q^2-1$. If $K \cong Sp_6(q)$ and W_M is an extension of the trivial module by the natural module, then any element in W_M is centralized by $Sp_4(q)$. If W_M is the spin module, then elements are centralized by $L_3(q)$ or $\Omega^-(6,q)$, and in all cases $p \mid q^2-1$. If $K \cong Sp_4(q)$, then any element is centralized by $L_2(q)$ and $p \mid q^2-1$. If finally $K \cong L_4(q)$ and W_M is the natural module, then any element in W_M is centralized by $L_3(q)$, if W_M is the orthogonal module, then elements are centralized by $L_2(q)$ or $Sp_4(q)$, further $p \mid q^2-1$. hence in all case (1) holds.

Let first $m_p(K)=1$. If $K \cong G_2(q)$, then a good E centralize $[V_M,K]$ and we would be in 3.43(3) or (4). If $K \cong U_4(q)$, then p does not divide q^2-1, so $p \mid q^2+1$ or $q2+q+1$. In the latter we may argue as in the $G_2(q)$–case. So assume that $p \mid q^2+1$ and we have $[V_M,K]=W_M \oplus W_M^g$, g a p–element. Now we have still some p–element ρ centralizing $[V_M,K]$. In particular any element in W_M is centralized by some good E and we may argue as above. Let $K \cong Sp_6(q)$, then we must have $e(G) \geq 4$ and then by 14.25 we also have $m_p(C_M) \geq 4$. But then W_M is centralized by a good E. So let next $K \cong L_4(q)$. then $p \mid q^2+1$ or q^2+q+1. The same applies for $K \cong Sp_4(q)$. If there is some field automorphism ρ of order p. Then as there is some p–element centralizing W_M, we get that any element in W_M is centralized by a good E for $K \not\cong Sp_4(q)$. If $K \cong Sp_4(q)$, we must have a conjugate of K as otherwise $N_G(S)$ contains a good p–element, which contradicts 14.2. So we have that K is centralized by a good E, then W_M is centralized by a good E. Hence we have

(2) Either $C_{V_M}(K)$ is centralized by a good E or any element in W_M is centralized by a good E. If $m_p(K)=1$ the latter holds.

Let now $[C_{V_{M_{\alpha-1}}}(K_{\alpha-1}),W_M] \neq 1$. Then we have elements in W_M which are centralized by a good E in $M_{\alpha-1}$. By (1) they are also centralized by some good p–element in M. Hence we would get with 5.5 that $p=3$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. But in all these case we are now considering, all 3–elements are good. So we have that

$$[C_{V_{M_{\alpha-1}}}(K_{\alpha-1}),W_M]=1.$$

In particular we have that W_M acts nontrivially on $W_{M_{\alpha-1}}$ and so

(3) $[V_M,K]$ acts nontrivially on $[V_{M_{\alpha-1}},K_{\alpha-1}]$.

Let first $K \cong G_2(q)$. Then W_M is the natural module. Further by 3.18 $V_{M_{\alpha-1}}$ induces an offender of order q^3. Hence in both modules W in $[V_M,K]$ we have that $|W : C_W(V_{M_{\alpha-1}})| \geq q^3$. This shows that we have a subgroup $R \leq [V_M,K]$, $|R|=q^3$, $R \leq O_2(\hat{M}_{\alpha-1})$ but $C_R(V_{M_{\alpha-1}})=1$. We have that $[[V_{M_{\alpha-1}},K_{\alpha-1}],R] \leq Y_{M_{\alpha-1}}$. By (2) any element in W_M is centralized by a good p–element. Hence with 5.5 we get that $Y_{M_{\alpha-1}} \cap W_M = 1$ and so $R \cap W_M = 1$. Hence in $[V_M, K]/W_M$ we have that R corresponds to a complement of $C_{[V_M,K]/W_M}(V_{M_{\alpha-1}})$. By 14.7 we see that for $r \in R^\sharp$ we get $[V_{M_{\alpha-1}}, r] = Y_{M_{\alpha-1}}$. But obviously there are elements $r \in R$ such that $|[V_{M_{\alpha-1}}, r]W_M/W_M| = q^2$ while $Y_{M_{\alpha-1}}$ covers $C_{[V_M,K]/W_M}(V_{M_{\alpha-1}})$.

Let next $K \cong U_4(q)$. Then we get that $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M}_{\alpha-1})| = q^4$ by 3.18. As above we get some R with $|R| = q^4$. Again there is $r \in R$ such that $|[V_{M_{\alpha-1}}, r]W_M/W_M| \neq q^4$, a contradiction.

Let next $K \cong Sp_6(q)$. As we have symmetry, we may assume that $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M})| \geq |V_M : V_M \cap O_2(\hat{M}_{\alpha-1})|$. So let $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M})| = q^3$. Then by 3.44 we get again a group R of order q^3 as above. And so again $R \cap W_M = 1$. Let R come from a natural module. As we must have that $C_{V_{M_{\alpha-1}}/O_2(\hat{M}) \cap V_{M_{\alpha-1}}}(r) = 1$ for all $1 \neq r \in R$, this gives that $V_{M_{\alpha-1}}$ just consists of elements t with $|[V, t]| = q^3$ for the natural module V. But this is not possible.

So we have that R comes from the spin module. Now we have that $V_{M_{\alpha-1}}$ just has elements t with $|[V, t]| = q^4$, where V is now the spin module. But then $V_{M_{\alpha-1}}$ cannot induce an F–module offender.

So we have that $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M})| \geq q^4$. Now with 3.44 we get R as above with $|R| = q^4$ and then again that $|[V, t]| = q^4$ for all $t \in V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})$. This now implies that $V_{M_{\alpha-1}}$ is in $O_2(X)$, where X is the point stabilizer of K on the natural module. But $O_2(X)$ contains a subgroup X_1 of order q^2 with $|[V, x]| = q^2$ for all $x \in X_1^\sharp$. As $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| \geq q^2$ and $|O_2(X)| = q^5$, this is not possible.

Let next $K \cong L_4(q)$. As $V_{M_{\alpha-1}}$ acts quadratically on the orthogonal module, we have that $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| \leq q^3$. Suppose equality. Then on both natural modules it induces a group of transvections to a hyperplane. So we get as above a group R with $|R| = q^2$. Let now first W_M be the orthogonal module. As there are elements in W_M which are not centralized by a good E by (3), we get that p does not divide $q^2 - 1$. In particular $m_p(K) = 1$. By (2) any element in W_M is centralized by a good E, a contradiction again. So we just have natural modules. Further we cannot have a submodule which is invariant under some elementary abelian subgroup of order p^3. This shows $W_M = V_1 \oplus V_2$, both V_i natural modules and $V_2 = V_1^g$ for some p–element g. Let now R be as before. Assume that $R \cap W_M = 1$. Then for $t \in V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})$ we either have $[R, t] = [V, t]$, where V is the natural module or $[R, t] = 1$. We have that $|[V, V_{M_{\alpha-1}}]| = q^3$, so $V_{M_{\alpha-1}}$ is uniquely determined. But then $C_R(t) \neq R$ for all $t \in V_{M_{\alpha-1}}$ and $C_R(t) \neq 1$ for at least one $t \in V_{M_{\alpha-1}}$, a contradiction. So we have that $R \cap W_M \neq 1$. As then $Y_{M_{\alpha-1}} \leq W_M$, we even have $R \leq W_M$. But then $[V_{M_{\alpha-1}}, R] \cap V_1 \neq 1$, a contradiction.

Let now $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| \leq q^2$. Then $V_{M_{\alpha-1}}$ just consists of c_2–elements on the orthogonal module. But then group generated of the centralizes of such elements is $q^4(L_2(q) \times L_2(q))$, the point stabilizer in the natural module, which obviously does not act on $C_V(V_{M_{\alpha-1}})$, V the orthogonal module.

So let finally $K \cong Sp_4(q)$. Assume first $q > 2$ We have that $[V_M, K]$ involves two natural modules. Let first $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| = q$. Let further W_M be a nonsplit extension of a trivial module by the natural one. Then by 3.53 and (2) we have some $1 \neq x \in W_M \cap W_{M_{\alpha-1}}$ which is centralized by a good E in M and $M_{\alpha-1}$ as well, a contradiction. So we have that $|W_M| = q^4$. Now the argument as in the $L_4(q)$–case shows that $[V_M, K] = W_M \oplus W_M^g$, for some p–element $g \in M$. In particular $p \mid q^2 - 1$. But now any element in $[W_M, V_{M_{\alpha-1}}]$ is centralized by a good p–element in K, a contradiction.

So we have that $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| \geq q^2$. Assume equality. Then we get R with $|R| = q^2$. Now by 14.7 we have that $V_{M_{\alpha-1}}$ does not contain transvections on $[V_M, K]/W_M$. Hence also there are no a_2–elements in $V_{M_{\alpha-1}}$. Hence we now have that $X = \langle C_K(t) \mid t \in (V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M}))^\sharp \rangle \cong q^3 L_2(q)$. If $C_{V_M}(K) \neq 1$, then as $O_2(X)$ contains transvections, we see with 3.53 that $[V_M, V_{M_{\alpha-1}}] \cap C_{V_M}(K) \neq 1$. But this is not possible. Hence we get again $[V_M, K] = W_M \oplus W_M^g$, $g \in M$ a p–element and W_M the natural module. Further we have that $p \mid q^2 - 1$. Now in $[V_M, K]$ we have exactly $q + 1$ irreducible K–submodules. On $[V_{M_{\alpha-1}}, W_M]$ acts the group $L_2(q) \times Z_{q-1}$. Hence all nontrivial elements in this commutator are conjugate. Therefore they all are centralized by some p–element. The remaining elements are $(q^4 - 1) - (q + 1)(q^2 - 1) = (q^2 - 1)(q^2 - q) = (q^2 - 1)q(q - 1)$. On these acts $(L_2(q) \times Z_{q-1})Z_p$. In this group the centralizer is of order p. Hence we see that all elements in $[V_M, K]$ are centralized by a good E, a contradiction.

So we have now $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M})| > q^2$, i.e. there are transvections in $V_{M_{\alpha-1}}$. Further there is R with $|R| = q$ as before. Again $R \cap W_M = 1$. So as $|[V_M, K] : C_{[V_M,K]}(V_{M_{\alpha-1}})| \geq q^4$, we must have $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| = q^3$. Now $[V_{M_{\alpha-1}}, W_M]$ contains $C_{W_M}(K)$ and so we have that W_M is the natural module and $[V_M, K] = W_M \oplus W_M^g$ as before. But then we get the same contradiction that any element in $[V_M, K]$ is centralized by a good p–element.

Now we are left with $q = 2$. Then $p = 3$. Further any 3–element is good, $e(G) = 3$ and $[V_M, K] = W_M \oplus W_M^g$, W_M the natural module. If $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| = 2$, then $[V_{M_{\alpha-1}}, [V_M, K]]$ is centralized by a good 3–element, a contradiction. So we have that $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| \geq 4$

and then $R \neq 1$. Now $[R, V_{M_{\alpha-1}}] \leq [W_M, V_{M_{\alpha-1}}]$. If $C_{W_M}(K) = 1$, then again all elements in $[[V_M, K], V_{M_{\alpha-1}}]$ are centralized by a good 3–element, a contradiction. So we have $C_{W_M}(K) \neq 1$. Now $[W_M, W_{M_{\alpha-1}}] \cap C_{W_M}(K) = 1$. This shows that W_{M_α} induces an outer automorphism on $K' \cong A_6$. Now a quadratic fours group of this type always contains some transvection t. So $[R, t]$ is centralized by a 3–element, if $[R, t] \neq 1$. Hence we must have $[R, t] = 1$, i.e. $R \leq C_{V_M}(t)$. As $[R, V_{M_{\alpha-1}}]$ cannot contain some element, which is centralized by a 3–element, we get that $|V_{M_{\alpha-1}}/V_{M_{\alpha-1}} \cap O_2(\hat{M})| = 4$ and V_{M_α} corresponds to $\langle (1,2),(3,4) \rangle$. Now R projects as a fours group onto W_M and W_M^g. Hence we have that $|Y_{M_\alpha}| = 4$. As $3 \in \sigma(M)$ and all 3–elements are good, we have that P is of type 14.5(3) or (7). In both cases we have that $S/C_S(Y_P)$ is cyclic. Now for any $r \in R^\sharp$ we have that $|[V_{M_\alpha}, r]| = 4$. Then $[R, V_{M_{\alpha-1}}] \cap W_M \neq 1$, as $(3,4)$ acts as a transvection. But this contradicts $Y_{M_{\alpha-1}} \cap W_M = 1$.

Let now 3.43(5)(xxi) with $K \cong SL_3(q)$. Suppose first $m_p(K) = 2$. Then $p \mid q-1$ and so any element in W_M is centralized by a good p–element. Hence W_M acts nontrivially on $K_{\alpha-1}$. So we have that $[W_M, W_{M_{\alpha-1}}] \neq 1$. In particular there must be some p–element ρ. which does not normalize W_M. This shows $W_M^{\langle\rho\rangle} = W_1 \oplus \cdots \oplus W_r$, $r \leq 4$. But as $o(\rho) \mid q-1$ there is some module, which is normalized by ρ, a contradiction.

So we have that $m_p(K) = 1$. Let $[C_{V_{M_{\alpha-1}}}(K_{\alpha-1}), [V_M, K]] \neq 1$. As $Y_M \cap C_{V_{M_{\alpha-1}}}(K_{\alpha-1}) = 1$ by 5.5, we get that $|C_{V_{M_{\alpha-1}}}(K_{\alpha-1}) : C_{C_{V_{M_{\alpha-1}}}(K_{\alpha-1})}([V_M, K])| \leq q^2$. Now if W_M is a natural module, then $|W_M : C_{W_M}(C_{V_{M_{\alpha-1}}}(K_{\alpha-1})| \geq q$, i.e. W_M induces a $2F$–module offender. Let now $\hat{K}$ be a component of $M_{\alpha-1}/R_{M_{\alpha-1}}$ on which W_M induces such an offender. Then by symmetry we may assume that $\hat{K}$ is one of the components in 3.43, which we not have handled so far. This means 3.43(i) or (xxi) - (xxv).

In cases (xxiii) or (xxiv) we have that $m_p(\hat{K}) = 2$. Then we have by (2), (3) and 5.5 that $[\hat{K}, [V_{M_{\alpha-1}}, K_{\alpha-1}]] \neq 1$. So we have that $[V_M, K]$ acts nontrivially on $K_{\alpha-1}$ and $\hat{K}$ as well. Hence we get that we have an F–module, which is not the case in (xxiii) and (xxiv). Let (xxv). Then $m_p(\hat{K}) = 1$ and $e(G) = 3$. As $3 \mid |K|$, we get that $\hat{K} \cong L_3(2)$ and $m_3(K) = 1$. As W_M acts on $\hat{K}$, we have $q \leq 4$ and so $K \cong L_3(2)$ as well. As no element in W_M can be centralized by a good E, we now get $p = 7$ and there are exactly three natural modules in $[V_M, K]$. We further see that $[[V_{M_{\alpha-1}}, K_\alpha], \hat{K}] = 1$. Let now $x \in [V_M, K]$ and assume that $Y_{M_{\alpha-1}} \cap V_{M_{\alpha-1}} \neq 1$. Then there are elements in $Y_{M_{\alpha-1}}$ which are centralized by a good 7–element in M, contradicting 5.5. So we have that $|[V_{M_{\alpha-1}}, x]| \leq 4$, which contradicts the fact that there are at least three natural $\hat{K}$–modules in $V_{M_{\alpha-1}}$.

So we have that $\hat{K} \cong L_3(r)$ or $L_2(r)$. Let first $\hat{K} \cong L_3(r)$. We may assume that $r \geq q$. Suppose $r = q$. As $m_p(K) = 1$, we get that $q = 2$ or 4. Let $q = 4$, then as $\hat{K}$ acts nontrivially on $C_{V_{M_{\alpha-1}}}(K_{\alpha-1})$, we get that $K_{\alpha-1}\hat{K} = K_{\alpha-1} \times \hat{K}$, a contradiction. So we have $K \cong L_3(2) \cong \hat{K}$. Now there is some p–element centralizing W_M. Hence we have that $|[V_{M_{\alpha-1}}, x]| \leq 4$ for $x \in W_M$. This shows that $\hat{K}$ induces exactly two modules and so $[[V_{M_{\alpha-1}}, K_{\alpha-1}], \hat{K}] = 1$. But then $p = 3$ and $[V_M, K] = W_M \oplus W_M^g$ for sone 3–element g. But in this group now any element is centralized by a 3–element and so in M it is centralized by a good E, a contradiction. So we may assume that $r > q$. Hence we have that $[[V_{M_{\alpha-1}}, K_{\alpha-1}], \hat{K}] = 1$. Again if $x \in W_M$, we have that $|[W_M, V_{M_{\alpha-1}}]| = q$, or q^2. On the other hand it is a power of r. As $q < r$, we see that we have $r = q^2$. But then $m_3(\hat{K}) = 2$. This shows $3 \notin \sigma(M)$ and then $e(G) > 3$. But then any element in W_M is centralized by a good Ea contradiction.

So let now $\hat{K} \cong L_2(r)$. Then in any case $[V_{M_{\alpha-1}}, K_{\alpha-1}, \hat{K}] = 1$. Now again $m_3(K) = 1$ and as above we get that $r = q$ or $r = q^2$. But then $q - 1 \mid |\hat{K}|$. As $e(G) = 3$, we than have $m_p(K) = 2$, a contradiction.

So we have that W_M acts on some r–group and induces there a $2F$–module offender. There is $R \leq W_M$, $|R| = q$ such that $C_{V_{M_{\alpha-1}}}(R) = C_{V_{M_{\alpha-1}}}(r)$ for all $r \in R^\sharp$. This shows $q = 2$ and $K \cong L_3(2)$. How we have that in the natural module any element is centralized by a 3–element. As K is centralized by an elementary abelian p–group of order p^2, we get that $p \neq 3$, hence $p = 7 \in \sigma(M)$. Hence $[V_M, K]$ is centralized by a 7–element and $[V, K] = W_M \oplus W_M^g \oplus W_M^{g^2}$ for a 7–element $g \in M$. We now have that $V_{\alpha-1}$ induces a group of order 4 and so there is a group $R \leq [V_M, K]$, R of order 8 and all elements in R have the same centralizer, a contradiction.

So we have that $[C_{V_M\alpha-1}(K_{\alpha-1}), W_M] = 1$. Hence $[W_M, W_{M_{\alpha-1}}] \neq 1$. Let $[[V_M, K] \cap O_2(\hat{M}_{M_{\alpha-1}}, V_{M_{\alpha-1}}] = 1$. Then we even have an F–module and $|V_{M_{\alpha-1}} : V_{M_{\alpha-1}} \cap O_2(\hat{M})| = q^2$. So we have exactly two natural modules. As no element in the natural module can be centralized by a good E, we have that $[V_M, K] = W_M \oplus W_M^g$, g a p–element and so $o(g) \mid q + 1$. But as also $q + 1$ divides the order of $L_2(q)$ in fact all elements in W_M are centralized by a good E.

So we must have $[[V_M, K] \cap O_2(\hat{M}_{M_{\alpha-1}}, V_{M_{\alpha-1}}] \neq 1$. Further there are more than two modules in $[V_M, K]$ and p divides $q^2 + q + 1$. Now there are exactly three such modules and there is a group $R \leq [V_M, K]$, $|R| = q$ and $1 \neq [V_{M_{\alpha-1}}, R] \leq Y_{M_{\alpha-1}}$. Further we get $|[R, V_{M_{\alpha-1}}]| = q^2$. We also have that there is no p–element centralizing $[V_M, K]$. This shows that we must have an outer automorphism of order p on K and so $q \geq 8$ and $e(G) = 3$. This shows that 14.5(4),(5) (6) and (7) are not possible. Now in the other cases we get that $|Y_M| = q^2$ and there is a cyclic group U of order $q^2 - 1$ acting transitively on Y_M. Let $x \in U$, $o(x) = r$, r a prime dividing $q - 1$, $r > 3$. Then x normalizes K. Let $K\langle x\rangle \cong K \times Z_r$. Then $rin\sigma(M)$ and x is a good r–element. But $P = \langle P \cap M, N_P(\langle x\rangle)\rangle$, a contradiction. So x induces an outer automorphism. As $o(x) > 3$, this is a field automorphism. This now shows that $q = t^r$ and $r \mid t - 1$. But then $m_r(C_K(x)) = 2$. Hence again there is some elementary abelian group of order r^3, a contradiction.

Let next 3.43(5)(xxii). Suppose first that S normalizes K. As by 14.2 $N(S)$ does not contain a good p–element, we get that a good E centralizes K or $p = 3$ and a Sylow 3–subgroup is isomorphic to $Z_3 \wr Z_3$. In the former either some p–element just centralizes $[V_M, K]$ or p divides $q - 1$ and $[V_M, K]$ is the sum of two modules each centralized by some p-element.

In the latter we have that K has at least three conjugates under the action of a Sylow 3–subgroup. Hence if K is not normal, then we get that we have $L_2(q) \times L_2(q)$ and either some p–element just centralizes $[V_M, K]$ or $[V_M, \langle K^S\rangle] = [V_M, K]$ is the $O^+(4, q)$–module. Then any element in $[V_M, K]$ is centralized by some good p–element.

Let first $[[V_M, K] \cap R_{M_{\alpha-1}}, V_{M_{\alpha-1}}] \neq 1$. Then we have that $Y_{M_{\alpha-1}} \leq [V_M, K]$. Hence we have that no element in $Y_{M_{\alpha-1}}$ is centralized by a good p–element in M. This shows that p divides $q - 1$ and $[V_M, K] = W_M \oplus \tilde{W}_M$, where both W_M and $\tilde{W}_M$ are centralized by some good p–element. Now we can calculate the orbit lengths on $[V_M, K]$, which are $q^2 - 1$, $(q^2 - 1)p$ and $q(q^2 - 1)(q - 1)$. Hence again any element is centralized by some good p–element. So we may assume that $[[V_M, K] \cap R_{M_{\alpha-1}}, V_{M_{\alpha-1}}] = 1$. Hence there is a subgroup R in $[V_M, K]$, $|R|0q$, which induces an F–module offender on $V_{M_{\alpha-1}}$. This shows that we have the situation of 3.42. But then by symmetry we may assume that R acts faithfully on $F(M_{\alpha-1}/R_{M_{\alpha-1}})$. But all elements in R have the same centralizer in $V_{M_{\alpha-1}}$, which shows $q = 2$, a contradiction.

Let next 3.43(5)(xxiv). Then we see that a Sylow 3–subgroup of M is centralized by a good E. Hence all 3–elements are good. This shows that 3 does not divide the order of $P \cap M$. Assume that we have the orthogonal $O^+(4, q)$–module. Then we get that also a Sylow 3–subgroup of P is centralized by a good E in some conjugate M^g of M. As $P = \langle N_P(\langle \rho\rangle) \mid 1 \neq \rho \in \tilde{P}\rangle$, where $\tilde{P}$ is a Sylow 3–subgroup of P, we get $P \leq M^g$. But then with 5.4 we get $M = M^g$, a contradiction. So we have that Y_P is generated by two conjugates of Y_M, in particular no element in $Y_P \setminus Y_M$ is centralized by a good p–element in M. But we have that $C_{V_M}(S \cap K)$ is centralized by some 7–element from K, a contradiction.

So we are left with 3.43(5)(xxv). Let first $n > 3$. Then as in the case of $L_6(2)$, we get that P has the restricted structure. Further we have that $C_{[V_M,K]}(S \cap K)$ is centralized by a 3 -element ρ. Hence this ρ is in some M^g. As $e(G) = 3$ and $3 \notin \sigma(M)$ by 3.43(5)(xxv), we have that $\rho \in K^g$ and so it centralizes a good E in M^g. But then $M \cap M^g$ contains a good E, a contradiction.

Let $K \cong L_3(2)$. Then we have at most four modules involved. If $3 \in \sigma(M)$, we get a contradiction as above, as $C_{[V_M,K]}(S \cap K)$ is centralized by a good 3–element. Hence $p = 7 \in \sigma(M)$. Further there is a good p-element centralizing $[V_M, K]$. This gives that $[V_{M_{\alpha-1}}, V_M \cap O_2(\hat{M}_{\alpha-1})] = 1$. So $|[V_M, K] : [V_M, K] \cap O_2(\hat{M}_{\alpha-1})| = 8$. Suppose that $[V_M, K]$ centralizes $K_{M_{\alpha-1}}$. If $[K_{M_{\alpha-1}}, [t, V_{M_{\alpha-1}}]] \neq 1$ for some $t \in [V_M, K]$, then $Y_M \cap [t, V_{M_{\alpha-1}}] \neq 1$. But $[K_{M_{\alpha-1}}, [t, V_{M_{\alpha-1}}]] \leq [K_{M_{\alpha-1}}, V_{M_{\alpha-1}}]$ which is centralized by a good p–element in $M_{\alpha-1}$, a contradiction. So we have that $K_{M_{\alpha-1}}$ acts trivially on $[[V_M, K], V_{M_{\alpha-1}}]$. In particular that group intersect Y_M trivially. But then we have that $|V_{M_{\alpha-1}} : C_{V_{M_{\alpha-1}}}([V_M, K])| = 4$. Hence we get that $[V_M, K]$ induces a strong F–module offender on $V_{M_{\alpha-1}}$. Now we get first that it centralizes $F(\hat{M}_{\alpha-1}/O_2(\hat{M}_{\alpha-1}))$ by 3.21 and then 3.42 applies. But as we have an overoffender none of the groups is possible. So we get that $[V_M, K]$ acts nontrivially on $K_{M_{\alpha-1}}$. Let W_M be a natural module in $[V_M, K]$. Then we have that for $t \in W_M$ with $[t, K_{M_{\alpha-1}}] \neq 1$, we have that $|[V_{M_{\alpha-1}}, t]| = 4$, as $Y_M \cap [V_{M_{\alpha-1}}, t] = 1$ and $[V_{M_{\alpha-1}}, t] \leq W_M$ by quadratic action. But as there are at least three natural modules in $[V_{M_{\alpha-1}}, K_{M_{\alpha-1}}]$, we get $|[t, [V_{M_{\alpha-1}}, K_{M_{\alpha-1}}]]| \geq 8$, a contradiction.

So we are left with 3.43(5)(i). Hence $[V_M, K]$ is a nonsplit extension of a trivial module by the natural module for $K \cong L_2(q)$. If there is a good p–element, which induces a field automorphism on K, then S cannot normalize K. Hence some conjugate has to centralize $[V_M, K]$. If some good E centralizes K, there is some p–element centralizing $[V_M, K]$. Hence in any case there is some good p–element centralizing $[V_M, K]$. Now we argue as above. Let first $[K_{M_{\alpha-1}}, [[V_K, K], V_{M_{\alpha-1}}]] = 1$. As by 3.52 $[[V_M, K], V_{M_{\alpha-1}}]^\sharp$ contains elements which are centralized by a good E in M, we get that $K_{M_{\alpha-1}}$ is covered by M. But as $K_{M_{\alpha-1}}$ contains a good p–element, we now would get that $M \cap M_{\alpha-1}$ involves a subgroup isomorphic to $L_2(q) \times L_2(q)$ and so contains a good E, a contradiction. Hence $[V_M, K]$ acts nontrivially on $K_{\alpha-1}$. But then again by 3.52 in $[[V_M, K], V_{M_{\alpha-1}}]^\sharp$ there is also some element which is centralized by a good E in M and a good p–element in $M_{\alpha-1}$, a contradiction. □

fitting

Lemma 14.30 *We have that $V_{M_{\alpha-1}}$ does not induce a $2F$–module offender on $E(M/R_M)$ acting on $[V_M, E(M/R_M)]$.*

Proof: Suppose false. Then by 14.29 we have some component K of M/R_M such that $[K, V_{M_{\alpha-1}}] \not\leq K$. Then with 3.24 we get that $\langle K^{V_{M_{\alpha-1}}}\rangle = \Omega^+(4,q)$ and just orthogonal modules are involved in $[V_M, K]$. With 3.36 we now get that $[V_M, K]$ is the orthogonal module. In particular any element is centralized by some good p–element. This shows that $[V_M, K]/[V_M, K] \cap R_{M_{\alpha-1}}$ has to act faithfully on $[V_{M_{\alpha-1}}, K_{M_{\alpha-1}}]$. Further $Y_{M_{\alpha-1}} \cap [V_M, K] = 1 = Y_M \cap [V_{M_{\alpha-1}}, K_{M_{\alpha-1}}]$. But then the centralizer of $[V_M, K]$ in that group would have index at most $2q$, a contradiction as $q > 2$. □

By symmetry we now may also assume that V_M does not induce a 2F–module offender on some component of $M_{\alpha-1}/R_{M_{\alpha-1}}$. By 14.30 and 14.23 we have that $V_{M_{\alpha-1}}$ induces an $2F$–module offender on $F(M/R_M)$. Then there is a Sylow r–subgroup of $F(M/R_M)$ on which $V_{M_{\alpha-1}}$ induces a $2F$–module offender $\tilde{V}_{M_{\alpha-1}}$. Let F_r be a Sylow r–subgroup of the preimage on which $\tilde{V}_{M_{\alpha-1}}$ acts. Let $F_{\alpha-1}$ be the corresponding subgroup in $M_{\alpha-1}$. Set $F = [F_r, \tilde{V}_{M_{\alpha-1}}]$, let F_1 be the corresponding group in $M_{\alpha-1}$. Recall that $F_1 \leq C_M$. If also V_M induces a $2F$–module offender on $V_{M_{\alpha-1}}/Y_{M_{\alpha-1}}$, we will assume that always $V_{M_{\alpha-1}}$ is at least as good as V_M. In particular V_M cannot induce an F–module offender if $V_{M_{\alpha-1}}$ does not.

rinsigma

Lemma 14.31 *We have that $r \in \sigma(M)$.*

Proof: Assume that $r \not\in \sigma(M)$. Then by 2.1 we have that $|\tilde{V}_{M_{\alpha-1}}| \leq 8$.

If there is some elementary abelian p–subgroup E_1 of order at least p^3, $p \in \sigma(M)$ such that $[E_1, F_r] \leq R_M$, then there is also a good E in C_M with $[F_r, E] \leq R_M$. So let first assume that there is no such good E. Then as $r = 3$ or 5, we see with 2.3 that $m_r(F_r) = 2$ and $p = 3$. Further for a critical subgroup C of F_r we have that $\Omega_1(C) \cong E_{5^2}$ or an extraspecial group of order 5^3. We have that $N_M(\Omega_1(C))/C_M(\Omega_1(C))$ is isomorphic to a subgroup of $GL_2(5)$. As $\tilde{V}_{M_{\alpha-1}}$ acts on C, we get some 5–element $\omega \in \Omega_1(C) \setminus \Phi(\Omega_1(C))$ such that $|[V_M/Y_M, \omega]| \leq 4$. As an element ρ of order 3 acts nontrivially, we see that $|[V_M/Y_M, \Omega_1(C)]| \leq 2^8$. But in $GL(8,2)$ there is no element of order three acting nontrivially on a Sylow 5–subgroup. Hence we have that $[V_M/Y_M, \Omega_1(C)] = 1$. Now we have that $\langle \rho, \tilde{V}_{M_{\alpha-1}}\rangle$ acts. If there is some involution which inverts $\Omega_1(C)/Phi(\Omega_1(C))$, then in the first case we get that F_r is abelian, and so we get the contradiction $[F_r, V_M] = 1$. In the second case we get that $F_r = CU$, where $U = Z(F_r)$ is cyclic. Further we have that

$[\rho, U] = 1$. Hence $[F_r, \rho] \leq R_M$, a contradiction. So we have that there is no such element. This shows that $|\tilde{V}_{M_{\alpha-1}}| = 2$ and inverts ρ. This now gives that $|[\rho, V_M/Y_M]| \leq 2^4$. But then for some $g \in F_r$ we have that $U = \langle \rho, \rho^g \rangle$ involves an elementary abelian group of order 5^2 and $|[V_M/Y_M, U]| \leq 2^8$, which as above implies that $[U \cap F_r, V_M/Y_M] = 1$. Hence $[F_r, \rho] \leq R_M$, a contradiction.

So in any case there is a good E in C_M such that $[E, F_r] \leq R_M$. Suppose that $[E, [F, V_M]] = 1$. Let $x \in [F, V_M]^\sharp$. Suppose $[x, F_1] \leq R_{M_{\alpha-1}}$ and $[x, V_{M_{\alpha-1}}] \neq 1$. If $[C_{V_{M_{\alpha-1}}}(F_1), x] = 1$, then we have that $[[V_{M_{\alpha-1}}, F_1], x] \neq 1$. But this group is centralized by a good E in M and $M_{\alpha-1}$ as well, a contradiction. So we have $[C_{V_{M_{\alpha-1}}}(F_1), x] \neq 1$. Then we get $F_1 \leq M$ as commutators are centralized by a good E in M.

There is some good elementary abelian p–subgroup W in $C_{M_{\alpha-1}}$ with $F_{\alpha-1} = (F_{\alpha-1} \cap R_{M_{\alpha-1}})C_{F_{\alpha-1}}(W)$. As we are free in choosing F_r, we may assume that $C_{F_1}(W) \cap R_M \leq F_r$. Hence we may even assume that $F_r C_{F_1}(W)$ is an r–group normalized by $\tilde{V}_{M_{\alpha-1}}$. Now $C_{F_1}(W)$ acts on F. Let C be a characteristic subgroup on which $\tilde{V}_{M_{\alpha-1}}$ acts nontrivially. We have that F_1 acts on $\tilde{V}_{M_{\alpha-1}}$. As this group is of order at most 8 and acts quadratically, we see that $[F_1, \tilde{V}_{M_{\alpha-1}}] \leq R_M$. Let $y \in \tilde{V}_{M_{\alpha-1}}$ and $1 \neq u \in [\Omega_1(C), y]$ which is inverted by y which is centralized by $C_{F_1}(W)$. We see that there is some $1 \neq f \in C_{M_{\alpha-1}} \cap C_{F_1}(W)$ which centralizes u. But with 5.3 we have that $C_G(f) \leq M_{\alpha-1}$ and so $u \leq M_{\alpha-1}$. But u is inverted by some element in $V_{M_{\alpha-1}}$ contradicting $V_{M_{\alpha-1}} \leq O_2(M_{\alpha-1})$. So we have that C is not abelian and then we have that $[\Omega_1(C), y]$ is extraspecial. But then we get that $Z([\Omega_1(C), y])$ is centralized by y and there is a group of order r^2 in $[\Omega_1(C), y]$, which is normalized by $C_{F_1}(W)$. Hence C_{F_1} again centralizes some element u, which is inverted by y, a contradiction.

So we have $[x, F_1] \not\leq R_{M_{\alpha-1}}$ and then also $[x, F_{\alpha-1}] \not\leq R_{M_{\alpha-1}}$. If there is a fours group $V \leq [F, V_M]$, which acts on $F_{\alpha-1}$ nontrivially, we see with 2.1 that there are r–elements in $C_{M_{\alpha-1}} \cap F_{alpha-1}$, which are in M but are inverted by elements in V_M, a contradiction. So we have that $[V_M, F]$ is of order 4. Then set $\tilde{F}_1 = \langle F_1, F_1^x \rangle$. We get that $|[V_{M_{\alpha-1}}, \tilde{F}_1]| \leq 16$. As $\tilde{F}_1$ is centralized by a good E in $C_{M_{\alpha-1}}$, we get the same for $[V_{M_{\alpha-1}}, \tilde{F}_1]$, but then some element in $[V_M, F]$ is centralized by a good E in C_M and $C_{M_{\alpha-1}}$, a contradiction.

So we have that $[F, V_M]$ is not centralized by a good E in C_M. This gives $r = 5$ and $p = 3 \in \sigma(M)$. In particular this is now an exact $2F$–module offender and so we may assume that $V_{M_{\alpha-1}}$ centralizes all components and all further Sylow subgroups of the Fitting subgroup. As we do not have an F–module offender $[V_M, F]$ on $V_{M_{\alpha-1}}$ we see that for $x \in [V_M, F] \setminus R_{M_{\alpha-1}}$ we get $[x, V_{M_{\alpha-1}}] \cap Y_M \neq 1$. Suppose first that $[x, F_{\alpha-1}] \leq R_{M_{\alpha-1}}$. We have that F is generated by elements ρ with $|[V_M, \rho]| = 16$. Let $\rho_{\alpha-1}$ the corresponding element in $F_{\alpha-1}$, then we have that $[[V_{M_{\alpha-1}}, \rho_{\alpha-1}], x] = 1$. This shows that $[V_{M_{\alpha-1}}, x] = [x, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})]$. As $[x, V_{M_{\alpha-1}}] \cap Y_M \neq 1$, we see again that $F_1 \leq M$. As above we get a contradiction.

So we have that $[x, F_{\alpha-1}] \not\leq R_{M_{\alpha-1}}$. In particular we have a fours group X which acts faithfully on $F_{\alpha-1}R_{M_{\alpha-1}}/R_{M_{\alpha-1}}$. Hence by 2.1 there is $U \cong D_{10} \times D_{10}$ in $F_{\alpha-1}R_{M_{\alpha-1}}/R_{M_{\alpha-1}}$ containing X as a Sylow 2–subgroup. We have that $[V_{M_{\alpha-1}}, O_5(U)] = V_1 \oplus V_2$ with $C_{O_5(U)}(V_i) \neq 1$, $i = 1, 2$. Let $X = \langle x, y \rangle$. then we may assume that $[V_1, x] \neq 1 \neq [V_2, y]$. Now we have that $V_i \cap Y_M \neq 1$, $i = 1, 2$, which gives $O_5(U) \leq M$, a contradiction. □

*bsmall*1

Lemma 14.32 *If b is odd then $b = 1$.*

Proof: Suppose $b > 1$. Then by 14.23, 14.30 and 14.31 we get that V_{M_α} induces a $2F$–module offender on a Sylow r–subgroup of $F(M/R_M)$, where $r \in \sigma(M)$. Let $1 \neq t \in [F, V_M]$ such that $[[t, F], E] = 1$ for some good E in C_M. Then in particular $[t, V_{M_{\alpha-1}}] \cap Y_{M_{\alpha-1}} = 1$. If $[t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})] \neq 1$, then we have that $F_1 \leq M$. In particular F_1 contains no good E, which shows $m_r(F_1) \leq 2$. If F_1 is cyclic, then also F is cyclic. Again we may assume that FF_1 is a r–group. As F_1 contains a good r–element, we get that $\Omega_1(F) \leq M_{\alpha-1}$. But $\Omega_1(F)$ is inverted by some element in $V_{M_{\alpha-1}}$, a contradiction. Hence $m_r(F_1) = 2$. Let C be a critical subgroup of F. If C is cyclic, we get as before that $\Omega_1(C) \leq M_{\alpha-1)}$ and so $V_{M_{\alpha-1}}$ centralizes C, a contradiction. Hence $m_r(C) = 2$ and so $\Omega_1(C)$ is elementary abelian of order r^2 or extraspecial of order r^3. As above we get again some r–element in $\Omega_1(C)$ which is centralized by F_1 and inverted by some element in $\tilde{V}_{M_{\alpha-1}}$, a contradiction.

So we have that $[t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})] = 1$. Hence $[t, [F_{\alpha-1}, V_{M_{\alpha-1}}]] \neq 1$. Let now $[F, V_M]$ be generated by such elements t. Suppose that $[t, F_{\alpha-1}] \leq R_{M_{\alpha-1}}$, then there is some $t_{\alpha-1} \in V_{M_{\alpha-1}}$ such that $[t, t_{\alpha-1}] \neq 1$ and $[t_{\alpha-1}, F_1]$ is centralized by a good E. But then $[t, t_{\alpha-1}]$ is centralized by a good E in C_M and $C_{M_{\alpha-1}}$ as well. Hence for each such element we have that $[t, F_{\alpha-1}] \not\leq R_{M_{\alpha-1}}$. If there is a foursgroup X of this type in V_M, then we get in $F_{\alpha-1}X$ by 2.1 a subgroup $U \cong D_{2r} \times D_{2r}$. Hence we have an elementary abelian subgroup of order r^2 in $M \cap M_{\alpha-1}$, which gives that $r = 3$ and a Sylow 3–subgroup is isomorphic to $Z_3 \wr Z_3$. Then we get that $O_3(U)$ is not contained in an elementary abelian subgroup of order 27. In particular we get that $F_{\alpha-1}$ is extraspecial of order 27 and so $F_{\alpha-1} = [X, F_{\alpha-1}]$. But then there is a group U in $F_{\alpha-1}X$, where U is as above and $O_3(U)$ is good, a contradiction.

Let us collect :

($*$) If $[V_M, F]$ is generated by elements t such that $[t, F]$ is centralized by a good E, then any such t centralizes $C_{V_{M_{\alpha-1}}}(F_{\alpha-1})$ and $[t, F_{\alpha-1}] \neq 1$. Further there is no fours group of this type.

Let first $|\tilde{V}_{M_{\alpha-1}}| \geq 8$. Then by 2.1, quadratic action and 4.5 we get that in $[V_M, F]$ we have $V_1 \oplus V_2 \oplus \cdots \oplus V_s$ and each V_i is centralized by a good E in C_M. If $r = 5$ we get that $[V_M, F]$ is generated by fours groups X such that $[X, F]$ is centralized by a good E, contradicting ($*$). So $r = 3$ and $|V_i| = 4$, $i = 1, 2, \ldots, s$. Hence we have that $|[V_M, F]| = 4^s$. By ($*$) we have that $s \leq 3$. Hence $|F/C_F(V_M)| \leq 3^3$. Let $t_{\alpha-1} \in V_{\alpha-1}$ corresponding V_1. Then $|[\langle t_{\alpha-1}^{F_{\alpha-1}} \rangle, t]| = 4$ for $t \in V_i$, i suitable. Now this group is centralized by a good E in M and $M_{\alpha-1}$ as well, a contradiction.

So we may assume that $|\tilde{V}_{M_{\alpha-1}}| \leq 4$. Let first $r = 5$. Let $|\tilde{V}_{M_{\alpha-1}}| = 4$. Assume that there is $U \cong D_{10} \times D_{10}$ with $\tilde{V}_{M_{\alpha-1}}$ as a Sylow 2–subgroup and for $O_5(U) = \langle \omega_1, \omega_2 \rangle$ we have that $|[V_M, \omega_i]| = 16$ and $[V_i, \omega_i, \omega_{3-i}] = 1$, $i = 1, 2$. Let U_1 be a Sylow 5–subgroup of C_M containing U. If $m_5(C_{U_1}(\omega_1)) \geq 3$, then there is a good E in C_M with $[[V_M, \omega_1], E] = 1$, which contradicts ($*$) again. So we have that $m_5(C_{U_1}(\omega_1)) = 2$. Then $U = \Omega_1(C_{U_1}(U))$. By the action of U on V_M, we see that $N_{U_1}(U) \leq C(\omega_1) \cap C(\omega_2)$. Hence we have that $U_1 = C_{U_1}(U)$. As M/C_M has cyclic Sylow 5–subgroups, we have that for a Sylow 5–subgroup U_2 of M containing U_1 that $U \leq Z(\Omega_1(U_2))$. Hence all 5–elements are good. Now with 14.3 we get that $|M_0|$ is not divisible by 5, which shows that $U_2 = U_1$. But we have that $m_r(U_2) \geq 3$ as $r \in \sigma(M)$.

Hence there is no such U. In particular R_M has a nontrivial Sylow 5–subgroup. Now $|[O_5(U), V_M]| = 16$. Now we may apply 2.1 to $F_r/F_r \cap R_M$, which shows that we get some good E in C_M which centralizes $[V_M, U]$, a contradiction to ($*$).

Let now $|\tilde{V}_{M_{\alpha-1}}| = 2$. Let ω be some element of order 5, which is inverted by some element in $\tilde{V}_{M_{\alpha-1}}$ and $|[V_M, \omega]| = 16$. Then by ($*$) we have that $m_5(C_{U_1}(\omega)) = 2$, where again U_1 is a Sylow 5–subgroup of C_M containing ω. Now we have that $C_{U_1}(\omega) = \langle \omega \rangle \times U_2$, with cyclic U_2. If $[V_M, \Omega_1(U_2)] \neq 1$, we see that $N_{U_1}(\langle \omega \rangle \times U_2) = C_{U_1}(\omega)$. Hence $U_1 = \langle \omega \rangle \times U_2$. As above we get that all 5–elements are good and then with 14.3 that U_1 is a Sylow 5–subgroup of M contradicting $m_5(U_1) \geq 3$. Hence we have that $[V_M, \Omega_1(U_2)] = 1$. Now let $\tau \in N_{U_1}(\langle \omega, \Omega_1(U_2) \rangle) \setminus C_{U_1}(\omega)$. Now as $[V_M, \langle \omega, \Omega_1(U_2) \rangle] = [V_M, \langle \omega \rangle]$, we have that τ normalizes $[V_M, \omega]$ and so we may assume that τ centralizes $[V_M, \omega]$. We have that $C_{U_1}(V_M)$ is cyclic. If $C(V_M) \cap \langle \omega, U_2, \tau \rangle \leq U_2$, then we see that

$\omega^{U_1} \cap \langle \omega, \tau, U_2 \rangle \leq \Omega_1(\langle U_2, \omega \rangle)$. But then $\langle \omega, \Omega_1(U_2) \rangle$ is normal in U_1. Let now $[V_M, \tau] = 1$. Now let $x \in U_1 \setminus \langle \tau, \omega, U_2 \rangle$ with $\omega^x = \omega t$, $t \in \Omega_1(C(V_M))$. Hence $\omega^x \in \langle \omega, \Omega_1(U_2) \rangle$, as $C(V_M)$ does not contain a good E. This again shows that $\langle \omega, \Omega_1(U_2) \rangle$ is normal in U_1. This shows that $U_1 = \langle \omega, U_2, \tau \rangle$. As ${}_5(M) \geq 3$, we see that there is some μ of order 5 which is not in C_M but normalizes U_1. Hence we may assume that μ centralizes τ and $\Omega_1(U_2)$. Further we have that U_1 is a central product of U_2 with an extraspecial group of order 5^3. Then in particular $\tau \in F_r$. We now have that $|[\Omega_1(F_r), V_M]| \leq 2^8$. But on this group also μ acts, which contradicts the structure of $GL(8,2)$.

So we have that all elements of order 5 inverted by some element in $V_{M_{\alpha-1}}$ are contained in R_M. Let $t \in V_M$ and assume that $[V_{M_{\alpha-1}}, t] \cap Y_M \neq 1$. Then we have a good 5–element from $M_{\alpha-1}$ in M, a contradiction. Hence we get that all elements in V_M induce transvections to a hyperplane. But then V_M induces an F–module offender, a contradiction.

So we have $r = 3$. Assume first that 3 divides $|R_M|$. Let $t \in V_M$. If $[V_{M_{\alpha-1}}, t] \cap Y_M \neq 1$, we get a good 3–element ρ from $M_{\alpha-1}$ in M. Hence we have that $U_2 \cong Z_3 \wr Z_3$ is a Sylow 3–subgroup of M and $Z(U_2) \leq R_M$. Now we have that $[\tilde{V}_{M_{\alpha-1}}, \mu] = 1$. This shows that μ acts on $[F_r, \tilde{V}_{M_{\alpha-1}}]$. Hence we have that $|\tilde{V}_{M_{\alpha-1}}| = 2$. Hence we get that U_2 acts on $[V_M, F]$. We get that μ centralizes this group and so $U_2'\langle \mu \rangle$ centralizes this group. But we have that $F \cap U_2' \not\leq R_M$, a contradiction. Hence we get that V_M is generated by elements inducing transvections. If $|[V_M, F]| > 16$, then we get that we have an F–module offender V_M, a contradiction. So we have that $|[V_M, F]| = |\tilde{V}_{M_{\alpha-1}}|^2$ and both groups induce F–module offender. Now there is some $\rho \in F$ with $|[V_M, \rho]| = 4$. Then this group is centralized by a good E in M. Suppose that this E is not in C_M. Then in particular $|Y_M| \geq 4$. If all elements in the cosets of Y_M are conjugate, we have some commutator with some element in $[V_M, \rho]$, which in fact is centralized by a good E, and then we get the same contradiction above. So we have 14.5(6) or (7). Then $\tilde{Y}_P = Y_P$. So let $d(\beta, \alpha) = b - 2$, then $[V_{M_{\alpha-1}}, Y_{M_\beta}] = 1$. Hence ρ centralizes a subgroup of index two in Y_{M_β}. Then $\rho \in M_\beta$ and so we have that $Z_3 \wr Z_3 \cong U_2$ again. Further $Z(U_2) \leq R_M$. But then we see that $\langle \rho, Z(U_2) \rangle$ is a good E in M_β, a contradiction.

So we have shown that R_M is a $3'$–group. Let now first $|\tilde{V}_{M_{\alpha-1}}| = 4$. Then we have a subgroup $U \cong \Sigma_3 \times \Sigma_3$. By quadratic action we get that $[V_M, O_3(U)] = V_1 \oplus V_2$ and $O_3(U) = \langle \rho_1, \rho_2 \rangle$, where $V_i = [V_M, \rho_i]$ and $[V_i, \rho_{3-i}] = 1$, $i = 1, 2$. This shows that $N_{U_1}(O_3(U)) = C_{U_1}(O_3(U))$ and so all elements in $O_3(U)$ are good. Let $O_3(U_{\alpha-1})$ be the corresponding group. Let $t \in V_1$ and assume that $[t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})] \neq 1$. Then there is a good 3–element from M in $M_{\alpha-1}$. This shows that $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Now we have that $|[t, V_{M_{\alpha-1}}] : [t, V_{M_{\alpha-1}}] \cap Y_M| \leq 2$. As $Y_M \cap C_{V_{M_\alpha}}(F_{\alpha-1}) = 1$, we have that t induces a transvection. But then t inverts a 3–element, which centralizes $[V_{M_{\alpha-1}}, F_{\alpha-1}]$. But then also $[t, V_{M_{\alpha-1}}]$ is centralized by a good E, which gives the contradiction $M = M_{\alpha-1}$. So we have that $[t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1}))] = 1$. Let next $[t, F_{\alpha-1}] \le R_{M_{\alpha-1}}$. Let $O_3(U_{\alpha-1}) = \langle \mu_1, \mu_2 \rangle$, where $\mu_i \sim \rho_i$, $i = 1, 2$. We may assume that $[t, [V_{M_{\alpha-1}}, \mu_1]] \neq 1$. Then we have that $|[t, [V_{M_{\alpha-1}}, \mu_1]]| \ge 4$. Hence we have that $Y_M \cap [t, [V_{M_{\alpha-1}}, \mu_1]] \neq 1$. Hence again there is a good 3–element μ_2 from $M_{\alpha-1}$ which is contained in M. This again shows that we have $Z_3 \wr Z_3$ as a Sylow 3–subgroup. We have that μ_2 is not good in M. As $C_{U_1}(O_3(U)) = U_2$ is elementary abelian of order 9, we see that $\tilde{V}_{M_\alpha}$ acts on U_2 and centralizes μ_2, where $U_1 = U_2\langle \mu_2 \rangle$. But there is no such subgroup in $GL(3, 3)$.

So we have that $[t, F_{\alpha-1}] \neq 1$. Let first $|[V_M, \rho_i]| = 4$, $i = 1, 2$. Let $\rho_{\alpha-1}$ be the corresponding element in $F_{\alpha-1}$. Again we see that there is some good E in C_M centralizing $[V_M, \rho]$. Set $U_{\alpha-1} = \langle \rho_{\alpha-1}, \rho_{\alpha-1}^t \rangle$. Then we have that $|[U_{\alpha-1}, V_{M_\alpha}] \le 16$. As $R_{M_{\alpha-1}}$ is a $3'$–group, we see that $U_{\alpha-1}$ is elementary abelian. If the order is 3, we get even that $M \cap M_{\alpha-1}$ contains a good E. Hence the order is 9. In that case we have that the commutator is of order 16 and as before we get that $C_{C_{M_{\alpha-1}}}(U_{\alpha-1})$ contains a good E and then there is a good 3–element centralizing $[V_{M_{\alpha-1}}, U_{\alpha-1}]$, which implies that this element is in M. Now we get that we have $Z_3 \wr Z_3$ as a Sylow 3–subgroup. But then a fours group direct a group of order three acts on F_r, which contradicts the structure of $GL(3, 3)$.

So we have that $|[V_M, \rho_1]| = 16$. Now we have a fours group $V \le V_1$, which acts faithfully on $F_{\alpha-1}$. Then we get some $\tilde{U} \cong \Sigma_3 \times \Sigma_3$ with V as a Sylow 3–subgroup and all elements are good. As $[V_{M_{\alpha-1}}, t] \cap Y_M \neq 1$, since $|[V_{M_{\alpha-1}}, t]| \ge 4$, otherwise V_M would induce an F–module offender, we see again that M contains $O_3(\tilde{U})$, a contradiction.

So we have that $|\tilde{V}_{M_{\alpha-1}}| = 2$. Let $\rho \in F_r$, ρ be inverted by $\tilde{V}_{M_{\alpha-1}}$. Let $m_3(C_{U_1}(\rho)) = 2$, where U_1 is a Sylow 3–subgroup of C_M. Then we have that $C_{U_1}(\rho) = \langle \rho \rangle \times U_2$, with cyclic U_2. If $[[V_M, \rho], \Omega_1(U_2)] = 1$, then we get that under $N_{U_1}(C_{U_1})(\rho)$ $\langle \rho \rangle$ is normal. Hence $U_1 = C_{U_1}(\rho)$. But then we see that also ρ is in the center of U_2, a Sylow 3–subgroup of M containing U_1. In particular all 3–elements are good. But then by 14.3 we have that $U_1 = U_2$, a contradiction. Hence we have that $[\Omega_1(U_2), [V_M, \rho]] \neq 1$. Now we have that $\langle \rho, \Omega_1(U_2) \rangle$ acts faithfully on $[V_M, \rho]$. But then $N_{U_1}(\langle \Omega_1(U_2), \rho \rangle)$ has to centralize ρ, the same contradiction. So we have in any case that $C_G(\rho) \le M$ and contains a good E in C_M. Hence if $|[V_M, \rho]| = 4$, then this group is centralized by a good E in C_M.

Let $t \in [V_M, \rho] \setminus R_{M_{\alpha-1}}$. Let $\rho_{\alpha-1}$ be the corresponding element in $M_{\alpha-1}$. Let $[C_{V_{M_{\alpha-1}}}(F_{\alpha-1}), t] \neq 1$. Then we may assume that $\tilde{V}_{M_{\alpha-1}} \le$

$C_{M_{\alpha-1}}(F_{\alpha-1})$. Now $[t, V_{M_{\alpha-1}}] = [t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})](Y_M \cap [t, V_{M_{\alpha-1}}])$. Suppose that $[t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})] \cap Y_M \neq 1$, then $\rho_{\alpha-1} \in M$. This again implies that $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Now $\rho_{\alpha-1}$ has to centralize $[\tilde{V}_{M_{\alpha-1}}, F_r]$ and so this group is of order 3. Hence $[F_r, \tilde{V}_{M_{\alpha-1}}] = Z(U_1)$. But then U_1 acts on $[V_M, \rho]$, which is of order at most 16, a contradiction. So we have that $[t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})] \cap Y_M = 1$. Then t induces a transvection on $C_{V_{M_{\alpha-1}}}(F_{\alpha-1})$. If t centralizes $F_{\alpha-1}$, then 3 divides the order of $C_{M_{\alpha-1}}(F_{\alpha-1})^\infty$ and so all 3–elements are good. We have in $C_{M_{\alpha-1}}(F_{\alpha-1})$ some $L_n(2)$, $Sp(2n, 2)$, $\Omega^\pm(2n, 2)$ or A_n on which t acts. Hence in any case we have that $[C_{V_{M_{\alpha-1}}}(F_{\alpha-1}), t]$ is centralized by some elementary abelian group of order 9. As $[V_M, \rho]$ is centralized by a 3–element we have a 3-element in $M \cap M_{\alpha-1}$, a contradiction as all 3–elements are good. So we have that $[t, F_{\alpha-1}] \neq 1$. If $|[V_M, \rho]| = 16$, we even have a fours group V which acts faithfully on $F_{\alpha-1}$. But then we have some $\tilde{U} \cong \Sigma_3 \times \Sigma_3$ with V as a Sylow 3–subgroup. As no 3–element from $\tilde{U}$ can be in M, we get that $|[V_{M_{\alpha-1}}, t]| = 2$. But then as before we see that there is a good 3–element from M which is in $M_{\alpha-1}$. Again we have that this centralizes V and acts on $F_{\alpha-1}$, a contradiction to the structure of $GL(3, 3)$. So we are left with $|[V_M, \rho]| = 4$. Then $[V_M, \rho]$ is centralized by a good E in C_M. Set $\tilde{U} = \langle \rho_{\alpha-1}, \rho_{\alpha-1}^t \rangle$. Then we have that $|[V_{M_{\alpha-1}}, \tilde{U}]| \leq 16$ and so it is centralized by some 3–element μ, which then is in M. Hence we get that $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Again μ has to centralize $[F_r, \tilde{V}_{M_{\alpha-1}}]$, which then has to be $Z(U_1)$. But then $[V_M, Z(U_1)]$ is of order 4 and normalized by U_1, a contradiction.

So we have that $[t, C_{V_{M_{\alpha-1}}}(F_{\alpha-1})] = 1$. Suppose that $[F_{\alpha-1}, t] \leq R_{M_{\alpha-1}}$. Now on $C_{V_{M_{\alpha-1}}}(\rho_{\alpha-1})$ we have that t has to induce at most transvections, otherwise $\rho_{\alpha-1} \in M$ and we get a contradiction as before. As $Y_M \cap Y_{M_{\alpha-1}} = 1$, we see that $|[t, V_{M_{\alpha-1}}]| \leq 8$. We have that $L = \langle t^{M_{\alpha-1}} \rangle \leq C_{M_{\alpha-1}}(F_{\alpha-1}R_{M_{\alpha-1}}/R_{M_{\alpha-1}})$ Further L acts on $[\rho_{\alpha-1}, V_{M_{\alpha-1}}]$. Suppose this action is nontrivial. Then L induces a subgroup of $A_5 = L_2(4)$. Suppose 3 divides $|L|$. Then all 3–elements are good. Further we have that L centralizes $C_{V_{M_{\alpha-1}}}(\rho_{\alpha-1})$. This shows that we have a 3–element which centralizes $[t, [V_{M_{\alpha-1}}, \rho_{\alpha-1}]]$. This then is in M. As all 3–elements are good, this is a contradiction. Hence we have that L induces F_{10}. We now have that $|[V_M, \rho]| = 16$. Hence there is a second element t_1, which now has to act nontrivially on $F_{\alpha-1}$. Then we get some subgroup $\tilde{U} \cong F_{10} \times \Sigma_3$, with $\langle t, t_1 \rangle$ as a Sylow 2–subgroup. But then we get that the Σ_3 is contained in M, a contradiction. So we have that $[t, [V_{M_{\alpha-1}}, \rho_{\alpha-1}]] = 1$. Then we have that t induces transvections on $V_{M_{\alpha-1}}$. Hence $L \cong L_n(2)$, $Sp(2n, 2)$, $O^\pm(2n, 2)$ or Σ_n. Now all 3–elements are good and $[V_{M_{\alpha-1}}, t]$ is centralized by a good E in $M_{\alpha-1}$. As $[V_M, \rho]$ is centralized by a 3–element in M, we get a contradiction as before.

Hence we have that $[F_{\alpha-1}, t] \neq 1$. If we have a fours group V acting faithfully on $F_{\alpha-1}$, then there is $\tilde{U} \cong \Sigma_3 \times \Sigma_3$ with V as a Sylow 2–subgroup. If $|[O_3(\tilde{U}), V_{M_{\alpha-1}}]| > 16$, we get some $\mu \in O_3(\tilde{U})$ with $|C_{[V_{M_{\alpha-1}}, O_3(\tilde{U})]}(\mu)| \geq 16$ and this group contains some element from Y_M. Hence we get $\mu \in M$, a contradiction. So we have that $|[O_3(\tilde{U}), V_{M_{\alpha-1}}]| = 16$. Now $O_3(\tilde{U}) = \langle \mu_1, \mu_2 \rangle$ such that $|[V_{M_{\alpha-1}}, \mu_i]| = 4$. As $|\tilde{V}_{M_{\alpha-1}}| = 2$, there is some $x \in [V_{M_{\alpha-1}}, O_3(\tilde{U})]$ such that $[V, x] \leq Y_M$. But then $Y_M \cap [V_{M_{\alpha-1}}, \mu_i] \neq 1$ for at least one i. Then $\mu_{3-i} \in M$, a contradiction.

So we have that $|[V_M, \rho]| = 4$. Now $[V_M, \rho]$ is centralized by a good E in C_M. Again set $\tilde{U} = \langle \rho_{\alpha-1}, \rho_{\alpha-1}^t \rangle$. Then we have that $|[V_{M_{\alpha-1}}, \tilde{U}]| \leq 16$ and so it is centralized by some 3–element μ, which then is in M. Hence we get that $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Again μ has to centralize $[F_r, \tilde{V}_{M_{\alpha-1}}]$, which then has to be $Z(U_1)$. But then $[V_M, Z(U_1)]$ is of order 4 and normalized by U_1, a contradiction.
□

b2q

Lemma 14.33 *If $b = 1$, then we have 14.5(1) or (2) with $q > 2$.*

Proof: Assume false. By 14.11 and 14.32 we may assume that $b = 1$. Then in particular $Y_P \not\leq O_2(M)$. Further $Y_P \leq C_M$. This gives that $[O_2(M), Y_P] \not\leq Y_M$. So we have that P is as in 14.5(1), (2), (4) or (5).

Assume 14.5(4). Then $|\tilde{Y}_P| = 4$. Further $V_M \not\leq O_2(P)$ and $V_M C_P/C_P$ is a Sylow 2–subgroup of $E(P/C_P)$. But then $[V_M, \tilde{Y}_P] = 1$ and so V_M is elementary abelian, which shows that V_M acts quadratically on Y_P, a contradiction.

Suppose now that in 14.5(1) and (2) we have $q = 2$. In that case (2) is just a special case of (5). So assume (5). Then $|\tilde{Y}_P| = 4$. Further we have $P/O_2(P) \cong \Sigma_3 \wr Z_2$, otherwise we could have chosen P of type (3). If $V_M O_2(M)/O_2(M)$ is contained in the transvection group, there is some element $x \in Y_P \setminus O_2(M)$ such that $|[x, V_M]| = 2$. In particular V_M is elementary abelian. If $V_M O_2(M)/O_2(M)$ is not in the transvection group, we get $|Y_P \cap O_2(M)| = 8$ and $|[x, V_M]| = 4$. If we are in (1), then $|\tilde{Y}_P| = 8$ and $|[Y_P, V_M] Y_M/Y_M| = 4$. In all cases there is some element $x \in Y_P \setminus O_2(M)$ with $|[V_M, x] Y_M/Y_M \leq 4$ and $x O_2(M) \in Z(S/O_2(M))$. In the case of V_M being abelian, we get that x has to be nontrivial on V_M and $O_2(M)/V_M$ as well, as $[V_M, O_2(M)] = Y_M$. Hence in any case we have that $[x, C_{O_2(M)}(V_M) V_M] = [x, V_M]$.

We are going to show that $[x, V_M]$ is centralized by a good E. Then we get $P \leq M$, a contradiction. For the rest of this proof we will assume that there is no such E. As $|Y_M| = 2$, we also have that $M = C_M$. Let first $\rho \in M$, $o(\rho)$ odd, $[\rho, V_M] = 1$ and $\rho^x = \rho^{-1}$. As $[C_{O_2(M)}(V_M) V_M, x] = [V_M, x]$, we get that $[\rho, C_{O_2(M)}(V_M)] = 1$ and so by the $A \times B$–lemma, qe have that $[\rho, O_2(M)] = 1$, a contradiction. So we have that $x \in O_2(\langle C_M(V_M), x\rangle)$.

Let first P be a Sylow p–subgroup of $F(M/O_2(M))$ with $[P, x] \neq 1$. Then we have $p \leq 5$. Assume $p = 5$ and $p \in \sigma(M)$. Then $|[P, x]| = 5$. If $[P, x] \leq E$, E elementary abelian of order p^3, then $m_p(C_E([V_M, x])) \geq 2$, a contradiction. Let now R be a Sylow 5–subgroup of M containing P, then we have that $C_R([P, x]) \cong [P, x] \times Z$, where Z is cyclic with $[Z, [[P, x], V_M]] = 1$. Now as $[P, x] \not\leq Z(R)$, we have that $[\Omega_1(Z), V_M] = 1$. But we have that $[[P, x], C_{O_2(M)}(V_M)V_M/V_M] = 1$ and so also $[\Omega_1(Z), C_{O_2(M)}(V_M)] = 1$, yielding $[\Omega_1(Z), O_2(M)] = 1$, a contradiction. So we have that $5 \notin \sigma(M)$. Let C be a critical subgroup of P. We have that $[P, x] \leq C$ and so $[C, x] = [P, x]$. In particular we must have that C is elementary abelian. Let $r \in \sigma(M)$. Then we have that $m_r(N_M(C)/C_M(C)) \leq 1$ by 2.3. If $m_5(C) = 3$, we have that there is some elementary abelian group of order r^3, centralizing C, and then also some good elementary abelian group of order r^2 centralizing $[V_M, x]$. So we have that $M - 5(C) = 2$. Further we may assume that there is no elementary abelian group of order r^3 centralizing C. But in any case there is some good E centralizing C. Then we may assume that $r = 3$, otherwise E would centralize $[[P, x], V_M]$. But now we get that some 3–element acts on C nontrivially and so $|[C, V_M]| = 2^8$. But there is no $(Z_5 \times Z_5)Z_3$ in $GL(8, 2)$.

So we now have that $p = 3$. Let first $3 \in \sigma(M)$. Then by 5.4 not all 3–elements can be good, so we have that $m_3(M) = 3$. Suppose that x acts nontrivially on an extraspecial group C. Then $Z(C) \leq [x, C]$. Hence C acts on a 4–space, which gives thet $[[V_M, x], Z(C)] = 1$. But then also $[V_M, Z(C)] = 1$. As $|[C, x]| > 3$ and $[C_{O_2(M)}(V_M)V_M, x] = [V_M, x]$, that $[Z(C), O_2(M)] = 1$, a contradiction. Let next $C \cong Z_3 \times 3^{1+2}$ and assume that x acts on C. Then we get that $\langle\rho\rangle = [C, x] \leq Z(C)$. If $|[\rho, V_M]| = 16$, then there is some elementary abelian subgroup of order 9 in C which centralizes $[V_M, \rho]$ and then also $[V_M, x]$ a contradiction. So we have that $|[\rho, V_M]| = 4$. As $[\rho, V_M]$ is centralized by an extraspecial group in C, we have that $[V_M, x] \not\leq [\rho, V_M]$. But some element $u \in [V_M, x] \setminus Y_M$ is centralized by some good E. This now implies that we have 14.5(5). Then $\langle u\rangle = Y_{M^g}$ for some $g \in P$. Hence we have that $\mathbb{Z}_3 \wr \mathbb{Z}_3$ is a Sylow 3–subgroup of M, a contradiction.

Let now C a critical subgroup as above. Then we have that C is elementary abelian. Suppose first that $|[C, x]| = 3$. Then we get that $x \notin \Phi(S)$, in particular we have that $Y_P \not\leq \Phi(O_2(P))$. If Y_P is irreducible, this implies $O_2(P) = Y_P$ and so $|S| \leq 2^7$, a contradiction. So we have 14.5(5) with at least two modules involved. So let $U \leq Y_P$ be a P–module, which is contained in $\Phi(O_2(P))$. Then we may assume that $[U, C] = 1$. As U is not in $O_2(M)$, we see that U has to act nontrivially on some component K of $M/O_2(M)$. If 3 divides $|K|$, then all 3–elements are good, a contradiction, so $K \cong Sz(q)$. But $|[V_M, y]| \leq 4$ for $y \in X$, contradicting 3.50. This shows that $Y_P = O_2(P)$ and so $|O_2(P)| \leq 2^{12}$ and then $|S| \leq 2^{15}$. But in any case we see that $Y_P \cap V_M$ is a characteristic elementary abelian subgroup of V_M, which gives $V_M \leq Y_P$, a contradiction.

So we have that $|[C, x]| = 9$. Then $[C, x] = \langle \rho_1, \rho_2 \rangle$ with $|[V_M, \rho_i]| = 4$, $i = 1, 2$. Hence $m_3(C_M(\rho_i)) = 2$, $i = 1, 2$. In particular $C = [C.x]$. As all other elements in C have a commutator of order 16 with V_M, we get that $C = \Omega_1(C_R(C))$, C a Sylow 3–subgroup of M. But $m_3(R) = 3$, a contradiction.

Hence we have that $3 \notin \sigma(M)$. Let C be a critical subgroup of P. By 2.3 there is a good E centralizing C. Choose $\rho \in C$ with $\rho^x = \rho^{-1}$. Then we have no good E centralizing $[\rho, V_M]$. This shows that $|[\rho, V_M]| = 16$ and $p = 5 \in \sigma(M)$. But now we must have a 5–element acting nontrivially on C, which shows with 2.2 that C is extraspecial of order 3^5. But then $m_3(M) = 3$ and so $m_5(M) \geq 4$, which gives an elementary abelian group of order 5^3, which centralizes C and then a good E centralizing $[V_M, x]$, a contradiction. So we have shown

$(*)$ $[Y_P, F(M/O_2(M))] = 1$.

Let now K be a component with $[K, x] \neq 1$. As $xO_2(M) \in Z(S/O_2(M))$, we get $K^x = K$. As $|[V_M, x]| \leq 4$, we get with 3.33 that $K \cong L_n(2)$, $Sp(2n, 2)$, $\Omega^{\pm}(2n, 2)$, A_n, $SU(n, 2)$, $G_2(2)'$, $SL_n(4)$, $Sp(2n, 4)$, $3A_6$, or $3U_4(3)$. In any case we have that 3 divides $|K|$. Further we know that not all 3–elements can be good if $3 \in \sigma(M)$. Let first $3 \in \sigma(M)$. Then we have that 3 does not divide $|C_M(K)|$ and so $m_3(K) = 3$. This shows $K \cong L_6(2)$, $L_7(2)$, $Sp_6(2)$, $\Omega^-(8, 2)$, $U_4(2)$, A_9, A_{10}, A_{11}, $SL_4(4)$ or $Sp_6(4)$. But by 1.17 in that group all 3–elements are good. So we have that $3 \notin \sigma(M)$. Now we have that there is no good E in $C_M(K)$, as K can induce at most two nontrivial irreducible modules, which then have to be centralized by E. So for $p \in \sigma(M)$ we have that $m_p(K) \geq \max(2, \mathrm{m_3(K)})$. But we easily check that none of the groups above satisfies this condition. □

According to 14.33 we now assume for the remainder of this chapter that we have 14.5(1) or (2) both with $q > 2$.

goop

Lemma 14.34 *Let $p \in \sigma(M)$ and assume that all p–elements are good, then p does not divided $q^2 - 1$.*

Proof: Suppose false. If we are in 14.5(1), then there is some p–element $\omega \in P \cap M$. But $P = \langle M \cap P, N_P(\langle \omega \rangle) \rangle \leq M$, a contradiction. So we have (2). Then P contains an elementary abelian p–subgroup R of order p^2. We have that $P = \langle N_P(\langle \omega \rangle) \mid \omega \in R^\sharp \rangle$. Hence $P \leq M^g$ for some $g \in G$. As $S \leq M \cap M^g$, we get $M = M^g$ with 9.1, a contradiction. □

centgood

Lemma 14.35 *There is no $Y_M \neq xY_M \in [Y_P, V_M]Y_M/Y_M$, which is centralized by a good E in M.*

Proof: Suppose false. Then we may assume that x is centralized by a good E in M. As $C_P(x) \leq M \cap P$, we see that P is as in 14.5(2). Further we have that x is conjugate to some element in Y_M. Hence $C_G(x) \leq M^g$ for some $g \in P$. This shows that M and M^g share a good E, which gives $M = M^g$ and so $x \in Y_M$, a contradiction. □

b2

Lemma 14.36 *We have $b = 2$.*

Proof: We have that $|Y_P/Y_P \cap O_2(M)| = q$ and there is some group of order $q-1$ in $P \cap M$ acting on this group. In fact this group is not in C_M. Next we see that $|[O_2(M)/Y_M, Y_P]| = q^2$, $[O_2(M)/Y_M, x] = [O_2(M)/Y_M, Y_P]$ for all $x \in Y_P \setminus O_2(M)$ and finally $C_{O_2(M)/Y_M}(x) = C_{O_2(M)/Y_M}(Y_P)$ for all $x \in Y_P \setminus O_2(M)$. Finally $[Y_P, O_2(M)] \leq V_M$.

Let $\rho \in C_M(V_M/Y_M)$ with $\rho^x = \rho^{-1}$ for some $x \in Y_P$. Then we have that $[\rho, O_2(M)] \leq V_M$ and so $[\rho, O_2(M)] = 1$. Hence we have that $Y_P \leq O_2(\langle C_M(V_M/Y_M), Y_P \rangle)$. Now we see with 5.3 that Y_P centralizes $F(M/O_2(M))$.

Hence there is some component K with $[K, Y_PO_2(M)/O_2(M)] \neq 1$. Assume that M is not exceptional with respect to some p. Let first $[K, Y_PO_2(M)/O_2(M)] \not\leq K$. Then because of the strong action, we get a contradiction with 3.24. So we have that K is normalized by Y_P. Further

(i) Y_P acts faithfully on K.

Let us first assume that K is not a group of Lie type in characteristic two. As Y_P induces a quadratic group of order at least 4 we get $K \cong A_n$, $3U_4(3)$ or some sporadic group by 3.30, 3.31 and 3.32. In any case, as otherwise all 3-elements are good and P either contains some 3-element from M or an elementary abelian 3-subgroup of order 9, we have $3 \notin \sigma(M)$. Hence $K \cong A_n$, $n \leq 11$, M_n or J_2. Further $[K, V_M]$ is not centralized by a good E. This with 3.43 now shows that we have A_n or J_2, where in the cases of J_2 and A_{10}, A_{11} we have $5 \in \sigma(M)$. But then in case of the alternating groups we have $e(G) > 3$ and again $[V_M, K]$ is centralized by a good E. So we have J_2 . Now Y_P induces a foursgroup. But then $K = \langle C_K(i) \mid 1 \neq i \in Y_PO_2(M)/O_2(M) \rangle$ acts on $[Y_P, V_M]$ a contradiction. So we are left with A_n, $n \leq 7$. As $q > 2$, then $[V_M, K]$ always is irreducible, so it is centralized by a good E.

So we have that $K = G(r)$ is a group of Lie type in characteristic two. Let U be the projection of $Y_PO_2(M)/O_2(M)$ onto K.

(ii) ρ normalizes K.

Suppose false. Then we have at least three conjugates K_1, K_2, K_3 under $\langle\rho\rangle$. Suppose first that $m_3(K) \geq 2$. Then we get that $3 \in \sigma(M)$ and all 3–elements are good. But this contradicts 14.34. So we have that $m_3(K) \leq 1$. Further $K/Z(K) \cong Sz(r)$, $L_2(r)$, $U_3(r)$ or $L_3(r)$ by 1.1. Suppose that U is not contained in a root group of K, then we have that $K \cong L_3(r)$. Further we now have a strong quadratic module for K, i.e. V_M just involves natural modules. As $r^2 \geq q$, we get at most 4 of them. Suppose that U is in a root group. Then by (i) we have that $q \leq r$. Now as $|[V_M, Y_P]| = q^2$, we get that $q = r$ for $K \cong Sz(r)$ or $U_3(r)$ by 3.50 and $r \leq q^2$ in the remaining cases. Hence in the first two cases there is just one nontrivial irreducible module in V_M, while in the last two cases there wight be two of them. In particular in all cases we may assume that $[K_3, [V_M, K_1]] = 1$. Suppose that $[[V_M, K_1], K_2] \neq 1$, then we have that $[[V_M, K_2], K_3] = 1$ and so $[K_1 \times K_2, [V_M, K_3]] = 1$. So we may assume that in all cases $[[V_M, K_1], K_2 \times K_3] = 1$. By 14.35 we have that $K_2 \times K_3$ contains no good E. This in the first place shows that we have exactly three conjugates under $\langle\rho\rangle$. Further all Sylow p–subgroups, p odd, of K are cyclic, which shows that $K \cong L_2(r)$, $Sz(q)$ or $L_3(2)$. Further we must have $e(G) > 3$. Let now E be some elementary abelian p–group of order p^4 in M. Then we have that $|C_E(K)| \geq p^3$. As p does not divide the order of K, we see that $C_E(K)$ also centralizes $[V_M, K]$, and so we get a contradiction with 14.35. This proves (ii).

(iii) U is contained in a root group R of K.

Let R be some root group in $Z(S \cap K)$ with $R \cap U \neq 1$. Then $U = U^{\langle\rho\rangle} \leq R$. So we may assume that $U \cap R = 1$. In particular $K \cong Sp(2n, r)$ or $F_4(r)$. In both cases we have that $U \leq Z(S \cap K)$ and so $q = |U| \leq r$. Hence U contains some element $x \neq 1$, which is contained in some $\Omega^-(4, r)$ and hence inverts some element of order $r^2 + 1$. In particular $|[V_M, x]| \geq r^2$. This shows $q = r$. Let now p be a Zsigmondy prime dividing $q - 1$ or $p = 7$ in case of $q = 64$. Let $\omega \in M \cap P$, $o(\omega) = p$ and $C_{Y_M}(\omega) = 1$. By 1.15 we get that ω induces an inner automorphism on K. This shows that p divides $|C_M(K)|$. If $p \in \sigma(M)$, then all p–elements are good, a contradiction to 14.34. So $p \notin \sigma(M)$, in particular $m_p(K) \leq 2$, which gives $K \cong Sp(4, q)$. Further we have $e(G) \geq 4$.

Suppose first that $Y_P \not\leq \Phi O_2(P)$. Then by 3.36 we have that $O_2(P) = Y_P$ and so $|O_2(M)/Y_M| = q^4$ and then this is the natural module for K. If

$Y_P \le \Phi(O_2(P))$, then we get that $O_2(P)O_2(M)/O_2(M) \cap K$ is not abelian and so intersect with a root element nontrivially. But as $[Y_P, V_M, O_2(P)] = 1$, we get a quadratic fours group which intersects a root group in a group of order 2. By 3.25 we get that there are just natural modules in V_M. As $|[V_M, Y_P]| = q^2$, we again get just one.

Now in any case we have shown that V_M/Y_M involves exactly one nontrivial irreducible module, the natural one. Suppose there is some good E, for some $s \in \sigma(M)$, centralizing K. As s cannot divide $q-1$, we get that $[E, [V_M, K]] = 1$. But then also $[V_M, Y_P]Y_M/Y_M$ is centralized by E, contradicting 14.35. As $e(G) \ge 4$, we get that $m_s(K) = 2$ and so s divides $q+1$. Now in K any element in $[V_M, K]$ is centralized by a good s–element. As there is some good s–element centralizing K and s does not divide $q-1$, we get that in $[V_M, K]Y_M/Y_M$ any element is centralized by a good E contradicting 14.35. This proves (iii).

Let K be not of rank one or $L_n(r)$. Let P_R be the parabolic corresponding to R. Assume that $[P_R, [V_M, Y_P]] = 1$. Then we have the corresponding $V(\lambda)$ in $[V_M, K]$. As by 14.35 $[V_M, Y_P]$ cannot be centralized by a good E and $r > 2$ by (i) and (iii), we now get with 3.29 $K \cong Sp(6, r)$, $Sp_4(r)$, $U_4(r)$. We have that $M \cap P$ acts on $[V_M, Y_P]$ and also on $[V_M, K]$. This shows that for any irreducible module V in $[V_M, K]$, we get that $|[V, Y_P]| = q$ or q^2. Hence $r = q$ or q^2

Assume now that P_R acts nontrivially on $[V_M, Y_P]$. Then as $q \le r$, we get that $q = r$ and we have that $SL_2(r)$ is induced. This gives $K \cong \Omega^\pm(2n, q)$, $Sp(2n, q)$ or $G_2(q)$. Hence we have

(iv) $K \cong L_n(r)$, $Sp(6, r)$, $Sp(4, r)$, $U_4(r)$, $U_3(r)$ or $Sz(r)$, or $[P_R, [V_M, Y_P]] \ne 1$ and $K \cong \Omega^\pm(2n, q)$, $Sp(2n, q)$ or $G_2(q)$.

(v) Suppose that $(q-1)^2$ divides the order of K and $K \not\cong L_n(r)$, then $e(G) \ge 4$.

We choose a Zsigmondy prime dividing $q-1$ or 7 in case of $q = 64$. As there is some group of order $q-1$ acting transitively on Y_M, we get with 1.15 that p divides $|C_M(K)|$, or $p = 3$ and $K \cong \Omega^+(8, q)$. Suppose the former. Hence $p \notin \sigma(M)$ by 14.34. As $m_p(KC_M(K)) = 3$, we get that $e(G) \ge 4$. In case of $\Omega^+(8, q)$, we get that $3 \in \sigma(M)$ and all 3–elements are good, which contradicts 14.34.

Let $K \cong U_4(r)$, then we have the natural module V and so $|[V, Y_P]| = r^2$ which gives $r = q$. By (v) we have $e(G) \ge 4$ and by 14.34 there is no good prime which divides $q-1$. Now by 14.35 there is no good E which centralize

K. So we get $m_p(K) \geq 2$ for $p \in \sigma(M)$ and so p has to divide q^2-1. Now all p–elements are good. But this contradicts 14.34.

Let $K \cong Sp(6,r)$, then either the natural module or the exterior square is involved. In both cases we see that $m_p(K) = 1$ for any $p \in \sigma(M)$ and $e(G) \geq 4$. and so K is centralized by some good E. As p cannot divide $r-1$, we get that a good E centralizes $[V_M, K]Y_M/Y_M$ contradicting 14.35.

So let next $K \cong Sp(4,r)$. Then in $[V_M, K]$ just natural modules are involved. So we get $r = q$ or $r = q^2$. If $r = q$, we get with (v) that a $e(G) \geq 4$ and with 14.34 there is no $p \in \sigma(M)$ with p divides $q-1$. Hence $m_p(C_M(K)) \neq 0$ for $p \in sigma(M)$. In particular all p–elements are good. This again shows that p does not divide q^2-1 and so $m_p(K) = 1$. Hence there is a good E, which centralizes K and also $[V_M, K]$, contradicting 14.35. So we have that $r = q^2$. Then just one nontrivial irreducible K–module is in V_M. Now again $e(G) \geq 4$. Let $p \in \sigma(M)$. Then p does not divide $r-1$. In particular any p–element in $C_M(K)$ has to centralize $[V_M, K]$ and so there is no good E centralizing K. Hence we must have some p–element ω inducing a field automorphism on K. Now as K is normal in $M/O_2(M)$, we see that p divides $|N_M(S)|$. As all p also divides $|C_M(K)|$, we see that all p–lements are good, which now contradicts 14.2.

Assume now that $[P_R, [V_M, Y_P]] \neq 1$. By (v) we get $e(G) \geq 4$ and no $p \in \sigma(M)$ divides $q-1$. Further $(q-1)^3$ does not divide $|K|$, which gives $K \cong \Omega^{\pm}(6,q)$, $Sp(4,q)$ or $G_2(q)$. We see that the modules are strong quadratic and so there is just one, which is the natural one and so defined over $GF(r)$. Hence by 14.35 no good E centralizes K, which gives that $m_p(Aut_M(K)) = 3$. Hence $m_p(K) \geq 2$ for $p \in \sigma(M)$. This shows that p divides q^2-1 and all p–elements are good, contradicting 14.34.

So let now $K \cong L_n(r)$. Let just natural modules be involved. Then some element in $[V_M, Y_P]Y_M/Y_M$ is centralized by $SL_{n-1}(r)$. Hence the $SL_{n-1}(r)$ cannot contain a good E by 14.35. So we have that $n \leq 4$. If $n = 4$ no $p \in \sigma(M)$ divides $r-1$. In particular $e(G) > 3$. Now $m_p(C_M(K)) \neq 1$. So all p–elements are good. If $r = q$, we have that p does not divide q^2-1 by 14.34 and so $m_p(K) = 1$. But then there is a good E which centralizes K and $[V_M, K]Y_M/Y_M$ as well, contradicting 14.35. So we have $r = q^2$ and just one natural module is involved. Hence any p–element centralizing K will centralize $[V_M, K]Y_M/Y_M$. So we get with 14.35 that there is a p–element which has to induce some field automorphism on K, contradicting 14.2. So we have $n \leq 3$.

Let $K \cong SL_3(r)$. Let first $m_p(K) \leq 1$. Now we have that p divides the order of $C_M(K)$ and so all p–elements are good. If we have two modules involved, we get $q = r$ and so by 14.34 p cannot divided $r^2 - 1$. This shows that any p–element, which centralizes K must centralize $[V_M, K]Y_M/Y_M$. The same is true if there is just one natural module involved, as p does not divided $r - 1$. By 14.35 there is no good E centralizing K. In particular we have some p–element, which induces a field automorphism. and so again there is some $xY_M \in [V_M, Y_P]Y_M/Y_M$, which is centralized by a good E, contradicting 14.35.

Let now $p \in \sigma(M)$ such that p divides $r - 1$. Suppose there is some subgroup $K \times \langle\omega\rangle$, $o(\omega) = p$. Then we have an elementary abelian p – group of order p^3 which acts on $[V_M, Y_P]Y_M/Y_M$ which contradicts 14.35. So there is some p–element which induces an outer automorphism on K. If this is a field automorphism, then we get again some good E which centralizes some $xY_M \in [V_M, Y_P]Y_M/Y_M$. So we have that $p = 3$. By 14.34 we have that $e(G) = 3$ and not all 3–elements are good. Suppose that $[V, K]$ is not irreducible. Then we get that $r = q$. Hence 3 divides $|P \cap M|$. So we have the assertion of 14.4 that either 3 divides $|P \cap M|$ or $[V_M, K]$ is irreducible. Now 14.4 provides us with a contradiction.

Let $n = 2$. Then $[V_M, K]$ involves at most two natural modules. Hence by 14.35 there is $p \in \sigma(M)$, which divides $|K|$. We also get that p divides $|C_M(K)|$. If $p \neq 3$ or K has at most two conjugates in M, we get that all p–elements are good. By 14.34 we have that p does not divide $q^2 - 1$. In particular we have just one natural module V involved. But now an elementary abelian subgroup of order p^3 acts on $[V, Y_P]$. If there is a natural submodule we get that this group acts on $[V_M, Y_P]$. If the extension is nonsplit we get the same conclusion with 3.52 as $|U| > 2$. But then we get a contradiction to 14.35.

So we are left with $p = 3$ and we have exactly three conjugates of K, K_1,K_2,K_3. If $[[V_M, K_1], K_2 \times K_3] = 1$, a good E centralizes some element $xY_M \in [V_M, Y_P]Y_M/Y_M$, contradicting 14.35. Hence $[V_M, K_1] = [V_M, K_2]$. But then $[K_3, [V_M, K_1 \times K_2]] = 1$, a contradiction.

So assume now that $V(\lambda_2)$ is in $[V_M, K]$. Then $n \geq 4$ and as in the case of $n = 4$ this is the orthogonal module, a case we handled before, we get $n \geq 5$. Now there is $p \in \sigma(M)$ which divides $r - 1$. But then there is a good E centralizing $C_{[V_M,K]}(S \cap K)$, a contradiction. The same argument applies for $V(\lambda_3)$ and $K \cong L_6(r)$. So by 3.29 we are left with the case that the tensor-product module is involved. Now for $x \in R^\sharp$ we have that $|[V_M, x]| \geq r^{n-1}$. As root groups do not act quadratically, we also see that $q < r$ and so $|[V_M, Y_P]| < r^2$, which now shows $n = 2$. Further now $r = q^2$. Suppose that there is a good E centralizing K. By 14.35 we have that $[E, [V_M, K]] \neq 1$, so p divides $q - 1$. As p divides $C_M(K)$ all p–elements are good. But this contradicts 14.34. Hence there is no such E and then $e(G) = 3$. Further there is some field automorphism of K of order p. By 14.2 there are no good p–elements in $N_G(S)$ so there must be a conjugate of K under S. But $\rho \notin K^S$, so there are good p–elements in $M \cap P$, a contradiction to 14.34.

Assume now that $K \cong Sz(r)$. Then $q = r$. As $O_2(P)$ centralizes $[Y_M, Y_P]$, we see that $|S \cap K : O_2(P) \cap K| = q$. But in P we see that $S/O_2(P)O_2(M)$ does not contain an elementary abelian subgroup of order 8.

So we are left with $K \cong SU_3(q)$. Again $r = q$. Let K not be normal. Then some conjugate of K centralizes $[V_M, K]$, as V_M involves just one nontrivial irreducible module by 3.50. But then some good E centralizes $[V_M, K]$, contradicting 14.35. So K is normal in $M/O_2(M)$. Suppose next that some good p–group E centralizes K. Again by 14.35 we must have that p divides $q^2 - 1$. But as p divides the order of of $C_M(K)$, and K is normal in $M/O_2(M)$, all p–elements are good, which contradicts 14.34. Hence there is no such good E. Assume first that $m_p(K) = 1$ for $p \in \sigma(M)$. Then p also divides $C_M(K)$ and so all p–elements are good. Now we must have an outer p–automorphism on K. By 14.34 $p \neq 3$, so it is a field automorphism. As K is normal in $M/O_2(M)$, we get that $N_G(S)$ contains a good p–element, contradicting 14.2. So we have that $m_p(K) = 2$ and then p divides $q + 1$. By 14.34 we have that p cannot divide $|C_M(K)|$. If there is a field automorphism of order p, we argue as before. So we have $p = 3$ and a diagonal automorphism of order three is induced. By 3.29 we know that V_M/Y_M is the natural K–module. Now we get a contradiction with 14.4.

If P involves $\Omega^+(4, q)$, we get some subgroup of order $(q-1)^2$ in $P \cap M$, which has to centralize $Z(K)$. But as $Z(K)$ acts fixed point freely on $[V_M, Y_P]/Y_M$ and 3 does not divide $q - 1$, this is not possible.

So we have that $E(P/C_P \cong L_2(q^2)$. So we have a group Z in $P \cap M$, which acts transitively on $[V_M, Y_P]Y_M/Y_M$. This shows that either $Z(K) \leq P$ or $Z(K)Z$ contains a 3–element τ centralizing $(Y_P \cap O_2(M))/Y_M$. Hence we have that τ induces an automorphism on K which centralizes $Y_P O_2(M)/O_2(M)$. Then τ also centralizes $(V_M/Y_M)/C_{V_M/Y_M}(Y_P)$. As $V_M' = Y_M$, we get that $[Y_M, \tau] = 1$. Hence we have that τ centralizes $Y_P \cap O_2(M)$. Let T be the maximal subgroup of S normalized by τ. Then we have that $|S : T| = 2$. as τ normalizes $S \cap K$. As no outer 2–automorphism of K can centralize $[V_M, Y_P]$, we get that $O_2(P)O_2(M)/O_2(M) \leq KC_M(K)$. Hence we get that τ normalizes $O_2(P) = C_T(Y_P)$. Then τ normalizes $\Omega_1(Z(O_2(P))) = Y_P$ and so centralizes Y_P. Set now $L = C_G(Y_P)P$. As $Z(K)Z = \langle \tau \rangle Z$, we get that in both cases $Z(K) \leq L$. As $L_2(q)$ is generated by the Sylow 2–normalizer and the normalizer of an element of order 3 in this Sylow 2–normalizer, and $N_G(Z(K)) \leq M$, we get that 3 divides the order of $C_G(Y_P)$.

As $Z(K) \not\leq C_G(Y_P)$, we get that a Sylow 3–subgroup W of L is not cyclic. Further L is generated by $M \cap L$ and $N_L(W)$. This gives that $N_G(L) \not\leq M$. Hence M has a Sylow 3-subgroup isomorphic to $Z_3 \wr Z_3$. Now all 3–elements in K are in a subgroup $K_1 \cong SU_3(2)$ and so all non central 3–elements of K are conjugate in K. But then all 3–elements in M are good, which contradicts 14.34.

So we now have that M is exceptional with respect to p. By 3.41 we have that $K \cong L_2(r)$, $r = 2^{2m}$ and $|Y| = 9$, or $K \cong L_3(r)$, $r = 2^{2m+1}$ and $|Y| = 3^2$. Let first $K \cong L_2(r)$. Again by 3.41 we have two natural modules in V_M. Hence $|[V_M, Y_P]Y_M/Y_M| = r^2$ and so $r = q$. But now in P there are 3–elements, which does not centralize Y_M, as 3 divides $q - 1$. But in M both, Y and K centralize Y_M and so a Sylow 3–subgroup centralizes Y_M.

So we have $K \cong SL_3(r)$ and we have a direct sum of four natural modules. In particular $|[V_M, Y_P]Y_M/Y_M| \geq r^4$. We have that $q = |Y_P| \leq r^2$. As $|[V_M, Y_P]Y_M/Y_M| = q^2$, we get that $q = r^2$. Now again 3 divides $q - 1$ and we get a 3–element in P which acts nontrivially on Y_M, the same contradiction as before. $\square$

15 The amalgam (M, P), $b = 2$

In this chapter we will assume that $b = 2$. By 14.11 we have that P induces $L_2(q)$ on the natural module Y_P. So $Y_P \leq O_2(M)$. Set $V_M = \langle Y_P^M \rangle$ as before. Let R be a Sylow p–subgroup of $C_M(V_M)$. Then $R \leq C_G(Y_P)$. Let P_1 be a Sylow p–subgroup of $C_G(Y_P)$ with $R \leq P_1$. As $N_G(Y_P) = C_G(Y_P)N_{N_G(Y_P)}(P_1)$ and $P \leq N_G(Y_P)$, we have $N_G(P_1) \not\leq M$. So assume $m_p(P_1) \geq 2$, then by 5.1 we have that $p = 3$, P_1 is elementary abelian of order 9 and a Sylow 3–subgroup of G is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$. As 3 divides the order of $L_2(q)$, we now get that $N_G(P)$ contains a subgroup of order 27 from M. But then it contains also a good E and so $P \leq M$, a contradiction. So we have $m_p(P_1) = 1$. Now $\Omega_1(P_1) = \Omega_1(R)$ and so $N_G(P_1) \leq M$. So we have shown

CVM

Lemma 15.1 *$C_M(V_M)$ is a p'–group for any $p \in \sigma(M)$.*

The following important lemma will be used without saying all over the places in this chapter.

Pstruk

Lemma 15.2 *Set $\hat{P} = \langle V_M, V_M^g \rangle$. Then $\hat{P}O_2(P)$ contains a Sylow 2–subgroup of $C_G(Y_M)$. Further $P = \hat{P}S$.*

Proof: Set $R/O_2(P) = O_{2'}(P/O_2(P))$. Then R acts on Y_M and so $R \leq M$. Hence we have that $[R, V_M] \leq R \cap V_M \leq O_2(P)$. Set $P_1 = \langle V_M^P \rangle$. Then we have that $[R, P_1] \leq O_2(P)$. By minimality of P we have that $P = P_1RS$. Hence again the minimality of P and the fact that $L_2(q)$ has no odd Schur extensions gives that $R = O_2(P)$. Hence $P_1 = \hat{P}$ as there are exactly $q + 1$ conjugates of V_M in P since there are exactly $q + 1$ conjugates of Y_M in Y_P. So we have that $P = \hat{P}S$ and $O_2(P)\hat{P}$ contains a Sylow 2–subgroup of $C_P(Y_M)$. □

As $[Y_P, V_M] = Y_M$, we see that $V_M' = Y_M$. Set $Y = (V_M \cap O_2(\langle V_M, V_M^g \rangle))(V_M^g \cap O_2(\langle V_M, V_M^g \rangle)) = (V_M \cap O_2(P))(V_M^g \cap O_2(P))$. Hence $Y \trianglelefteq P$. Set $U = V_M \cap V_M^g$. If $U = Y$ then $V_M \cap O_2(P) \trianglelefteq P$ and so $[V_M \cap O_2(P), P] = Y_P$ and $[O_2(P), O^2(P)] = Y_P$. This now implies with 3.36 that $|O_2(P) : Y_P| \leq q$. Then $V_M = Y_PY_P^h$, for a certain $h \in M$. In particular there are exactly two maximal elementary abelian subgroups Y_P and Y_P^h in V_M. Now $O^2(M)$ normalizes Y_P and so $Y_P \trianglelefteq \langle M, P \rangle$, a contradiction.

So we have $U \neq Y$ and by 3.50 Y/U is a direct sum of natural modules.

exep

Lemma 15.3 *If M is exceptional with respect to p, then $q > 2$ and p divides $q - 1$.*

Proof: Suppose that M is exceptional with respect to p. As p–elements in the component of $M/O_2(M)$ are fixed point freely on $O_2(M)$, there is some p-element which acts fixed point freely on Y_M. But $|Y_M| = q$ and there is a transitive cyclic group of order $q-1$ on Y_M. Hence p divides $q-1$. In particular M is not exceptional for $q=2$. $\square$

Uabelian

Lemma 15.4 *$U' = 1$.*

Proof: We have $U' \leq V_M' \cap (V_M^g)' = Y_M \cap Y_M^g = 1$. $\square$

centU

Lemma 15.5 *If $x \in U$ with $C_G(x) \leq M$, then $x \in Y_M$.*

Proof: By way of contradiction we may assume $x \notin Y_P$. We have $[U, \hat{P}] = Y_P$. Hence $[x, \hat{P}] \leq Y_P$. We have $C_{\hat{P}}(x) \leq M \cap \hat{P}$. Hence $x^{\hat{P}}$ is divisible by $q+1$. In particular $[x, Y] = 1$. If $\langle Y_P, x\rangle$ would be an indecomposable module, we would get that x has exactly $q/2(q+1)$ conjugates. But then $C(x) \not\leq M$. Hence we have that the extension splits and this implies that $Z(\hat{P}) \cap \langle x, Y_P\rangle \neq 1$. But $\hat{P} \trianglelefteq P$ and $P = \hat{P}S$. Now $Z(P) \neq 1$, a contradiction. $\square$

From now on we fix the following notation : Set $\tilde{M} = N_M(S \cap C_M(V_M/Y_M))$. As seen above $C_M(V_M/Y_M)$ has p'–order. Hence we have that $m_p(\tilde{M}) = m_p(M)$. So replacing M by $\tilde{M}$ in what follows does not change arguments. As $M = \tilde{M}C_M(V_M/Y_M)$, we have that $V_M = \langle Y_M^{\tilde{M}}\rangle$. Further we will need the elements of order $q-1$ in P which normalize a Sylow 2-subgroup of $E(P/O_2(P))$. These are in M. But they also normalize $S \cap C_M(V_M/Y_M)$ as $C_S(Y_P/Y_M)$ also centralizes Y_M by 15.2 and so $C_S(V_M/Y_M) = C_{O_2(P)}(V_M/Y_M)$ by 15.2. Hence these elements are in $\tilde{M}$. The advantage of $\tilde{M}$ over M is that $O_2(\tilde{M})$ is a Sylow 2–subgroup of $C_{\tilde{M}}(V_M/Y_M)$.

For what follows, we now define an important subgroup X. We have that Y/U is a direct sum of natural modules. Hence the group X to be defined is one with $X \leq V_M^g$, $X \cap O_2(M) \leq U$, $|XU/U| = q$ and $XO_2(M)/O_2(M) \trianglelefteq S/O_2(M)$. Let $\nu \in M \cap P$, some element of order $q-1$. We will choose the pair X, ν such that $[XU/U, \omega] = XU/U$. Now we choose X such that $X = [X, \nu]$. Further ω is some power of ν such that the order of ω is a Zsigmondy prime or for $q = 64$ it is 9. In any case we have that $N_G(\langle\omega\rangle) \not\leq M$.

Furthermore let K be some component of $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$ or a Sylow r–subgroup of $F(\tilde{M}/C_{\tilde{M}}(V_M/Y_M))$ with $[X, K] \neq 1$.

So in what follows we denote by X any subgroup of Y/U of order q, which is invariant under $S\langle\nu\rangle$.

VMstructure

Lemma 15.6 *We have that $Z(V_M) = V_M' = \Phi(V_M) = Y_M$. Further if H is a hyperplane of Y_M, then V_M/H is extraspecial.*

Proof: As $[V_M, Y_P] = Y_M$ and $V_M = \langle Y_P^M \rangle$ we get that $Y_M = V_M' = \Phi(V_M)$. Let H be a hyperplane in Y_M. Set $\bar{V}_M = V_M/H$. Then $\bar{V}_M' = \Phi(\bar{V}_M) = \bar{Y}_M$. Let Z_M be the preimage of $Z(\bar{V}_M)$. Then for $h \in M$ we have that $[Z_M, Y_P^h] \le H$. Suppose that $t \in Z_M \setminus O_2(P^h)$. Then we have that $C_{Y_P^h}(t) = Y_M$ and so $[Y_P^h, t] = Y_M$. Hence we have shown that $[Z_M, Y_P^h] = 1$ for all $h \in M$ and so $Z_M \le Z(V_M)$, i.e $Y_M = Z(V_M)$ and $Z(\bar{V}_M) = \bar{Y}_M$. In particular V_M/H is extraspecial. □

centY

Lemma 15.7 $C_{V_M/Y_M}(Y) = Y_P/Y_M$.

Proof: Suppose false. Let first $s \in C_U(Y) \setminus Y_P$. We have that $\hat{P}$ acts on $\langle Y_P, s \rangle$. If this modules splits we may assume $s \in Z(\hat{P})$ and so $Z(P) \ne 1$, a contradiction. So we have a nonsplit extension. Then we may assume that $|s^{\hat{P}}| = 2(q+1)$. In particular $[V_M, s] \ne 1$. This gives some Y_P^h, $h \in \tilde{M}$, such that $s \in P^h \setminus C_{Y_P^h}$. Then we have that $|Y_P^h : C_{Y_P^h}(s)| = q$, in particular $|V_M : C_{V_M}(s)| = q$ and the same is true for any conjugate of V_M in P. But then all 2–elements in P, which centralize s are in $O_2(P)$, a contradiction. Thus we have that $C_U(Y) = Y_P$. Let now $s \in U \setminus Y_P$, with $[Y, s] \le Y_M$. As $[s, V_M^g] \le Y_M^g$, we get that $[\langle Y_P, s \rangle, V_M^g \cap Y] = 1$. But then the action of $\hat{P}$ on Y implies that $[Y, s] = 1$, a contradiction. So we have

$$(1) \qquad C_{U/Y_M}(Y) = Y_P/Y_M.$$

Let $U_M \le V_M$, $[U_M, Y] \le Y_M$, $U_M \cap U \le Y_M$, $[\omega, U_M] = U_M$, $|U_M/U_M \cap Y_M| = q$. We may choose X in such a way that XU_MU/U is the natural $\hat{P}$–module. Now $[Y, XU_M] \le Y_P$. As V_M normalizes $C_{V_M}(Y)$, we get for $t \in X$ that $[t, V_M] \le Y_PU_M$. In particular $[X, [t, V_M]] \le [X, Y_PU_M] \le [Y, Y_PU_M] \le Y_M$. So we have that

$$(2) \qquad X \text{ acts quadratically on } V_M/Y_M, \text{ for } t \in X, |[V_M/Y_M, t]| \le q^2.$$

$$(3) \qquad Z(Y) = Y_P.$$

By 15.6 we have that $V_M \cap O_2(P) = C_{V_M}(Y_P)$ and $Y_P = Z(V_M \cap O_2(P))$. Hence $Z(Y) \cap V_M = (Z(Y) \cap Z(V_M))Y_P$ and $Z(Y) \cap V_M^g = (Z(Y) \cap Z(V_M^g))Y_P$. As $Z(V_M) \cap Z(V_M^g) \leq Z(\hat{P}) = 1$, we see that

$$Z(Y) = (Z(Y) \cap Z(V_M)) \times (Z(Y) \cap Z(V_M^g)).$$

Now by 3.50(iii) $Z(Y)$ is a direct sum of natural modules and so $[Z(Y), V_M] = Z(Y) \cap Z(V_M)$. In particular $Z(Y) \leq O_2(\tilde{M})$, as $O_2(\tilde{M})$ contains all 2–elements which centralize V_M/Y_M. This shows $[Z(Y), V_M] = Y_M$. Hence $Z(Y) \cap Z(V_M) = Y_M$, and so $Z(Y) = Y_P$.

This now implies

$$(4) \qquad [t, V_M/Y_M] \cap Y_P/Y_M \neq 1 \text{ and } q < |[t, V_M/Y_M]| \leq q^2, \text{for } t \in X.$$

Suppose that $[t, V_M/Y_M] \cap Y_P/Y_M = 1$. Then we have with (2) that $[t, V_M/Y_M] = 1$, but this contradicts the choice of X.

$$(5) \qquad q > 2.$$

Suppose $q = 2$. Set $\langle t \rangle = X$. We first show $[K, Y] \leq K$. Recall that $[t, K] \neq 1$. As $t \in Z(S/O_2(M))$, we see that $[K, t] \leq K$. We assume that there is some $y \in Y$ with $K^y \neq K$. In particular $t \neq y$. Now $C_{K \times K^y}(y) = K_1$ acts on $\hat{V}_M = [V_M, y] \leq O_2(P)$. Furthermore $[t, \hat{V}_M] \leq Y_P/Y_M$. By (3) we have that t induces a transvection on some nontrivial irreducible K_1 - module W in V_M. Now 3.16 implies $K_1 \cong L_n(2)$, $Sp_{2n}(2)$, $\Omega_{2n}^{\pm}(2)$, or A_n, and W is the natural module. In any case a 3 - element in K_1 centralizes Y_P/Y_M. Let $3 \in \sigma(M)$. Now by 1.17 all 3–elements are good. So we get a contradiction with 5.5. So $3 \notin \sigma(M)$. But as $K_1 \cong K$, we see $m_3(K) = 1$, or Sylow 3–subgroups are extraspecial of width two. Whence $K \cong K_1 \cong L_3(2)$, A_5, $3A_6$ or $3A_7$. But K_1 just induces one nontrivial module and so a good E centralizes Y_P/Y_M, again a contradiction. So we have shown that $[K, Y] \leq K$.

Now by 3.33 and (3) we have $K/Z(K) \cong A_n$, $L_n(s)$, $Sp_{2n}(s)$, $\Omega_{2n}^{\pm}(s)$, $s \leq 4$, $U_n(2)$, $G_2(2)'$, $U_4(3)$, or K is solvable.

Suppose first that K is solvable. Let K_1 be in $\tilde{M}$ such that $K_1 C_{\tilde{M}}(V_M/Y_M) = K$ and $K_1 \cap C_{\tilde{M}}(V_M/Y_M)$ is a Sylow r–subgroup of $C_{\tilde{M}}(V_M/Y_M)$. Then we have that $\tilde{M} = N_{\tilde{M}}(K_1)C_{\tilde{M}}(V_M/Y_M)$. Set $\hat{M} = N_{\tilde{M}}(K_1)$. Then we have that $m_p(\hat{M}) = m_p(M)$ for all $p \in \sigma(M)$ by 15.1. Further we may assume that $M \cap P \leq N_M(K_1O_2(\tilde{M})/O_2(\tilde{M}))$. Let C be a critical subgroup of K_1. As $t \in O_2(M^g)$, we see that $[C_C(V_M/Y_M), t] = 1$. Assume $r = 5$. Then we get that $|[C,t]| = 5$. Further $|[V_M/Y_M, [C,t]]| = 16$. As $[C,t]$ is normal in C, we have that $[C,t]$ is centralized by a good E, if $5 \in \sigma(M)$. If $5 \notin \sigma(M)$ the same is true by 2.3. Hence some good p–element centralizes Y_P, a contradiction to 14.2.

So we may assume that $r = 3$. Let C be as before, then we get that $[C,t]$ is of order three, elementary abelian of order 9 or extraspecial of order 27. Further we have that $C = C_C(t)[C,t]$. Suppose $|[C,t]| = 3$ and $Y_P/Y_M \not\leq [V_M, [C,t]]$. Then we have that $|[V_M, [C,t]]| = 4$. Further $V_M = [V_M, [C,t]](V_M \cap O_2(P))$. We have that Y normalizes $[C,t]$ and so $Y = C_Y([C,t])X$. But then we have that $X = Y$ and then $|V_M| \leq 2^5$, which gives that some good E even centralizes V_M, contradicting 14.2. So we have in any case that $Y_P/Y_M \leq [V_M, [C,t]]$. Let first $3 \notin \sigma(M)$. Then we get with 2.3 that a good E must centralize C and so $[C,t]$. Hence a good p–element centralizes $[V_M, [C,t]]$, which contradicts 14.2.

So we have that $3 \in \sigma(M)$. In particular by 15.1 we have that $K_1 = K$ intersects $C_{\tilde{M}}(V_M/Y_M)$ trivially. If $[C,t]$ is elementary abelian, we have that $|C| \leq 9$, as C acts on $[V_M/Y_M, [C,t]]$ and this group is of order at most 16, but no good 3–element can centralize this group. By 5.11 all elements in C are good, which means that there is some elementary abelian group of order 27 centralizing $[C,t]$. This gives that $[V_M, [C,t]]$ is centralized by a good p–element, contradicting 14.2. So we are left with $[C,t]$ extraspecial of order 27. Then $|[V_M/Y_M, [C,t]]| = 64$. As C is of class two, we now have that $C = [C,t]$. So we have that $N_{\tilde{M}}(C)/C_{\tilde{M}}(C)C$ is isomorphic to a subgroup of $GL_2(3)$. We have that $Y \cap C_{\tilde{M}}(C)$ acts trivially on $[V_M/Y_M, C]$. But $[V_M, C] \not\leq O_2(P)$, so we have that $Y \cap C_{\tilde{M}}(C) = 1$. So we have that $|Y| \leq 4$ and so $V_M = [V_M, C]$. We further have that $YO_2(M)$ is normal in S, so we have that Y acts on a Sylow 3–subgroup and then t acts on a characteristic elementary abelian subgroup of order 27. Now as above we see that some good 3–element centralizes Y_P/Y_M, a contradiction.

So we have that K is not solvable. Suppose first that $3 \in \sigma(M)$. Then if 3 divides $|C_{\tilde{M}/C_{\tilde{M}}(V_M/Y_M)}(K)|$, then all 3–elements are good. In the other cases we get with 5.11 and 1.17 that all 3–elements in K are good. But there is no good 3–element which centralizes $C_{V_M/Y_M}(S \cap K)$ by 14.2. Hence we get $K \cong L_2(4)$. Then $|[V_M/Y_M, K]| = 16$ as in this case we have the natural module. Now $[V_M, K]$ is normalized by $N_{\tilde{M}/C_{\tilde{M}}(V_M/Y_M)}(K)$, and so by some elementary abelian group of order 27. As $|[[V_M, K], t]| = 4$, we have that $Y_P/Y_M \leq [V_M, K]/Y_M$ and so we get a contradiction with 14.2 again.

Hence we may assume $p > 3$ for $p \in \sigma(M)$. Now $K/Z(K) \cong A_n, n \leq 11$, $L_n(4), n \leq 4$, $L_n(2), n \leq 7$, $Sp_{2n}(s), n \leq 3, s \leq 4$, $\Omega_8^-(s), s \leq 4$, $U_4(2)$, $G_2(2)'$.

Suppose first $m_p(K) = 1$. Then there is good E centralizing K. But as $[V_M, K]$ involves at most two nontrivial irreducible modules and $p \not| 4-1$, we get $[E, [V_M, K]] = 1$, a contradiction to $Y_P \leq [V_M, K]$.

So we have $m_p(K) > 1$. This implies $K \cong A_{10}$, A_{11}, $L_4(4)$, $Sp_4(4)$, $Sp_6(4)$, $p = 5$ in this cases, or $K \cong L_6(2)$, $L_7(2)$, $p = 7$.

As $m_p(K) = 2$, then there is some $\nu \in C_{\tilde{M}/O_2(\tilde{M})}(K)$, $o(\nu) = p$. Again $[\nu, [V_M, K]] = 1$, a contradiction. This proves (4).

As $q > 2$ by (4), we have a quadratic fours group X on V_M/Y_M. This implies that $K \cong G(r)$, r even, $3 \cdot U_4(3)$, sporadic, alternating or solvable by 3.26. If K is solvable, so $[K, \omega] \leq K$. We will prove the same for K a component.

Suppose $[\omega, K] \not\leq K$. Let $K_1 \times \ldots \times K_s = K^{\langle \omega \rangle}$. Suppose $s > 3$. Then as ω centralizes a diagonal, we get with 5.3 that all Sylow subgroups for odd primes in K are cyclic, so have $K_1 \cong L_2(r)$, $Sz(r)$, r even, J_1 or $L_3(2)$. Furthermore let $\langle \nu \rangle \leq P \cap M$, $o(\nu) = q - 1$. Let $\mu \in \langle \nu \rangle$, with $[K_1, \mu] \leq K_1$, μ of prime order. Then, as also $N_G(\langle \mu \rangle)$ is not in M, we get that $o(\mu)$ is coprime to $|K_1|$. Hence μ has to induce a field automorphism on K_1, so $K \cong L_2(r)$ or $Sz(r)$. As $C_{K_1}(\mu)$ is not a 2 - group, and for every odd prime u, $u \Big| |K_1|$, we get $u \in \sigma(M)$. This again contradicts 5.3, recall that ω is in some odd frobenius group and so $C_{O_2(M)}(\omega) \neq 1$. Hence we have $K^{\langle \nu \rangle} = K_1 \times \ldots \times K_{q-1}$. Now $t \in X$ acts on $E \cong E_{p^{q-1}}$. But then some $F \cong E_{p^2}$ centralizes some $x \in Y_P \setminus Y_M$, a contradiction.

So we have $s = 3$, and then $q = 4$ or $q = 64$. Let first $q = 64$. As $[\omega^3, Y_M] \neq 1$, we get $\omega^3 \not\in K_1 \times K_2 \times K_3$. We have $[\omega^3, K_1] \leq K_1$. Further we have that $N_G(\langle \omega^3 \rangle) \not\leq M$. Suppose that 3 divides the order of K_1. If $[\omega^3, K_1] = K_1$, we get that ω^3 centralizes some 3–element in K_1. The same is true if $[K_1, \omega^3] = 1$. Hence in any case ω^3 would centralize an elementary abelian group of order 3^4, a contradiction. So we get that K_1 is a $3'$ - group. This shows $K_1 \cong Sz(r)$. As $[t, K_1] \neq 1$, we get with 3.50 and (3) $r \leq q$. Let now $\mu \in M \cap P$ with $o(\mu) = 7$. Then $[\mu, K_1] \leq K_1$, otherwise we would get that $N_G(\langle \omega^3 \rangle) \leq M$ by 5.3. As $r \leq 64$, we see that ν cannot induce an outer automorphism on K_1, so we get $\langle \mu, K_1 \rangle \cong Z_7 \times K_1$, as $[\mu, Y_M] \neq 1$ and $[K_1, Y_M] = 1$. But then $m_7(\langle \mu, K_1, K_2, K_3 \rangle) = 4$, which shows $7 \in \sigma(M)$, and $N_G(\langle \mu \rangle) \leq M$, a contradiction.

So we are left with $q = 4$. Now $m_3(K) \leq 1$ and so $K \cong L_2(r)$, $L_3(r)$, $U_3(r)$, or $Sz(r)$, r even. We have $|[V_1, t]| \leq 16$. Now 3.50 implies $r \leq 16$ for $K \cong L_2(r)$ or $L_3(r)$, $r \leq 4$ for $K \cong U_3(r)$ or $Sz(r)$. This shows $K \cong L_2(r), r \leq 16$, $L_3(2)$ or $U_3(4)$. But as $C_{\langle K_1^{\langle\omega\rangle}\rangle}(\omega) \cong K$, we get with 5.3 that $K \cong L_2(r)$, $r \leq 16$, or $L_3(2)$. As $[\omega, S]O_2(M)/O_2(M) = YO_2(M)/O_2(M)$ is elementary abelian, $K \cong L_3(2)$ is not possible.

As $[K_1, t] \neq 1$ and $t \in Z(S/C_S(V_M/Y_M))$, we see that there are at most two irreducible K_1–modules in $[V_M, K_1]$. Hence we may assume that $[V_M, K_1, K_3] = 1$. The action of ω then also shows $[V_M, K_1, K_2] = 1$ too. We have $[K_i, t] \leq K_i$, $i = 1, 2, 3$. We now have that $|[[V_M/Y_M, K_1], t]| \geq 4$. As $|[V_M/Y_M, t]| \leq 16$ by (3), we may assume that $[[V_M, K_3], t] = 1$. In particular $[t, K_3] = 1$, and $[V_M, t, K_3] = 1$. As $Y_P/Y_M \cap [V_M, t] \neq 1$, we get $K_3 \leq M^g$ and so we have $[X, K_3] \leq O_2(M^g)$, which shows $[K_3, X] = 1$. But then also $[X, K_1] = 1$, a contradiction.

So we have $[K, \omega] \leq K$ and then also $[Y, K] \leq K$, as $[YO_2(M)/O_2(M), \omega] = YO_2(M)/O_2(M)$. Now as $[X, K] \neq 1$ and $[X, \omega] = X$, we get $[\omega, K] \neq 1$. As $[\omega, Y_M] \neq 1$, it either induces an outer automorphism, or an inner automorphism normalizing a Sylow 2–subgroup of K. Hence we see that K is solvable or $K \cong G(r)$ or by 3.26 ω is a 3–element and $K \cong A_n$, $3U_4(3)$ or a sporadic group and all 3–elements are good. Hence we have that K is solvable or $K \cong G(r)$.

Let first $K \cong G(r)$. As $t \in Z(S/O_2(M))$, we have that t is in some root subgroup R or we have $K \cong Sp(2n, r)$ or $F_4(r)$. The action of ω now implies that even $X \leq R$ or we have one of the two exceptional cases. Hence we have $r \geq q$ or in the exceptional cases we have $r^2 \geq q$. But as we may assume that no element is in a root group, we also get $r \geq q$ in that cases.

Let $U = C_K(t)$. Let K not be of rank 1 and not be $L_3(r)$. Assume first that $[U, [V_M, t]] = 1$. Then $U \leq M^g$. Hence U contains no good E. This first shows that $K \cong L_4(r)$, $Sp(2n, r)$, $n \leq 3$, $\Omega^-(8, r)$, $U_4(r)$, $G_2(r)$, ${}^2F_4(r)$, ${}^3D_4(r)$. Then besides in the case of $Sp(2n, r)$ we have that V_M is a strong quadratic module, so we get with 3.25 that $K \not\cong G_2(r)$, ${}^2F_4(r)$, $\Omega^-(8, r)$ or ${}^3D_4(r)$.

Assume that K induces at most two nontrivial irreducible modules in V_M. Let $g \in \tilde{M}$ with $K^g \neq K$. Then we have that $[V_M, K, K^g] = 1$. We see that there are p–elements in K which are good and $[V_M, K]$ is centralized by such elements. Hence we have that $Y_P \not\leq [V_M, K]$. But then $Y_P \cap C_{V_M}(K) \neq 1$, a contradiction. So we have that K is normalized by $\tilde{M}$. Then by 5.18 no good p–element inducing a field automorphism on K or $p = 3$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M If now $m_p(K) \leq 1$, then we have that no outer automorphisms are induced by good p–elements. So K is centralized by a good E. As Y_P/Y_M is not centralized by a good p–element, we see that K has to induce two nontrivial modules. So we have

$(*)$ If $m_p(K) \leq 1$, then K induces at least two nontrivial irreducible modules in V_M.

Let first $K \cong U_4(r)$. Then $[V_M, K]$ involves just the natural module. In particular we get $|[V_M, t]| = q^2 = r^2$. If $m_p(K) \geq 2$ then by 5.11 and 1.17 all p–elements in K are good. But $[V_M, t]$ is centralized by some $L_2(q)$ and so by some p–element. Hence $m_p(K) \leq 1$, contradicting $(*)$.

Let $K \cong Sp(6, r)$. Then we see that for $p \in \sigma(M)$ we have that p does not divide $r^2 - 1$ as $L_2(r)$ centralizes $[V_M, t]$. In particular $m_p(K) \leq 1$ and $e(G) \geq 4$. But as we have at most two nontrivial modules in $[V_M, K]$ we see that some good p–element centralizes $[V_M, K]$, a contradiction.

Let now $K \cong Sp(4, r)$. Suppose that t is in some root group. Then as before we see that p does not divide $r^2 - 1$, as otherwise p–elements in K are good by 5.11. Now $m_p(K) \leq 1$. Then there is a good E centralizing K and as p does not divide $r^2 - 1$, we get that E centralizing $[V_M, K]$. So we have that t is not in a root group. Then we get $r = q$. As there is no field automorphism acting fixed point freely on X, we get that ω induces an inner automorphism on K. Hence there is an abelian subgroup of order $o(\omega)^3$ containing ω. As $N_G(\langle \omega \rangle) \not\leq M$, this gives $e(G) > 3$ and so there is a good p–element ρ centralizing K. As p does not divide $r - 1$, we see that $[\rho, [V_M, K]] = 1$, since $[V_M, K]$ involves just one nontrivial irreducible module, the natural one, a contradiction.

So let finally $K \cong L_4(r)$. By 3.29 we have that just natural and dual modules are involved. If there are two of them, then we have that $r = q$ and ω is inner. So $P \cap M$ contains a good p–element, a contradiction. So we have that $[V_M, K]$ involves exactly one nontrivial irreducible module. This shows that K is not centralized by a good E, and so $m_p(K) \geq 2$ for $p \in \sigma(M)$. Hence p divides $r^2 - 1$. Now any p-element is good and $[V_M, t]$ is centralized by a good p–element, a contradiction.

So assume now that $O^2(U)$ acts nontrivially on $[V_M, t]$. As $|[V_M/Y_M, t]| \leq r^2$, this shows $r = q$ and $L_2(q)$ is induced on $[V_M/Y_M, t]$. Then $[O_2(C_K(t)), [V_M/Y_M, t]] = 1$ and so V_M is strong quadratic. We have $K \cong Sp(2n, q)$, $G_2(q)$ or $\Omega^{\pm}(2n, q)$ by 3.25.

Let first $K \cong Sp(2n, r)$ and t not in a root group. Then we see that $n \leq 3$. The case $Sp(4, r)$ was handled before. So let $n = 3$. As V_M is strong quadratic, we get that V_M involves the natural module just once. But now there are elements in $Y_P \setminus Y_M$ which are centralized by some $Sp(4,r)$. This shows that $m_p(K) = 1$ for $p \in \sigma(M)$, contradicting $(*)$.

From now on we have $t \in R$, R a root group.

Let again first $K \cong Sp(2n,r)$. Then as we have a strong quadratic module, we have that V_M involves exactly one nontrivial irreducible K–module, some $V(\lambda)$. If $n > 2$, then $Y_P/Y_M \cap C_{V_M}(S \cap K)$ is centralized by some $L_3(r)$, $Sp(4,r)$ or $L_2(r) \times L_2(r)$. As $r > 2$, we have that there are no good p–elements whose order divides $r-1$. So we have $n = 2$ and just the natural module is involved. Again any element is centralized by some $L_2(r)$ and so $m_p(K) \leq 1$, as otherwise all p–elements would be good. But this contradicts $(*)$.

Let $K \cong \Omega^{\pm}(2n,r)$. Then just the natural module is involved as in the half spin module V we have that $|[V,t] > q^2$. Now we have that some $1 \neq x \in Y_P/Y_M$ is centralized by some $\Omega^{\pm}(2n-2,r)$. Hence this group cannot contain a good p–element. As $r > 2$, we are left with $K \cong \Omega^-(8,r)$, $\Omega^-(6,r)$ or $\Omega^+(6,r)$.

Let first $K \cong \Omega^-(8,r)$. Then some $\Omega^-(6,r)$ centralizes some element in $Y_P \setminus Y_M$. So p cannot divide r^2-1. Now $m_p(K) \leq 1$ and we get a contradiction with $(*)$. Let next $K \cong \Omega^-(6,r)$. If p does not divide r^2-1 we may argue as before using $(*)$. So p has to divide r^2-1. Now elements in $Y_P \setminus Y_M$ are centralized by some $L_2(r^2)$ and so by some good p–element by 5.11 and 1.17, a contradiction. Let finally $K \cong \Omega^+(6,r)$. Now some element in $Y_P \setminus Y_M$ is centralized by $L_2(r) \times L_2(r)$. Hence we get that p does not divide r^2-1 by 5.11 and 1.17. This shows $m_p(K) \leq 1$, a contradiction to $(*)$.

So let finally $K \cong G_2(r)$, then V_M just involves the 6–dimensional module. Further some element in $Y_P \setminus Y_M$ is centralized by $L_2(r)$ and so we have that p does not divide r^2-1 for $p \in \sigma(M)$. Otherwise by 5.11 there is a good E in K. As not any p–element in K can be good, we have that there is no p–element centralizing K. Hence we must have an outer automorphism, which is of order p. By 5.18 we now get that $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M and so a Sylow 3–subgroup of K is extraspecial of order 27. But then this is also a Sylow 3–subgroup of $G_2(2)$ and in $G_2(2)$ all subgroups of order 9 are conjugate, which gives that all 3–elements are good, a contradiction. This in turn implies $m_p(K) \leq 1$, a contradiction to $(*)$.

Let now $K \cong L_3(r)$. Let K_1 be some subgroup $L_2(r)$ in K with $X \leq K_1$. Suppose first that there is exactly one nontrivial irreducible K_1–

module involved. But then for one of the parabolics P_1 in K we have $[C_{[V_M,K]}(O_2(P_1)), O^2(P_1)] \neq 1$. Now we have that $[O^2(P_1), [V_M, K]] = C_{[V_M,K]}(O_2(P_1))$. Now we get that $[C_{[V_M,K]}(O_2(P_1)), O^2(P_1)] \cong O_2(P_1)$ and then $|[V_M/Y_M, K]| \leq q^3$. So $[V_M, K]$ is the natural K–module. But then we get that no $p \in \sigma(M)$ divides $r^2 - 1$, which shows that we have $m_p(K) \leq 1$, contradicting $(*)$. If now $q < r$, then by 3.50 and $|[V_M/Y_M, X]| \leq q^2$ we get that there is just one nontrivial irreducible K_1–module in V_M, a contradiction. So we have $r = q$. Now X is a root group and then ω cannot induce a field automorphism. This shows that $K\langle\omega\rangle \cong K \times Z_u$, $u = o(\omega)$, or $o(\omega) = 3$ and ω induces a diagonal automorphism. Suppose the former. As $N_G(\langle\omega\rangle) \not\leq M$, and $o(\omega)$ divides $r - 1$, we get that $e(G) > 3$. As p does not divide $r - 1$ and p divides $|C_M(K)|$, we see that all p–elements are good. As $m_p(K) \leq 1$, we see that K is centralized by some good E and so some good p–element, a contradiction. So we have $o(\omega) = 3$ and ω induces a diagonal automorphism on K. Let $p \in \sigma(M)$, $p > 3$. Then there is a good E centralizing K and then some good p–element also centralizes $[V_M, K]$. So we have that $\sigma(M) = \{3\}$. As now not all 3–elements can be good, we have $e(G) = 3$. As K is normal in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$, we get with 5.11 that all 3–elements in K are good. Now $C_{[V_M/Y_M,K]}(S \cap K) = Y_P/Y_M$ or of order 16. In the latter we have that $K \cong SL(3, 4)$ and we have two natural modules involved. As all 3–elements in $SL(3, 4)$ are good, we have that $K\langle\omega\rangle \not\cong GL(3, 4)$, as there all 3–elements are centralized by some elementary abelian group of order 27. But then both 3–dimension modules are the same and so $[V_M, t]$ is centralized by some 3–element in K, a contradiction. So we have that $C_{[V_M/Y_M,K]}(S \cap K) = Y_P/Y_M$. If $K \cong L_3(4)$, then there are 3–elements centralizing K and so all 3–elements would be good. Hence we have $K \cong SL(3, 4)$ and then a good E normalizes Y_P/Y_M and then a good 3–element centralizes Y_P, a contradiction.

Assume now $K \cong U_3(r)$ or $Sz(r)$. Then V_M involves just the natural module and $r = q$. As there is no good E which centralizes $[V_M, K]$, we see that K is normal in M. If there is some good p–element τ centralizing K, $\tau \notin Z(K)$, then $[[V_M, K], \tau] \neq 1$, so $p = o(\tau)$ divides $r^2 - 1$. If p divides $r - 1$, we get that all p–elements are good, as a Sylow p subgroup of M has some element from K and some from $C(K)$ in its center. and so as p divides $|P \cap M|$ we have a contradiction. So we now get $K \cong U_3(r)$ and p divides $r + 1$. Now choose $1 \neq y \in C_{YO_2(M)/O_2(M)}(K)$. Then $[y, [V_M, K]] = 1$. We get some $w \in C_M(K)$, $o(w)$ odd, which is inverted by y and also centralizes $[V_M, K]$. Hence we have that $w \in M^g$, but we may choose $y \leq O_2(M^g)$, a contradiction. So we have that $C_{YO_2(M)/O_2(M)}(K) = 1$ and so $YO_2(M)/O_2(M) = X$. But then Y/U is the natural module and by (2) we get $|V_M| \leq r^5$, a contradiction.

Let now $K \cong L_2(r)$. If $[V_M, K]$ is irreducible, then as there is no good p–element in $N_G(S)$ by 14.2, we always get some good p–element centralizing $[V_M, K]$, a contradiction to 5.5 or $p = 3$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. But then we must have three conjugates of K, K_1 and K_2. As $[V_M, K]$ is irreducible they centralize all $[V_M, K]$. But now the 3–elements in K are good and $[V_M, K]$ is centralized by $K_1 \times K_2$, a contradiction. So we have two nontrivial modules in V_M. With 3.50 we get that these are natural modules, so $r = q$. Let first $p \in \sigma(M)$ such that p does not divide $r^2 - 1$. Then we see that there is a good E, which centralizes $[V_M, K]$. So we always may assume that p divides the order of $L_2(r)$, in particular M is not exceptional with respect to p. Then all p–elements are good. Now p has to divide $r + 1$ as otherwise p divides $|P \cap M|$. Further again there is no good E in $C_M(K)$ as otherwise some good p–element centralizes $[V_M, K]$. Hence some p–element induces a field automorphism and so we have a conjugate K_1 of K such that $[V_M, K \times K_1]$ is the orthogonal $\Omega^+(4, r)$–module. We further see that $e(G) = 3$. But as $\omega \notin K \times K_1$, and there is no elementary abelian subgroup of order $|o(\omega)|^3$, we get a contradiction.

So we are left with K to be solvable. But X acts quadratically and $|X| > 2$, so by 2.1 we get some dihedral group $D = D_1 \times \cdots D_s$ with X a Sylow 2–subgroup of D. We may assume that $t \in D_1$. Then by (3) we have that $Y_P/Y_M \cap [V_M/Y_M, t] \neq 1$. Hence this is centralized by $D_2 \times \cdots \times D_s$. So we may assume that $O^2(D_2) \leq M^g$ with $X \leq O_2(M^g)$, a contradiction. □

omegap

Lemma 15.8 *Let* $1 \neq \mu \in M \cap \tilde{P}$, *$o(\mu)$ odd, such that* $N_G(\langle \mu \rangle) \not\leq M$. *If* μ *centralizes an elementary abelian group of order* p^3 *in* M *for* $p \in \sigma(M)$, *then* $C_M(\mu)$ *is solvable and* $V_M \cap V_M^g = Y_P$, *i.e.* $[O_2(M)O_2(P), \mu] = V_M(V_M^g \cap M)$.

Proof: Assume false. Then we may apply 5.3. This first yields that $C_M(\mu)$ is solvable. Further we get that $C_{O_2(M)}(\mu) = 1$. As $[\mu, V_M \cap V_M^g] = Y_P$, we get $Y_P = V_M \cap V_M^g$. As $[O_2(P), \mu] \leq Y$, the rest follows.

Xquadratic

Lemma 15.9 *Let* W *be some* K*-module in* $V_M \cap O_2(P)/Y_M$, *then* X *acts quadratically on* W.

Proof: Let $x \in X$ and $y \in W$. Then we have that $[x, y] \in U$. As $[x^2, W] = 1$, we have that x commutes with $[x, y]$. Let $\nu \in P$, which acts irreducibly on $X/X \cap U$. Write $[x, y] = uv$, where $u \in Y_P$ and $[v, \nu] = 1$. Then $[x, v] = 1$ and we get that $[X, v] = 1$ and then also $[X, [x, y]] = 1$ and so we get that $[W, X, X] = 1$. □

goodaction

Lemma 15.10 *Let* $q > 2$. *Then* $[K, \omega] \leq K$.

Proof: Suppose $K_1 \times \ldots \times K_y = K^{\langle \omega \rangle}$, $y \geq 3$. Let first $y > 3$. Then for any odd prime p which divides the order of K, we have $p \in \sigma(M)$. Assume that $m_p(K) > 1$ for some odd prime. As $C_{K^{\langle\omega\rangle}}(\omega) \cong K$, we get with 5.3 that we are in case (v), as $N_G(\langle\omega\rangle) \not\leq M$. In particular we get that $K \cong A_p$. But then $m_3(K) > 1$, but $3 \neq p$, a contradiction. So we have that $m_p(K_1) = 1$ for every odd prime p and so $K_1 \cong L_2(r)$, $Sz(r)$, $L_3(2)$ or J_1. Now let $\langle\nu\rangle \leq M \cap P$, $\omega \in \langle\nu\rangle$, $o(\nu) = q-1$. Let $\mu \neq 1$, $\mu \in \langle\nu\rangle$, $[K_1, \mu] \leq K_1$, μ of prime order. Then $[K_i, \mu] \leq K_i$, $i = 1, \ldots, y$. As $N_G(\langle\mu\rangle) \not\leq M$, we may apply 15.8. As ν acts fixed point freely on Y/Y_P, we now see that $C_S(\omega) = C_S(\nu)$. Hence μ centralizes a Sylow 2–subgroup of $C_{K^{\langle\omega\rangle}}(\omega)\tilde{K}$. As μ induces an automorphism on this group, centralizing a Sylow 2–subgroup, we now see that $[\mu, \tilde{K}] = 1$. But then $[\mu, K_i] = 1$ for all i, contradicting 15.8 as $N_G(\langle\mu\rangle) \not\leq M$. So we have $K_1^{\langle\nu\rangle} = K_1 \times \ldots \times K_{q-1}$.

Suppose next $y = 3$ and $q = 64$, so $o(\omega) = 9$. As $[Y_M, \omega^3] \neq 1$, we have $\omega^3 \notin K_1 \times \ldots \times K_y$. Suppose that 3 divides the order of K. Then ω^3 is contained in an elementary abelian subgroup of order 3^4, a contradiction. Hence $3 \not\big| |K_1|$. This implies $K_1 \cong Sz(r)$. As $[K, X] \neq 1$ and $[X, \omega^3] = X$, we have that $[\omega^3, K_1] \neq 1$. Then ω^3 induces a field automorphism and so $r = r_1^3$. Hence $7 \Big| |K_1|$. Let now $\mu \in \langle\nu\rangle$, $o(\mu) = 7$. Suppose $[\mu, K_1] \leq K_1$. Then $7 \Big| |C_{K_1}(\mu)|$ and so there is $E \cong E_{7^4}$, $\mu \in E$, $E \leq M$, a contradiction to $N_G(\langle\mu\rangle) \not\leq M$. Hence $K_1^{\langle\nu\rangle} = K_1 \times \ldots \times K_{21}$. But then μ has three orbits on $K_1^{\langle\nu\rangle}$ and again is contained in an elementary abelian group of order 7^4, a contradiction. Hence in any case

(1) $K^{\langle\nu\rangle} = K_1 \times \ldots \times K_{q-1}, \langle\nu\rangle \leq M \cap P, o(\nu) = q-1, K_1 \cong L_2(r), Sz(r), L_3(2)$ or J_1.

We have $[\nu, S \cap K^{\langle\nu\rangle}] = [(S \cap \hat{P})O_2(P), \nu] = V_M YY \cap K^{\langle\nu\rangle} = Y \cap K^{\langle\nu\rangle}$. As this is abelian, we have with (1) that $K \cong L_2(r)$ or J_1.

(2) $$q = 4.$$

Suppose $q > 4$. Now chose some 2–element $a \in K_1$ and some p–element $w_1 \in K_1$ inverted by a. As $q > 4$, we have $p \in \sigma(M)$. Set $W = \langle w_1^{\langle\nu\rangle}\rangle$ and $A = \langle a^{\langle\nu\rangle}\rangle$. Then A acts on W. If $x \in A$ with $[a, \nu] = 1$, then a has to invert W. So $|C_A(\nu)| = 2$ and $B = [\nu, A]$ is of order 2^{q-2}. Further as seen before $B \leq Y$. By 2.1 we have that BW contains a direct product of $q-2$ dihedral groups with B as a Sylow 2–subgroup. Hence there is $t \in Y \cap K^{\langle\nu\rangle}$ such that $C_{\langle K^{\langle\nu\rangle}\rangle}(t) \geq F \cong E_{p^{q-3}}$. Further we may assume that $p > 3$. Now F acts on $[t, V_M]$. We have $|[t, V_M] : [t, V_M] \cap V_M^g| = q$. Furthermore by 15.7 $[t, V_M] \cap V_M^g \not\leq Y_M$.

Suppose that there is no $E \leq F$, $E \cong E_{p^2}$ with $[E, [t, V_M]] = 1$. Then there is some E with $|C_{[t,V_M]}(E)| \geq 8^{q-5}$, as $[[t, V_M], x] \geq 8$ for any p–element x in $C_(t)$ not centralizing $[t, V_M]$. As $q \geq 8$, we have $8^{q-5} > q$ and so in any case there is $E \cong E_{p^2}$, $E \leq F$ with $C_{[t,V_M]}(E) \cap V_M^g \not\leq Y_M$ by 15.7. Choose $y \in C_{V_M^g}(E) \setminus Y_M$. As $p \in \sigma(M)$, we have that $C_G(y) \leq M$ contradicting 15.5.

$$(3) \qquad \sigma(M) \cap \pi(K) \neq \emptyset$$

Otherwise there is some $F \cong E_{p^4}$ in $\tilde{M}$ centralizing $K_1 \times K_2 \times K_3$. Let $1 \neq t \in Y \cap K^{\langle \nu \rangle}$. Then we have that $|[V_M, t] : [V_M, t] \cap V_M^g| = 4$ and $[V_M, t] \cap V_M^g \not\leq Y_M$. Now there is $F_1 \leq F$, $|F_1| = p^3$ such that $C_{[V_M,t]}(F_1) \neq 1$. As before we see that $C_{[V_M,t]}(F_1) \leq Y_M$, which shows that $p = 3$. But 3 divides the order of K_1.

$$(4) \qquad C_Y(K_1 \times K_2 \times K_3) = 1.$$

Let $y \in C_Y(K^{\langle \nu \rangle})$, $y \neq 1$. Set $Y_1 = \langle y^{\langle \nu \rangle} \rangle$. Then $K^{\langle \nu \rangle}$ acts on $[Y_1, V_M]$ and $|[Y_1, V_M] : C_{[Y_1,V_M]}(t)| \leq 16$, for $t \in Y \cap K^{\langle \nu \rangle}$. Let $[[Y_1, V_M], K_1 \times K_2 \times K_3] = 1$. By 15.7 we have that $[Y_1, V_M] \cap V_M^g \not\leq Y_M$. But a good E centralizes $[V_M, Y_1]$, a contradiction as above. So $[K_1, [Y_1, V_M]] \neq 1$. As involutions in J_1 invert elements of order 11, we see that $K_1 \not\cong J_1$. Now (1) and 3.50 imply $K_1 \cong L_2(r)$, $r \leq 16$. But we may assume that $[t, K_1] \neq 1 \neq [t, K_2]$ and so t inverts an elementary abelian group of order $17^2, 7^2$ for $r = 16$, $r = 8$, respectively. As $|[Y_1, V_M] : C_{[Y_1,V_M]}(t)| \leq 16$, this is impossible. Hence $K \cong L_2(4)$. Now we see that in $[Y_1, V_M]$ there are at most two irreducible K_1–modules involved. Hence we may assume $[[V_M, Y_1], K_1, K_3] = 1$. Then also $[[V_M, Y_1], K_3, K_1] = 1$. As ω acts on $\{K_1, K_2, K_3\}$ we get $[[[V_M, Y_1], K_1], K_2 \times K_3] = 1$. As $[t, [K_1, [V_M, Y_1]]] \leq V_M \cap V_M^g$, we get a contradiction.

Now by (1), (2) and (4) $|Y : Y \cap O_2(M)| \leq r^2$, for $K \cong L_2(r)$ and 64 for $K \cong J_1$. Let $x \in V_M \setminus Y_M$. By 15.6 we have that $[x, V_M] = Y_M$. We have that $V_M^g \cap O_2(M) = V_M^g \cap V_M$ is elementary abelian and so again by 15.6 $|V_M/Y_M| \leq q^2|Y : Y \cap O_2(M)|^2 \leq 16r^4$ for $K \cong L_2(r)$ and 2^{16} for $K \cong J_1$. Let $r = 2^m$. Then $(\tilde{M}/C_{\tilde{M}}(V_M/Y_M))^{(\infty)} \lesssim O^{\pm}(4m + 4, 2)$ for $K \cong L_2(r)$, and $O^{\pm}(16, 2)$ for $K \cong J_1$. If $L_2(r) \times L_2(r) \times L_2(r)$ acts on V_M/Y_M, then $|V_M/Y_M| \geq r^6$, and so $4m + 4 \geq 6m$. This implies $m = 2$ and we have $L_2(4) \cong K_1$. So we have $|V_M/Y_M| = 2^{12}$. Further $V_M/Y_M = [V_M/Y_M, K_1] \oplus [V_M/Y_M, K_2] \oplus [V_M/Y_M, K_3]$. Now we have that Y is a Sylow 2–subgroup of $K^{\langle \nu \rangle}$. Hence there is some $1 \neq t \in Y$ with $[t, K_3] = 1$ and so $[[t, V_M], K_3] = 1$. As $C_{[t,V_M]/Y_M}(Y) \neq 1$, we get with 15.7 that $[t, V_M] \cap Y_P \not\leq Y_M$. But then K_3 is also in M^g, contradicting

$Y \leq O_2(M \cap M^g)$.

So we are left with $K_1 \cong J_1$. Then $K_1 \times K_2 \times K_3 \lesssim \Omega^{\pm}(16,2)$. But 7^3 divides the order of $K_1 \times K_2 \times K_3$ but not of $\Omega^{\pm}(16,2)$. □

KYnormal

Lemma 15.11 *We have* $[K,Y] \leq K$.

Proof: If $q > 2$, then by 15.10 we have that $[K,\omega] \leq K$, so $[K,Y] \leq K$, as $[Y,\omega] = Y$. So let $q = 2$. Then in particular we have that K is nonsolvable. By 15.3 M is not exceptional.

Let $y \in Y$ with $K^y \neq K$. As $Y \trianglelefteq S$, we see that K possesses abelian Sylow 2 - subgroups. Hence $K \cong L_2(r)$, ${}^2G_2(q)$ or J_1. Now $C_{K^y \times K}(y) \cong K$ acts on $[y,V_M]$. If $1 \neq t \in C_{K^y \times K}(y) \cap Y$, then $|[[y,V_M],t]| \leq 4$. By 15.7 $Y_P/Y_M \leq [y,V_M]$ and so $C_{K^y \times K}(y)$ acts faithfully on $[y,V_M]$. Application of 3.33 shows $K \cong L_2(4)$.

Let first $C_Y(K) \neq 1$. We have $[C_Y(K),K^y] = 1$. Let $\tilde{y} \in C_Y(K)^{\sharp}$, $W = [V_M,\tilde{y}]$. Let W_1 be some quasi irreducible KK^y–module in W, which we may assume to be centralized by $S \cap C_M(KK^y)$. We have $|W_1 : V_M^g \cap W_1| = 2$ and $|[V_M^g \cap W_1, y]| \leq 2$. Hence $|[W_1,y]| \leq 4$. Set $K_1 = C_{KK^y}(y)$. Then $[K_1,[W_1,y]] = 1$. But $Y \cap K_1 \neq 1$. This shows that $C_{Y_P/Y_M}(K_1) = 1$. In particular y induces a transvection on W_1, which is not possible. So we get that $[y,W_1] = 1$, and then $[W_1,K] = 1$, contradicting 15.7.

So assume now $C_Y(K) = 1$. Then we see $|Y : Y \cap O_2(M)| \leq 8$ and so $|V_M/Y_M| \leq 2^8$. Now $M/C_M(V_M/Y_M) \lesssim O_8^{\pm}(2)$. Furthermore $p = 3 \in \sigma(M)$. But then $3 \,\Big|\, |C_{M/O_2(M)}(K^y \times K)|$. As $5^2 \,\Big|\, |M/O_2(M)|$, we have $M/C_M(V_M/Y_M) \lesssim O_8^{+}(2)$. As 5^3 does not divide the order of $O^+(8,2)$, we have that K^yK is normal in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$. In particular all 3–elements are good. As no 3 - element centralizes Y_P we have that 27 divides $|(Y_P/Y_M)^M|$. As in $\Omega^+(8,2)$ no Sylow 5–subgroup centralizes a vector in the natural module, we see that $|(Y_P/Y_M)^M| = 135$. But as $E \cong E_{27}$ is contained in M, we get some 3 - element ρ with $|C_{V_M}(\rho)| \geq 2^5$. Now ρ centralizes some involution $i \in V_M \setminus Y_M$. As all involutions in $V_M \setminus Y_M$ are conjugate under M (there are exactly 270 such involutions), we get $i \sim j \in Y_P \setminus Y_M$, a contradiction.

So in all cases we have that $[K,Y] \leq K$. □

goodK

Lemma 15.12 *Let* $q > 2$*, then there is some* K *with* $[K,X] \neq 1$ *and* $C_Y(K) = 1$.

Proof: Choose any K and assume that always $C_Y(K) \neq 1$. By 15.10 we have $[K,\omega] \leq K$. As $[Y,\omega] = Y$, we see that $[K,Y] \leq K$. There is

$y \in C_Y(K)$, $y \neq 1$. Then we may even choose $Y_1 \leq C_Y(K)$ which is normalized by ω, $|Y_1| = q$ as ω acts fixed point freely on $Y/Y \cap O_2(M)$. Set $W = [V_M, Y_1]$. Assume $[[V_M, Y_1], K] = 1$. Then by 5.14 also $[Y_P/Y_M, K] = 1$. Hence there is some $t \in Y_P \setminus Y_M$ such that $C_M(t)$ covers K. But then K is covered by $M \cap M^g$ and $X \leq O_2(M^g)$, which contradicts $K = [K, X]$. Let now W_1 be some nontrivial quasi irreducible K–module in W, which we may assume to be centralized by $S \cap C_{\tilde{M}}(K)$. We have that $|[V_M, Y_1] : V_M \cap V_M^g \cap [V_M, Y_1]| = q$. We further have $|V_M \cap V_M^g : C_{V_M \cap V_M^g/Y_M}(X)| \leq q$ and so $|[V_M, Y_1]/Y_M : C_{[V_M,Y_1]/Y_M}(X)| \leq q^2 = |X|^2$. Now we have that X induces a 2F–module offender on W and so also on W_1. Further by 15.9 we have that X acts quadratically on W.

Let first K be some r–group. By 2.1 we have a subgroup D in KX, which is a direct product of dihedral groups D_i of order $2r$ with X as a Sylow 2–subgroup. Set $X_i = X \cap D_i$. We have that $[X_1, W_1] \leq V_M \cap V_M^g$. As $[X_1, W_1] \leq C_{V_M \cap V_M^g}(X)$ so $Y_P \cap [X_1, W_1]Y_M > Y_M$ by 15.7. Now $D_2 \times \cdots \times D_n$ centralizes some element in $Y_P \setminus Y_M$ and so is in M^g, but $X \leq O_2(M^g)$. As $q > 2$ we have $n \geq 2$, a contradiction.

As $|X| > 2$, we get with 3.31, 3.32 that K is alternating, M_i, J_2 Co_1 or Co_2 or $3U_4(3)$, or a group of Lie type in characteristic two. We have that ω acts nontrivially on K as it acts that way on X. So if K is not of Lie type in characteristic two, ω has to induce an inner automorphism. As ω is nontrivial on Y_M, it is an automorphism, which normalizes a Sylow 2-subgroup. So we get $K \cong J_2$. But then ω does not act nontrivially on a foursgroup in $Z(S \cap K)$.

So we have that K is a group of Lie type in characteristic two. Suppose that $X \cap R \neq 1$ for some root group R in $Z(S \cap K)$. Now ω induces an automorphism, which has to act nontrivially on this root subgroup R. Hence we have $q \leq r$. Further we have that $|W_1 : C_{W_1}(Y)| \leq q|Y : C_Y(K)|$ and $|C_{W_1}(Y)| = |Y_P/Y_M| = q$, so $|W_1| \leq q^2|Y : C_Y(K)|$.

Now there is some $1 \neq t \in R$, R a root subgroup of K. Hence $|[W, t]| \leq q^2 \leq r^2$. Now with 3.29 we get that $K \cong L_n(r)$, $Sp(2n, r)$, $\Omega^{\pm}(2n, r)$, $G_2(r)$, $Sz(q)$ or $U_n(q)$. Further the corresponding modules are given by 3.29.

Let U_1 be the parabolic in K with $[U_1, Y_P] = 1$. Then $U_1 \leq M \cap M^g$ and so $Y \cap K \leq O_2(U_1)$ and Y is normalized by U_1.

Suppose first that $K \cong G_2(r)$. Then W_1 involves the natural module and so $|W_1| \geq r^6$. Now $|Y : C_Y(K)| \leq r^3$ and so we get that $|W_1| \leq q^2 r^3 \leq r^5$, a contradiction.

Let $K \cong \Omega^{\pm}(2n, r)$, then the natural module is involved. We will handle $\Omega^{+}(6, r)$ as $L_4(r)$, so we may assume that $n \geq 4$ in case of $\Omega^{+}(2n, r)$. But now we have that Y_P is centralized by $\Omega^{\pm}(2n-2, r)$. If $K \not\cong \Omega^{-}(8, r)$ or $\Omega^{-}(6, r)$ there is some $p \in \sigma(M)$ which divides $r-1$. So we get some good E centralizing Y_P, a contradiction. Let now $K \cong \Omega^{-}(8, r)$ or $\Omega^{-}(6, r)$. Let $K \cong \Omega^{-}(8, r)$. Then we have $|W_1| \leq q^2 r^6$. This shows $q \geq r$. Hence we get $r = q$ and so $m_x(\langle K, \nu \rangle) = 4$ for some prime x dividing $q-1$, a contradiction.

Let next $K \cong \Omega^{-}(6, r)$, then $|W_1| \leq q^2 r^4 \leq r^6$ and so just the natural module is involved, in which case $r = q$. Let U_1 be as before. Then as $Y \cap K$ is normalized by U_1, we get that $|Y \cap K| = q^4$. We have that $[Y \cap K, V_M, Y \cap K] \leq V_M \cap V_M^g$. We also have $[Y \cap K, W_1, Y \cap K] = Y_P/Y_M$. As $\sigma(M) \cap \pi(U_1) = \emptyset$, we see that $e(G) > 3$ and $m_p(K) \leq 1$ for $p \in \sigma(M)$. This shows that K is normal in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$. So by 5.18 no p–element induces a field automorphism on K and so there is some $E \cong E_{p^3}$, $p \in \sigma(M)$ with $[K, E] = 1$. In particular there is some $F \leq E$ of order p^2 centralizing $[Y \cap K, W_1, Y \cap K] = Y_P/Y_M$, a contradiction.

Let $K \cong Sp(2n, r)$. If $n \geq 4$ then we see that there is $p \in \sigma(M)$ dividing $r-1$ and so as Y_P is centralized by $Sp(2n-2, r)$, $L_n(r)$ or $L_2(r) \times L_{n-1}(r)$, depending on the particular module, we get a good E centralizing Y_P, a contradiction.

Let $K \cong Sp(6, r)$. We have that $|W_1| \leq q^2 r^6 \leq r^8$. Hence either the natural or the spin module is involved. If we have the spin module, then we have $r = q$. But then we see that for $\nu \in P \cap M$, $o(\nu) = q-1$, we have that $m_x(K\langle \nu \rangle) = 4$ for some prime x dividing $r-1$, which is a contradiction. So we just have the natural module involved. Further $q < r$. This now gives $|Y : C_Y(K)| > r^4$. As Y is normal in the point stabilizer P_1, we see that $|Y : C_Y(K)| = r^5$ and $|Y \cap K| = r^5$. But ω acts fixed point freely on Y and so on $O_2(P_1)$, which gives that it acts fixed point freely on R and so r is a power of q. Hence we get that $r = q^2$ and $|W_1|$ is the natural module. But now again $m_x(K\langle \nu \rangle) = 4$ for some x dividing $q-1$.

Let $K \cong Sp(4, r)$, then we get $|W_1| \leq q^2 r^3$ and so again just the natural module is involved. Further $|Y : C_Y(K)| > r$. As $Y/C_Y(K)$ is normal in the point stabilizer, we get $|Y : C_Y(K)| = r^3$ and $|K \cap Y| = r^3$. Now again r is a power of q and so $r = q$ or q^2. We have that $m_x(K\langle \nu \rangle) = 3$ for some prime x dividing $q-1$. So we get that $e(G) \geq 4$. Further we have that $\sigma(M) \cap \pi(L_2(r)) = \emptyset$. Hence we get that $m_p(K) \leq 1$ for $p \in \sigma(M)$. As $N_G(S)$ does not contain a good p–element by 14.2, we have some good $F \cong E_{p^3}$ centralizing K. We have that $Y_M > [Y \cap K, V_M, Y \cap K] \leq V_M \cap V_M^g$ and so we get a good E centralizing some element in $V_M \cap V_M^g \setminus Y_M$, which contradicts 15.5.

Let next $K \cong L_n(r)$. If $n \geq 5$, then there are primes $p \in \sigma(M)$ dividing $r-1$. There is always some $L_2(r)$ centralizing Y_P, contradicting 14.2. So we have $n \leq 4$.

Let $K \cong L_4(r)$. Then we get $|W_1| \leq q^2r^4$. If we have the orthogonal module, we get $r = q$, a contradiction as then $m_x(K\langle \nu \rangle) = 4$ for some prime p which divides $q-1$. So we have the natural module and then $|Y : C_Y(K)| \leq r^3$ as $Y \cap K$ is normal in U_1. Further $r > q$ and so $|Y : C_Y(K)| = r^3$ and then $r = q^2$. But then again the same contradiction as above arises.

Let next $K \cong U_n(r)$, $n \neq 4$. Then we have the natural module. Now a subgroup isomorphic to $U_{n-2}(r)$ centralizes Y_p and so $|Y \cap K| \leq r$. This gives $|W_1| \leq rq^2 \leq r^3$, a contradiction.

So let next $K \cong Sz(r)$, then again $|Y \cap K| \leq r$ and so $|W_1| \leq rq^2 \leq r^3$, a contradiction.

Let $K \cong L_3(r)$. Then we get $|W_1| \leq q^2r^2$. So just the natural module is involved and so we see $q^2 = r$ or $q = r$ as before. Now the elements of Y induce transvections on W_1. We have that $C_{W_1}(Y) \leq Y_P/Y_M$, which is of order r, so $r = q$, $K \cong SL_3(q)$ and $|Y \cap K| = q^2$.

Let now $K \cong L_2(r)$. Then $|W_1| \leq q^2r$. This gives that just the natural or the orthogonal module is involved. Let first the orthogonal module be involved. Then $r = u^2$ and as X acts quadratically we have that $|X| \leq u$. Now $u^4 \leq |W_1| \leq q^2u \leq u^3$, a contradiction. Hence we have the natural module involved. Now again we have that $C_{W_1}(Y) \leq Y_P/Y_M$. This gives $r \leq q$ and then $r = q$ again.

So we have shown that $K \cong SL_3(q)$ or $L_2(q)$. Suppose first that K is normalized by S. Then there is some $X_1 \leq C_Y(K)$, X_1 normal in $S/S \cap C(V_M/Y_M)$, $|X_1| = q$ and ω acts on X_1. Hence X_1 plays the same role as X and so we have that there is a second component $K_1 \cong SL_3(q)$ or $L_2(q)$.

Let first $K \cong SL_3(q)$ and assume that $K_1 \cong SL_3(q)$ too. If $q \neq 4$, then $m_p(KK_1) \geq 4$ and so for some p which divides $q-1/gcd(3, q-1)$ we have a good p–element in $M \cap P$, a contradiction. So we have $K \cong K_1 \cong SL_3(4)$. Further we may assume that $m_3(KK_1) = 3$, as otherwise we may argue as before. As $o(\omega) = 3$ and ω centralizes an elementary abelian group of order 9 in KK_1, we have that $3 \notin \sigma(M)$. If $C_Y(KK_1) \neq 1$, we may repeat the argument above and get a third component $SL_3(q)$ or $L_2(q)$, contradicting $3 \notin \sigma(M)$. So we have that $|Y| \leq q^4$, which gives that $|V_M/Y_M| \leq q^{10}$. Now some $F \cong E_{p^4}$ acts on V_M faithfully, which gives that $p = 5$. But in

K we have that Y_P/Y_M is centralized by some $L_2(4)$ and so by some good 5–element, which contradicts 14.2. So we have that $K_1 \cong L_2(q)$. The same argument as before shows that $C_Y(KK_1) = 1$ and $q = 4$. So $|Y| \leq q^3$ and then $|V_M/Y_M| \leq q^8$. As before $3 \notin \sigma(M)$. As there is some $F \cong E_{p^4}$ acting faithfully on V_M, we get again $p = 5$ and a contradiction as $L_2(4)$ centralizes Y_P.

Hence we have $K \cong K_1 \cong L_2(q)$. If $C_Y(KK_1) \neq 1$, we would get a third component $K_2 \cong L_2(q)$ and then $m_x(\langle \nu, K, K_1, K_2\rangle) = 4$, a contradiction again. Thus $C_Y(KK_1) = 1$ and so $|Y| = q^2$. This now implies $|V_M/Y_M| \leq q^6$. As $m_x(\langle \nu, K, K_1\rangle) = 3$, we get $e(G) > 3$. As K and K_1 are normalized by S, there is no p–element, $p \in \sigma(M)$, which induces a field automorphism on K or K_1, see 5.18. Hence there is a good E centralizing KK_1. If $[[V_M, K], K_1] = 1$, we get a good p–element from K_1 which centralizes $[V_M, K] \cap Y_P > Y_M$, a contradiction. So we have that KK_1 induces $\Omega^+(4, q)$ on V_M. But even then there is a good p–element in $C(KK_1)$, which centralizes $[V_M, KK_1]/Y_M$. Recall that as KK_1 is normal in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$ all p-elements are good. But now we have a contradiction with 14.2.

So we are left with $K^S \neq K$. As X is normalized by S, we have that X acts nontrivially on each component in K^S and so by 15.10 ω normalizes each component. Let $K \cong SL_3(q)$. This shows that $K^S = KK_1$ and then as before we get that $q = 4$. If we have $K \cong L_2(q)$, we also have $K^S = KK_1$. In both cases we now get $e(G) \geq 4$. If there is a good E centralizing K^S, we may argue as before. Hence there must be some p–element, $p \in \sigma(M)$, which induces a field automorphism on K and is inverted by some element in S, not normalizing K. This shows $K \cong L_2(q)$, $q = r^p$. Still $|V_M/Y_M| \leq q^6$. As again all p–elements are good, we get that there is no p–element which centralizes some element in $Y_P \setminus Y_M$. In particular KK_1 has to induce $\Omega^+(4, q)$ on V_M. But there is some p–element τ centralizing KK_1. This now has to act nontrivially on $[V_M, KK_1]$, which gives that p divides $q - 1$. But then some $F \cong E_{p^4}$ act on Y_P/Y_M, a contradiction. This proves the lemma. □

From now on in case of $q > 2$ if we speak about K we always mean some K with $C_Y(K) = 1$.

Knonsolvable

Lemma 15.13 *If $q > 2$, we have that K is nonsolvable.*

Proof: We assume that K is a normal r–subgroup in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$. Let M be exceptional. As $[\omega, Y] = Y$, we see that $[Y, O_p(M/O_2(M))] = 1$. Hence by 15.7 we have that $O_p(M/O_2(M)$ acts on Y_P and there are p–elements which are in M^g, a contradiction with 5.5. So we have that M is not exceptional.

Let $t \in X^\sharp$. Then $|[V_M, t] : [V_M, t] \cap V_M^g| = q$. As $C_{V_M/Y_M}(Y) = Y_P/Y_M$

by 15.7, we see $Y_P \cap [V_M, t] \neq 1$. Let $|Y : Y \cap O_2(M)| \geq 2^5$. By 2.1 we have $m_r(K) \geq 5$ and so $r \in \sigma(M)$. Then as $[t, V_M, s] \leq V_M \cap V_M^g$ for $s \in Y$, we see that there is some $1 \neq u \in V_M \cap V_M^g \setminus Y_M$, u is centralized by $E \cong E_{r^2}$ by 2.1. But this contradicts 15.5. So we have $|Y| \leq 2^4$. Now $|V_M/Y_M| \leq 2^{16}$. Then $q \leq 16$. If $|Y| = 16$, we have $r \neq 3$, as otherwise $m_3(K) \geq 4$ and so all 3–elements are good, but 3 divides $|M \cap P|$. So let $|Y| = 16$. Then we must have that $m_r(O_{16}^{\pm}(2)) \geq 4$. This shows $r = 5$ and by 2.1 we have that YK contains $D = D_1 \times D_2 \times D_3 \times D_4$, all D_i dihedral groups of order 10. Assume $X = Y$. We have that $|[V_M/Y_M, x]| = |[V_M/Y_M, y]|$ for all $x, y \in X^\sharp$. Now there is some x inverting $O_r(D)$, so $|[V_M/Y_M, x]| = 2^8$. Let $1 \neq y \in X \cap D_1$, then $D_2 \times D_3 \times D_4$ acts on $[V_M/Y_M, y]$ and so some $O_r(D_i)$ centralizes $[V_M/Y_M, y]$ and so also some nontrivial element in Y_P/Y_M, a contradiction. Then we have $q = 4$. Then we may assume that ω centralizes one of the D_i But as $[Y, \omega] = Y$, this is not possible. Assume next $q = 8$, then $X = Y$. We now have that $|V_M/Y_M| \leq 2^{12}$. In particular we have that $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$ is a subgroup of $\Omega^+(12, 2)$. Now $Y\langle\omega\rangle$ is a group of order 56, which should normalize K. In particular we get K must be a 3–group of order 3^6. But the action of such a group is uniquely determined. and so there are exactly 6 subgroups in K whose commutator with V_M/Y_M is of order 4. In particular there is no element of order 7 in $\Omega^+(12, 2)$ acting on K. So we are left with $q = 4 = |Y|$. In this case $|V_M/Y_M| \leq 2^8$. Suppose first $r = 5$. Then we have that $K/C_K(V_M)$ is elementary abelian of order 25 and A_4 acts on this group, a contradiction to the structure of $GL_2(5)$. So we have $r = 3$. Further $L = \tilde{M}/C_{\tilde{M}}(V_M/Y_M) \cong 3^3A_4$ or 3^3S_4 and $\tilde{M}$ induces orbits of length 3,4,6 on the hyperplanes of $O_3(L)$. We consider the action of ω on Y. We had that either X is elementary abelian or X is a nonabelian group of order q^2 with a fixed point free automorphism of order $q - 1$. But now $q = 3$ and so X is abelian in particular we have that Y_PX is abelian of order 2^6.

Let first $|V_M/Y_M| = 2^6$. Let H be a hyperplane in Y_M. Then by 15.6 V_M/H is an extraspecial group of order 2^7, which now contains an abelian subgroup of order 2^5, a contradiction. So we have $|V_M/Y_M| = 2^8$. Let $C_{V_M/Y_M}(K) \neq 1$. As $Y\langle\omega\rangle$ act on this group we have that $C_{V_M/Y_M}(K) = Y_P/Y_M$. But then $M \cap M^g$ contains an elementary abelian subgroup of order 27, where $3 \in \sigma(M)$, a contradiction. So $C_{V_M/Y_M}(K) = 1$. Now there are exactly four hyperplanes in K which have nontrivial centralizer on V_M/Y_M. One of these K_1 say, is normalized by ω. As $|C_{V_M/Y_M}(K_1)| = 4$ and ω acts transitively on Y_M we see that the preimage is abelian. Now take V_M/H, H some hyperplane in Y_M. Then V_M/H is extraspecial, but here the centralizer C of K_1 cannot be abelian, as otherwise the K_1 also has to act trivially on $(V_M/H)/C_{V_M/H}(C)$. This proves the lemma. □

Knonsolv2

Lemma 15.14 *Let $q = 2$. Then there is a nonsolvable K with $[K, Y] \leq K$*

and some $1 \neq y \in Y$ *with* $[S \cap K, y] = 1$ *and* $[K, y] \neq 1$.

Proof: By 15.3 we have that M is not exceptional. Let now K be solvable. Choose $Y_1 \leq Y$ with Y_1 being maximal such that $C_{Y_1}(K) = 1$. By 2.1 we get a subgroup $D \cong D_1 \times \cdots \times D_x$, a direct product of dihedral groups, with Y_1 as a Sylow 2–subgroup. Let $\langle t_i \rangle = Y_1 \cap D_i$, $i = 1, \ldots, x$. Let first $x \geq 4$. We have that $[V_M, t_1, t_2, t_3]Y_M = Y_P$. But then D_4 centralizes Y_P and so is in M^g. But there we have $Y \leq O_2(M^g)$, a contradiction.

So we have shown that $|Y_1| \leq 8$. If $|Y_1| = 8$, then we get that D_3 has to act nontrivially on $[V_M, t_1, t_2]$, where t_3 induces transvections. Hence we get $r = 3$. Let $y \in C_Y(O_3(D))$. Then we have that $\langle t_2, t_3 \rangle$ induces transvections to Y_P on $[V_M, y, t_1]$, a contradiction, as this implies that some 3-element in $O_3(D)$ centralizes Y_P. So we have $C_Y(K) = 1$ and then $Y_1 = Y$. This now shows $|V_M/Y_M| \leq 2^8$ and so $3 \in \sigma(M)$. As 3^3 divides the order of M, we have $|V_M/Y_M| \geq 2^6$.

Let first $|V_M| = 2^7$. Then $M/C_M(V_M/Y_M) \lesssim O_6^-(2)$. As $V_M \cap V_M^g$ is elementary abelian we get $|V_M \cap V_M^g| \leq 8$. For $y \in Y \setminus O_2(M)$, we have $|[V_M, y]Y_M/Y_M| \leq 8$. But then some good 3 - element in D centralizes $[V_M, Y]$ which by 15.7 contains Y_P, a contradiction to 14.2.

Hence we have $|V_M| = 2^9$ and as $V_M \cap V_M^g$ is elementary abelian we get $M/C_M(V_M/Y_M) \lesssim O^+(8, 2)$. Let $Q = C_{O_2(\tilde{M})}(V_M)$. Then $Q \cap V_M = Y_M$, $Q \leq O_2(P)$. We have $[V_M^g \cap M, Q] \leq Q \cap V_M^g \leq Y_M$. Then $[D, Q] \leq Y_M$ and so $[O_3(D), Q] = 1$. Hence M is unique with $C_G(x) \leq M$ for every $x \in Q^\sharp$. Let $u \in Q$, $u \notin Q^g$. Then u acts on an elementary abelian group of order 3^3 in $O_3(M^g/C_{M^g}(V_{M^g}/Y_{M^g}))$. Suppose $[V_M^g \cap M, u] = Y_M$. Then we have $|[V_M^g, u]| \leq 4$. But then there is some good 3 - element ν in M^g with $[\nu, Y_M] = 1$. By 5.5 we may assume that M has a Sylow 3–subgroup isomorphic to $Z_3 \wr Z_3$. We have that $C_K(\nu) \neq 1$. In particular $K \cap M^g \neq 1$. But there is some $y \in Y$, which inverts K and so also inverts $K \cap M^g$, contradicting $Y \leq O_2(M^g)$.

So we have $[V_M^g \cap M, Q] = 1$. As $[Q, V_M \cap M^g] = 1$, we see that $V_M \cap M^g$ acts on $[V_M^g, Q]$. By 15.7 we get $[V_M^g, Q] \leq Y_M^g$. Now as $[V_M^g \cap M, Q] = 1$, we get $|Q : Q \cap Q^g| \leq 2$. As $M \neq M^g$, we have $|Q| = 2$ and so $Q = Y_M$. This implies $O_2(M) = V_M$. Now choose $x \in V_M^g \cap M$, such that x centralizes $E \cong E_9$ in $O_3(M/O_2(M))$. We have $|[x, V_M]Y_M| = 16$ and E normalizes $[x, V_M]Y_M$. Let $\rho \in E^\sharp$ with $[[x, V_M], \rho] = 1$. Then $\rho \in M^g$. But there is some $y \in Y$ with $\rho^y = \rho^{-1}$, a contradiction. So we have that E acts faithfully and then $[x, V_M]$ is elementary abelian. Then $\langle x, [x, V_M], Y_M \rangle = F$ is abelian of order 64. Furthermore F is the only elementary abelian group of order 64 in $\langle V_M, x \rangle$. This shows that $E \leq N_G(F)$ and so $N_G(F) \leq M$.

Now let $\rho \in P$ with $o(\rho) = 3$. Set $F_1 = \langle V_M \cap V_M^g, x, x^\rho \rangle$. Then $|F_1| = 2^7$. Hence $F \leq F_1$. As $|V_M \cap V_M^g : C_{V_M \cap V_M^g}(x)| = 2$ and $[\rho, V_M \cap V_M^g] = Y_P$, we see that $|Z(F_1)| = 2^4$. We have $|[\rho, F_1]| = 16$ and so $[\rho, F_1]$ also is abelian. Then $[\rho, F_1]Z(F_1)$ is abelian of order 64. Further for $t \in F_1 \setminus [\rho, F_1]Z(F_1)$ we have that $[[\rho, F_1]Z(F_1), t] = Y_P$. We then have that $F = [\rho, F_1]Z(F_1)$. This implies $\rho \in N_G(F) \leq M$, a contradiction.

Let now $|Y_1| \leq 4$. Let $C_Y(K) \neq 1$. Then choose $y \in C_Y(K)^\sharp$. Set $W = [V_M, y]$ and let W_1 be an irreducible D–submodule of W. We may assume that $[W_1, C_Y(D)] = 1$. So $W_1 \cap Y_P \not\leq Y_M$ by 15.7. Then we have that $|[t_1, W]| = 4$. So if $|Y_1| = 4$, we get $r = 3$ and then $|W_1| = 16$. Now $[W_1, t_1] = [W, t_1]$ and so $W = W_1 \oplus C_W(O_3(D))$. As $C_W(O_3(D))$ is Y–invariant we get with 15.7 that $C_W(O_3(D)) = 1$. This gives $|W| = 4$. Now $[W, C_Y(K)] = 1$ and so $|C_Y(K)| = 2$. This shows $|Y| = 8$. Now we get $|V_M/Y_M| = 2^8$. We see that in $O^+(8, 2)$ the centralizer of $O_3(D)$ is a $\{2, 3\}$–group and so we have that Y acts nontrivially on K, a contradiction. So we have that $|Y| = 4$ and then $|V_M/Y_M| = 2^6$. As now $3 \in \sigma(M)$, we see that this group is of minus type. But as $V_M \cap V_M^g$ is elementary abelian, we get $|Y| = 8$, a contradiction.

So let now $|Y_1| = 2$. Let first $C_Y(K) = 1$. Then we get that $|V_M| \leq 2^5$. But then V_M is centralized by some good p–element, contradicting 15.1. So we have $C_Y(K) \neq 1$. Let $y \in C_Y(K)$ and W and W_1 as before. Then we have that $|[W, t_1]| = 4$, which gives that $r = 3$ or 5. Let $r = 5$, then W is an irreducible module for D and so we see again that $|C_Y(K)| = 2$, which gives $|Y| = 4$ and so $|V_M| \leq 2^7$. Then we we have that V_M is of $-$–type and $3 \in \sigma(M)$. But in $O^-(6, 2)$ there is no 5-element centralizing an elementary abelian 3-group of order 3^3, a contradiction.

So we have $r = 3$. Hence we have that in any case K is a 3–group. Further we have that $C_Y(K) \neq 1$. In particular $C_Y(K)$ contains some $1 \neq y$ which is centralized by S. But now we have that y has to act nontrivially on some component K_1 of $C_{\tilde{M}/C_{\tilde{M}}(V_M/Y_M)}(K)$. By 15.11 we have that $[K_1, Y] \leq K_1$. □

From now on we always will assume that K is a component of $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$.

comlie

Lemma 15.15 *We have that K is a group of Lie type. But $K \not\cong L_4(2)$.*

Proof: We may assume that K is alternating or sporadic. In this context we will consider A_5 as $L_2(4)$ and A_6 as $Sp(4, 2)'$. But we include $3A_6$ and A_8 in the proof.

Let first $q > 2$. Then by 15.12 we have $C_Y(K) = 1$. As $[\omega, K] \neq 1$ and K has no outer automorphism of odd order, we see that ω induces an inner automorphism, which normalizes a Sylow 2–subgroup of K and acts nontrivially on the center of a Sylow 2–subgroup of K. Hence by 1.12 we get $K \cong A_5$ or J_1. The case of A_5 will be treated as $L_2(4)$. So let $K \cong J_1$, then $q \leq 8$ and $|V_M/Y_M| \leq 2^{12}$. As $K \not\leq GL(6, 2)$, we see that K induces exactly one nontrivial irreducible module W in V_M/Y_M. Hence by 15.7 we have that Y_P/Y_M is in W. As K is centralized by a good E, we get a good p–element centralizing Y_P/Y_M. By 5.5 this shows that $p = 3$ and $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. But then we must have three conjugates of K. Now we get a contradiction as $J_1 \times J_1 \times J_1$ is not isomorphic to a subgroup of $GL(12, 2)$.

So we have that $q = 2$. According to 15.14 choose $t \in Y$, with $[t, K] \neq 1$ and $[t, Z(S \cap K)] = 1$ and assume that $C_Y(K) \neq 1$. Choose $y \in C_Y(K)^\sharp$. Set $W = [V_M, y]$ and let $W_1 \leq [W, K]$ be a quasi irreducible submodule. We have that $|[[W, K], t]| \leq 4$ and so by 3.32 we see that K is alternating and by 3.33 we have that W_1 either involves the permutation module, or $n = 7$ or 8 and it is a 4–dimensional module, or we have $3A_6$ on a 6–dimensional module. Further there are at most two such modules involved in W. Let first $3 \in \sigma(M)$. Suppose $n \geq 13$. Then $m_3(K) \geq 4$ and so all 3–elements are good. Now any involution in K is centralized by an elementary abelian subgroup of order 9 in K and so $Y_P/Y_M \leq [W, t]$ is centralized by some good element of order three, contradicting 14.2. So we have that $n \leq 12$. If $n = 11$ or 12. Then by 1.17 again all 3–elements are good. Now t is centralized by an elementary abelian group of order 9 or by A_5. In both cases Y_P is centralized by a 3–element, a contradiction as before. So we have $n \leq 10$. Suppose that W is the permutation module. Then we have $n \geq 7$. Let $v \in C_W(S \cap K)$. If $n = 7$, then $C_K(v) \cong A_6$, Σ_5, or $\Sigma_3 \times \Sigma_4$. If $n = 8$, then $C_K(v) \cong E_{16}(\Sigma_3 \times \Sigma_3)$. If $n = 9$, then $C_K(v) \cong A_8$ and if $n = 10$, then $C_K(v) \cong \Sigma_8$. In any case Y_P is centralized by a good 3–element. So we have $K \cong 3A_6$ on the 6-dimensional module or A_8, A_7 on the 4-dimensional module. Now the corresponding centralizers are Σ_4, $E_8L_3(2)$ and $L_3(2)$. But also here we have 3-elements centralizing $C_V(S \cap K)$.

So assume now that $3 \notin \sigma(M)$. Then $n \leq 11$. Set $T = S \cap C(K)$. Then without loss we may assume that $W_1 \leq C_{V_M}(T)$. As $|W_1 : W_1 \cap V_M^g| \leq 2$, we get that $|W_1 : C_{W_1}(Y)| \leq 2|Y : C_Y(K)|$. As $|Y : C_Y(K)| \leq 2^5$, we get $|W_1 : C_{W_1}(Y)| \leq 2^6$. By 15.7 we have $C_{W_1}(Y) \leq Y_P/Y_M$, so $|W_1| \leq 2^7$. Hence $n \leq 8$. So we get $|W_1| = 16$ or $n = 8$, $|W_1| = 2^6$ and $|Y : C_Y(K)| = 16$ by 3.35. In any case we get that $W_1 \leq [V_M, y]$ for all $y \in C_Y(K)^\sharp$. Suppose $|C_Y(K)| = 2$ and so $2^3 \leq |Y| \leq 2^5$. This shows $|V_M/Y_M| \leq 2^{12}$. Hence V_M involves at most three modules. But then there is a good E which contains a p–element centralizing V_M, as $p \geq 5$ a contradiction. So we may assume that $|C_Y(K)| > 2$. Let $\langle y, y_1 \rangle$ be a fours group in $C_Y(K)$. Then $W_1 = [V_M, y] \cap [V_M, y_1] \leq V_M \cap V_M^g$. Hence t induces a transvection on W_1, which again gives that $K \cong A_8$ and $|W_1| = 16$. This now shows that $|Y| = 8$ and Y is the transvection group in K to Y_P/Y_M. Suppose that $W_1 = [W, K]$. By 3.35 we get that $W = W_1 \oplus C_W(K)$. As Y acts on $C_W(K)$ we get with 15.7 that $C_W(K) = 1$. Hence $W = W_1$. So we have that W involves a further nontrivial irreducible module, which then also has to be the $L_4(2)$–module. Suppose that $[C_Y(K), [W, K]] = 1$. Then $|[W, K] : C_{[W,K]}(Y)| \leq 8$. But then $|[W, K]| \leq 2^5$, a contradiction. Hence there is some $y_2 \in C_Y(K)$ with $[W, K] = W_2 \oplus W_2^{y_2}$, where W_2 is the natural $L_4(2)$–module for K. But we must have that $|W : C_W(y_2)| \leq 4$, a contradiction. So in any case we have shown that $C_Y(K) = 1$.

We have $|V_M/Y_M| \leq 4|YO_2(M)/O_2(M)|^2$. Let first K be sporadic. Then with 3.49 we get that $K/Z(K) \cong M_i$, J_2, HiS, Co_1 or Co_2. In the last two groups there is an elementary abelian group of order 3^3, $3 \in \sigma(M)$, in $C_K(t)$ and so some element in $V_M \cap V_M^g \setminus Y_M$ is centralized by a good E, contradicting 15.5. For M_{11}, M_{12}, J_2, HiS, M_{22}, M_{23}, M_{24}, we get $|YO_2(M)/O_2(M)| \leq 4$, 8,16,32,32,16,64, respectively and so we get the following upper bounds for $|V_M/Y_M|$: 2^6, 2^8, 2^{10}, 2^{12}, 2^{12}, 2^{10}, 2^{14}. As M_{11} and M_{12} possess elements of order 11, they cannot act on a group of order 2^8 nontrivially. As J_2 contains an elementary abelian subgroup of order 25 normalized by a dihedral group of order 12, it cannot act nontrivially on a 10–dimensional module over $GF(2)$. Further HiS contains a nonabelian subgroup of order 125, so it is not a subgroup of $GL(12, 2)$ and finally M_{23} contains an element of order 23, so it cannot act nontrivially on a 2–group of order 2^{10}. Hence only M_{22} and M_{24} are possible. But now there is some good p–element centralizing K and so it centralizes also $C_{[V_M,K]}(Y) = Y_P/Y_M$, a contradiction.

So let now $K/Z(K) \cong A_n$. Then $|Y| \leq 2^{\frac{n}{2}}$ and so $|V_M/Y_M| \leq 2^{n+2}$. Let first $3 \in \sigma(M)$. Then as $|K|_{2'} > 2^{n+2} - 1$, for $n > 8$, we see with 1.11 that we always have some 3-element centralizing Y_P, a contradiction. Let $n = 8$. Then $|V_M/Y_M| \leq 2^{10}$. If the permutation module is a submodule, we have that Y_P/Y_M is centralized by a good E, a contradiction. So we have the $L_4(2)$–module as a submodule. If this module is Y–invariant, again Y_P/Y_M is centralized by a good 3–element, a contradiction. Hence we have two $L_4(2)$–modules, which are interchanged by some element in $y \in Y$ and so Σ_8 is induced. We have that $C_K(y) \cong \Sigma_6$ or $Z_2 \times \Sigma_4$. Further as $|V_M/Y_M| \geq 2^8$, we get $|Y| \geq 2^3$. By 15.7 we have that $C_{V_M/Y_M}(K) = 1$, so $|V_M/Y_M| = 2^8$ by 3.36. Now there is a good 3–element in $\tilde{M}$ centralizing K. But the two K–modules in V_M are not isomorphic, hence a good 3–element centralizes V_M, contradicting 15.1. Now let $n \leq 7$, then $|V_M/Y_M| \leq 2^6$ and just one nontrivial irreducible K–module can be involved in V_M. But then we have a 3–element in $C_{\tilde{M}}(K)$ centralizing $[V_M, K]$, a contradiction.

So we may assume that $3 \notin \sigma(M)$. Then we have $n \leq 11$. Now there are at most two nontrivial irreducible modules in V_M, which shows that we must have $m_p(K) \geq 2$ for $p \in \sigma(M)$, so $K \cong A_{10}$ or A_{11}. As $m_3(K) = 3$, we get that $e(G) \geq 4$. As $|V_M/Y_M| \leq 2^{12}$ and $GL(12,2)$ does not contain an elementary abelian subgroup of order 5^4, we get a good 5–element centralizing V_M, contradicting 15.1. $\square$

Klie2

Lemma 15.16 *We have that K is a group of Lie type in characteristic two different from $L_4(2)$.*

Proof: By 15.15 we may assume that K is a group of Lie type in odd characteristic which is not also a group of Lie type in characteristic two, too.

Let first $q = 2$. As in 15.14 let $t \in Y$ with $[K,t] \neq 1$ and $1 \neq y \in C_Y(K)$, $W = [V_M, y]$ and W_1 be a nontrivial quasi irreducible K–module in W. Now $|[W,t]| \leq 4$ and so with 3.31 we get that $K \cong 3U_4(3)$. Further W_1 is the 12–dimensional module. But we may choose $t \in Z(S \cap K)$. Then $|[W_1,t]| = 16$, a contradiction. So we have that $C_Y(K) = 1$. Now by 3.48 we get that $K \cong 3U_4(3)$, $L_3(3)$, $U_4(3)$, $L_4(3)$ or $L_2(25)$. If $K \not\cong L_3(3)$ or $L_2(25)$, we have that $3 \in \sigma(M)$. We have that $U_4(3)$ cannot act on a group of order 2^{12} as it contains a subgroup $3^4L_2(9)$. In $L_4(3)$ we have $3^4(SL_2(3)SL_2(3))$ and so it also cannot act on such a group. If we have $3U_4(3)$, then t is centralized by some elementary abelian subgroup of order 3^3. We have $|[V_M,t] : V_M \cap V_M^g \cap [V_M,t]| = 2$. As $V_M \cap V_M^g \cap [V_M,t] \not\leq Y_M$, we get some $1 \neq x$ in $V_M \cap V_M^g \setminus Y_M$, which is centralized by a good E, contradicting 15.5. If we have $L_3(3)$ or $L_2(25)$, then $|Y| \leq 2^3$ and so $|V_M/Y_M| \leq 2^8$, but K contains an element of order 13, a contradiction.

So we have $q > 2$. By 15.12 we have $C_Y(K) = 1$. We have that ω induces an automorphism on K which normalizes a Sylow 2 - subgroup of K. As $[\omega, X] = X$, we have that $[\omega, K] \neq 1$. Let $q = 64$ and $\nu \in P \cap M$, $o(\nu) = 7$. Suppose that ν does not normalize K. Then $K^{\langle \nu \rangle} = K_1 \times \cdots \times K_7$, and so $3 \in \sigma(M)$ and all 3–elements are good. But $N_G(\langle \omega \rangle) \not\leq M$, a contradiction. So ν normalizes K and induces an automorphism which normalizes a Sylow 2–subgroup of K.

Let first $K \cong L_2(r)$ or ${}^2G_2(r)$. Then we have $|Y| \leq 8$. So we have $|V_M/Y_M| \leq 2^{12}$. Assume ${}^2G_2(r) \leq \Omega^{\pm}(12,2)$. This is just possible for $r = 3$, but ${}^2G_2(3)$ is isomorphic to $L_2(8)$, a group in characteristic two. In case of $L_2(r)$ we get $o(\omega) = 3$ and so $|X| = |Y| = 4$. Now $K \leq \Omega^{\pm}(8,2)$, which gives $K \cong L_2(17)$. But then ω would centralize K, a contradiction.

Suppose now that $\rho = \nu$ or ω induces a field automorphisms. Hence

$K = G(p^f)$ and we have $p \in \sigma(M)$. Let $K_1 = C_K(\rho)$. Then we have that K_1 is nonsolvable and $m_p(K_1) \geq 2$ as $K \not\cong L_2(r)$ or ${}^2G_2(r)$. But now application of 5.3 shows $N_G(\langle \rho \rangle) \leq M$, a contradiction.

So we have shown that neither ω nor ν induces a field automorphism. Application of 1.13 shows that $o(\omega) = 3$. Hence $q = 4$. Further ω induces an inner automorphism. As $[\omega, Y] = Y$, we get that $|Y| \leq 2^{m_2(K)}$.

This now implies that $m_3(K\langle \omega \rangle) \leq 3$ as otherwise $N_G(\langle \omega \rangle) \leq M$. Let $m_2(K) \leq 6$. We have that $|V_M/Y_M| \leq q^2|Y|^2 \leq 16 \cdot 2^{12} = 2^{16}$. Then we get $K \lesssim O^{\pm}_{16}(2)$. As K contains a nonabelian subgroup of order u^3 for some odd prime u, this implies $r = 3^f$. As $m_3(K) \leq 3$, we get with 1.1 $K \cong L_3(3)$, $U_3(3)$, $PSp_4(3)$, or $G_2(3)$. But as ω has to induce some automorphism which acts nontrivially on X which is contained in the center of a Sylow 2–subgroup, we get a contradiction in all cases, as always this center has order two.

So we have $m_2(K) > 6$. Application of 1.1 shows that $m_3(K) = 3$, $K \cong Sp_6(r)$, $\Omega^-_8(r)$, $L_6(r)$, $U_6(r)$, $L_7(r)$, or $U_7(r)$. Now $3 \not| \, r$ and $3 \not| \, r-1$, while $3 \not| \, r+1$ in case of $K \cong U_6(r)$ or $U_7(r)$. Hence never K possesses a diagonal automorphism of order 3. As ω does not induce a field automorphism, we see $\langle \omega, K \rangle \cong Z_3 \times K$. But now $m_3(Z_3 \times K) = 4$, a contradiction. □

*nocentq*2

Lemma 15.17 *Let $q = 2$, then $C_Y(K) = 1$.*

Proof: By 15.16 we have that K is of Lie type in characteristic two but not isomorphic to $L_4(2)$. We choose $y \in C_Y(K)^\sharp$ and define $W = [V_M, y, K]$ and W_1 to be a quasi irreducible submodule of W. We may assume that $[C_Y(K), W_1] = 1$. By 15.7 we have $C_{W_1}(K) = 1$, as otherwise $Y_P/Y_M \leq C_{W_1}(K)$ and so K is covered by $M \cap M^g$, but $Y \leq O_2(M^g)$. According to 15.14 let $t \in Y$ with $[K, t] = K$ and $[t, S] = 1$. As $|[W, t]| \leq 4$, we get with 3.29 that $K \cong L_n(r)$, $Sp(2n, r)$, $r \leq 4$, $\Omega^{\pm}(2n, 2)$, $U_n(2)$ or $G_2(2)'$. As $|W_1 : W_1 \cap V_M^g| \leq 2$ and Y acts as a transvection group to Y_P/Y_M on $V_M \cap V_M^g$ by 15.6, we get with 15.7 $|W_1| \leq 4|Y : C_Y(K)|$. So W_1 is given by 3.33.

Let first $K \cong G_2(2)'$. Then W_1 is the 6–dimensional module. But $|Y : C_Y(K)| \leq 8$, which gives $|W| \leq 2^5$, a contradiction.

Let $K \cong U_n(2)$, then we will assume that we have the natural module ($U_4(2)$ on the orthogonal module will be handled next). Now t corresponds to an unitary transvection, so $C_K(t)$ involves $U_{n-2}(2)$. If $n > 4$, then $3 \in \sigma(M)$, but always some 3–element centralizes $[W, t]$, a contradiction as $[W, t] \cap Y_P/Y_M \neq 1$. So we have $3 \notin \sigma(M)$ and $n = 4$. Then $|Y : C_Y(K)| \leq 2^4$ and so $|W| \leq 2^6$, a contradiction.

Let next $K \cong \Omega^{\pm}(2n,2)$. We then have the natural module W_1. We assume $n \geq 3$ and $K \not\cong \Omega^{+}(6,2)$. Then as $[W,t]$ is of order 4, t corresponds to a root element and so $C_K(t)$ involves $\Sigma_3 \times \Omega^{\pm}(2n-4,2)$. So we have always some element of order three which centralizes $[W,t]$. Hence we get that $3 \notin \sigma(M)$. So we get $K \cong \Omega^{-}(8,2)$ or $\Omega^{-}(6,2)$. Again we have that $|W_1 : C_{W_1}(Y)| \leq 2|Y : C_Y(K)|$, so we get $|W_1 : C_{W_1}(Y)| \leq 2^7$ or 2^5. As $|C_{W_1}(Y)| = 2$, we get $W_1 = W$. Assume there is $y \neq y_1 \in C_Y(K)^{\sharp}$. Then as $Y_P/Y_M \leq W_1$, we get that $W_1 \leq [V_M, y_1]$ as well. But then $W_1 \leq [V_M, y] \cap [V_M, y_1] \leq V_M \cap V_M^g$. Then t induces a transvection on W_1, a contradiction. So we have that $|C_Y(K)| = 2$, which gives that $|Y| \leq 2^7$, 2^5, respectively. Then $|V_M/Y_M| \leq 2^{16}$, 2^{12}. In particular there are exactly two natural modules involved. As $m_p(K) \leq 1$, we get a good E centralizing K. But $p \geq 5$ for $p \in \sigma(M)$, and so there is a good E centralizing V_M, a contradiction.

Let $K \cong L_2(4)$ and W_1 be the permutation module. Again we have $|C_Y(K)| = 2$ and so $|Y| \leq 8$. This gives $|V_M/Y_M| \leq 2^8$. Then we have that $|V_M/Y_M| = 2^8$ as there are two nontrivial K–modules involved. But then we have that $p = 3 \in \sigma(M)$ and all 3–elements are good. But in W_1, we have that Y_P/Y_M is centralized by a 3–element, contradicting 14.2.

Let now $K \cong Sp(2n,4)$ or $L_n(4)$. then W_1 is the natural module and t is a transvection. As $|C_{W_1}(Y)| = 2$, we get some $y_1 \in Y$, which induces a field automorphism on K. Then $C_K(y_1) \cong Sp(2n,2)$ or $L_n(2)$. We have that $C_K(y_1)$ acts on $[W_1, y_1]$. As $|[W_1, y_1]| \leq 4$, we get $C_K(y_1) \cong \Sigma_3$. So $K \cong L_2(4)$. Then $|Y : C_Y(K)| = 4$ and so $|W_1| = 16$, W_1 is the natural module. Again $|C_Y(K)| = 2$ and t induces a transvection over $GF(4)$ on W_1. So we get $|Y| \leq 8$ and $|V_M/Y_M| = 2^8$. Now $3 \in \sigma(M)$ and all 3–elements are good. As Y_P/Y_M is not centralized by a 3–element, we get that 27 divides $|(Y_P/Y_M)^M|$. As elements of order 5 in K act fixed point freely on V_M/Y_M we get that $|(Y_P/Y_M)^m| = 135$ and all involutions in $V_M \setminus Y_M$ are conjugate. Let now $E \leq M$, $E \cong E_{27}$. Then E contains some element ρ of order 3 such that $|C_{V_M}(\rho)| = 2^5$. But then ρ centralizes some involution in $V_M \setminus Y_M$ and so Y_P is centralized by a good 3–element in M, contradicting 14.2.

Let $K \cong Sp(2n,2)$. Then $Z(S \cap K)$ is centralized by some $Sp(2n-4,2)$. Let $n \geq 4$. Then $[W_1, t]$ is centralized by $Sp(2n-4,2)'$ and so by a good E, as $m_3(K) \geq 4$ and so $3 \in \sigma(M)$.

So we may assume that $K \cong Sp(6,2)$ or A_6. Now we have that $|W_1| \leq 2^8, 2^5$, respectively. If $|C_Y(K)| > 2$, we get again that $W_1 \leq V_M \cap V_M^g$ and so $Y/C_Y(K)$ has to induce transvections, so $|Y : C_Y(K)| = 2$. Then we have that $|Y| \leq 2^7$ or 2^4. This gives that $|V_M/Y_M| \leq 2^{16}$ or 2^{10}. This in fact implies that K induces at most two nontrivial irreducible modules in V_M/Y_M. In particular $C_{\tilde{M}}(K) \lesssim \Sigma_3$. Hence in any case $\sigma(M) = \{3\}$ and all 3–elements are good. But W_1 is either the natural module or the spin module and in both cases $C_{W_1}(S \cap K)$ is centralized by a 3–element, contradicting 14.2 as $C_{W_1}(S \cap K) \geq Y_P/Y_M$.

So we are left with $K \cong L_n(2)$. Then $C_K(t)/O_2(C_K(t)) \cong L_{n-2}(2)$. Suppose $3 \in \sigma(M)$. If $n \geq 5$, then we have that Y_P is centralized by a good 3–element, a contradiction. Hence in this case we must have $K \cong L_3(2)$. Now $|Y : C_Y(K)| \leq 2$ and so $|W_1| \leq 16$, i.e. W_1 involves just one natural module. Let $3 \notin \sigma(M)$, then $n \leq 7$. Let $n \geq 5$. Then $[W_1, t]$ is centralized by $C_K(t)$ and so Y_P is centralized by $C_K(t)$. By Smith lemma [Sm] we have that $Y_P/Y_M = [W_1, t]$ and so t induces a transvection and then W_1 is the natural module.

In any case we have that $n \leq 7$ and W_1 is the natural module. Let $|C_Y(K)| > 2$. Then we have that $W_1 \leq V_M \cap V_M^g$ and so $Y/C_Y(K)$ is the full transvection group. Let $n \neq 3$. If $W = W_1$, then we have with 3.36 that $[V_M, y] = W_1 \oplus C_{W_1}(K)$. But now 15.7 shows that $C_{W_1}(K) = 1$. This gives the contradiction $W_1 = [V_M, y]$. So we have that $W \neq W_1$ and there is some $y_2 \in C_Y(K)$ and so some module W_2 such that $W_2 \oplus W_2^{y_2} \leq W$. But we have that $|[W, y_2]| \leq 4$, a contradiction. This shows $[W, C_Y(K)] = 1$ and then $|W| \leq 2^{n+1}$, a contradiction.

So we have that $n = 3$ or $|C_Y(K)| = 2$. In case of $n = 3$, we have that $W = W_1$ and $|[V_M, y] : W_1| = 2$.

Let next $|C_Y(K)| = 2$. Assume further that K is normal in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$. As $3 \notin \sigma(M)$, there is some good E centralizing K. Set $\tilde{W} = \langle W_1^E \rangle$. We have that $C_E(Y_P) = 1$. So we get at least 9 conjugates of W_1 under E. Hence $|V_M/Y_M| \geq 2^{9n}$ and on the other hand $|Y/C_Y(K)| \leq 2^{12}$. This shows $|V_M/Y_M| \leq 2^{26}$, a contradiction. So we have that K is not normal in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$. As $3 \notin \sigma(M)$, we have that $K \cong L_3(2)$.

Now in any case we are left with $K \cong L_3(2)$. Further $|[V_M, y]| = 2^4$. Let $y_1 \in C_Y(K) \setminus \langle y \rangle$. Then $[[V_M, y], y_1] \leq W_1$ and so $[[V_M, y], y_1] = 1$. This shows that $[[V_M, y], C_Y(K)] = 1$. Set $U = \langle [V_M, y] \mid y \in C_Y(K) \rangle$. Then $[U, C_Y(K)] = 1$. As $[[V_M, y], K] = W_1$ for all $y \in C_Y(K)^\sharp$, we see that $[U, K] = W_1$. Further $C_U(K) = 1$ by 15.7, so we get with 3.36 that $U = [V_M, y]$ is of order 16. But we have that $|[C_Y(K), V_M]/W_1| \geq |C_Y(K)|$ and so $|C_Y(K)| = 2$. Now $|Y| \leq 8$ and $|V_M/Y_M| \leq 2^8$. As $m_p(M) \geq 3$ for $p \in \sigma(M)$, we get with 15.1 that $\sigma(M) = \{3\}$. Now we get that $K^{\tilde{M}} \lesssim L_3(2) \times L_3(2)$ and 3 divides $|K^{\tilde{M}}|$, so all 3–elements are good. But

$C_K(Y_P)$ contains a 3–element, a contradiction. □

qgreater2

Lemma 15.18 *We have $q > 2$.*

Proof: Suppose $q = 2$. By 15.17 we may assume that there is a component K of $\hat{M} = \tilde{M}/C_{\tilde{M}}(V_M/Y_M)$ with $[K, Y] \leq K$ and $C_K(Y) = 1$. Further by 15.16 we have $K \cong G(r)$, $r = 2^n$, a group of Lie type but $K \not\cong L_4(2)$.

Let first K be a rank 1 group $L_2(r)$, $U_3(r)$ or $Sz(r)$. Then $|Y| \leq r$ and so $|V_M| \leq 4r^2 \leq r^3$. Hence only $L_2(r)$ is possible. Further we have that K is normal in $\hat{M}$. So if $p \in \sigma(M)$ and p divides $|K|$, then all p–elements are good, as a Sylow p–subgroup of M contains an elementary abelian group of order p^2 in its center. Further by 5.18 we have that there is no field automorphism of order p of K. Hence there is a good E centralizing K. But then there is a good p–element centralizing $[V_M, K]$ and so Y_P, a contradiction.

By 3.51 we now have that $Y \leq O_2(YP_1)$, P_1 some minimal parabolic. Hence $C_{V_M}(O_2(YP_1)) = Y_P/Y_M$. This shows that P_1 does not contain a good p-element. Going over the groups of Lie type, we now get that $K \cong L_n(r)$, $n \leq 4$, $L_n(2)$, $5 \leq n \leq 7$, $U_n(r)$, $n \leq 5$, $Sp(2n, r)$, $n \leq 3$, $\Omega^-(8, r)$, $G_2(r)$, ${}^3D_4(r)$, or ${}^2F_4(r)$. Let $p \in \sigma(M)$, then we get that p does not divide $r^2 - 1$, $r^6 - 1$ for ${}^3D_4(r)$. Hence we see that $m_p(K) \leq 2$. By 14.2 we have no good p–element in $N_G(S)$, hence we see that always some good p–element centralizes K. In particular $Y \not\leq K$, and $C_{V_M}(O_2(P_1)) \neq Y_P/Y_M$. Hence we have that $[V_M, K]$ involves a direct sum of at least two isomorphic nontrivial irreducible K–modules. Now we go over the cases above. Recall that $|V_M/Y_M| \leq 4|Y|^2$.

Let $K \cong \Omega^-(8, r)$. Then we get $|V_M/Y_M| \leq 4r^{12} < r^{15}$, which by 3.45 can never involve two nontrivial modules.

Let $K \cong {}^3D_4(r)$ or ${}^2F_4(r)$, then $|V_M/Y_M| \leq 4r^{10} < r^{12}$, which by 3.45 also is not possible.

Let $K \cong G_2(r)$, then $|V_M/Y_M| \leq 4r^6 \leq r^8$, a contradiction again with 3.45.

In case of $K \cong U_5(r)$ or $U_4(r)$, we get $|V_M/Y_M| \leq 4r^8 \leq r^{10}$, which also is not possible, as by 3.45 minimal modules have order at least r^6.

Let $K \cong Sp(6, r)$, then we get $|V_M/Y_M| \leq 4r^{12} \leq r^{14}$. Now as p cannot divide $r^2 - 1$, we see that $m_p(K) \leq 1$ and so there is a good E centralizing K, which shows that we must have at least three modules. But by 3.45 minimal modules have dimension at least 6, a contradiction.

Let $K \cong Sp(4,r)$. Then we get $|V_M/Y_M| \leq 4r^6 \leq r^8$. This is just possible for $r = 2$. Now we have exactly two irreducible modules, but $p > 3$ as otherwise P_1 contains a good 3–element, and so they are centralized by a good p–element, a contradiction.

Let next $K \cong L_3(r)$. Then $|V_M/Y_M| \leq 4r^4 \leq r^6$, which shows $r = 2$ and we have exactly two modules. As $p > 3$, we get a contradiction as in the case of $Sp(4,r)$.

Let $K \cong L_4(r)$. Then $|V_M/Y_M| \leq 4r^8 \leq r^{10}$. As p does not divide $r^2 - 1$, we have $m_p(K) \leq 1$, so some good E centralizes K and we have at least three modules involved, a contradiction.

Let $K \cong L_5(2)$. Then $|V_M/Y_M| \leq 2^{14}$. As $p > 3$, we have that $m_p(K) \leq 1$ and so some good E centralizes K, which gives a contradiction as above.

Let $K \cong L_6(2)$, we get $|V_M/Y_M| \leq 2^{20}$. Let $p = 3$. Then by 1.17 all 3–elements are good, but P_1 contains a 3–element. Hence we have $p > 3$ in particular $e(G) \geq 4$. So we have a good E in $C_M(K)$. In particular we now have that there are at least three modules involved. This gives $p = 7$ and then some good p–element centralizes $[V_M, K]$, a contradiction.

Let finally $K \cong L_7(2)$, then $|V_M/Y_M| \leq 2^{26}$. Again $p > 3$ by 1.17 and so $e(G) \geq 4$. Hence we have a good E which centralizes K and then also some good p–element centralizes $[V_M, K]$, a contradiction. □

Kstruk

Lemma 15.19 *We have a component K of $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$ with $[Y,K] \leq K$, $[\omega, K] \leq K$ and $C_Y(K) = 1$, which is isomorphic to $L_n(r)$, $3 \leq n \leq 4$, $U_n(r)$, $4 \leq n \leq 5$, $Sp(2n,r)$, $n \leq 3$, $G_2(r)$, ${}^3D_4(r)$, ${}^2F_4(r)$, $\Omega^-(8,r)$, with $r = 2^n \geq q > 2$. Further there is a minimal parabolic P_1 of K such that $Y \leq O_2(P_1Y)$, and P_1 is normalized by ω. Finally $O^{(\infty)}(P_1)$ does not contain a good p–element for $p \in \sigma(M)$.*

Proof: By 15.18 we have that $q > 2$. Now by 15.12 we have some K with $[K, Y\langle\omega\rangle] \leq K$ and $C_Y(K) = 1$. By 15.16 we have that $K = G(r)$, $r = 2^n$, a group of Lie type in characteristic two.

Let first $K \cong L_2(r)$, $Sz(r)$ or $U_3(r)$. As $[X,\omega] = X$ and $|X| = q$, we have that $q \leq r$. Further we have that $|V_M/Y_M| \leq q^2|Y/Y \cap O_2(\tilde{M})|$. Then we have $|Y : Y \cap O_2(M)| \leq r$ and so $|V_M/Y_M| \leq q^2r^2 \leq r^4$. This first gives $K \not\cong U_3(r)$. Further as V_M is non abelian, we have that $K \not\cong Sz(r)$ by 3.55. So we are left with $K \cong L_2(r)$. In particular there are at most two nontrivial irreducible modules involved. If we have some good field automorphism, we get with 14.2 that there is $s \in S$ with $[K, K^s] = 1$ and then $|V_M/Y_M| = r^4$

and $r = q$. If $[[V_M, K], K^s] = 1$, then as $Y_P \cap [V_M, K] > Y_M$, we get that K^s is covered by M^g and so $[Y, K^s] = 1$. But $[X, K] \neq 1$ and $X^s = X$, so $[X, K^s] \neq 1$, a contradiction. So we have that V_M/Y_M is the orthogonal $\Omega^+(4, q)$–module. As $m_x(\langle K^S, \nu \rangle) = 3$ for any prime x dividing $o(\nu) = q - 1$, we get that $e(G) > 3$, and so again there is some good p–element, p does not divide $q - 1$, which centralizes $\langle K^S \rangle$ and then also V_M/Y_M, a contradiction. So we may assume that K is normal in $M/O_2(M)$ and there is a good E centralizing K. Then as there are at most two nontrivial modules in V_M, we get that p has to divide $q - 1$. But now the center of a Sylow p–subgroup of M is noncyclic and so all p–elements are good. But now p divides $|P \cap M|$, a contradiction.

So the Lie rank of K is at least two. Then by 3.51 there is some minimal parabolic P_1 of K such that $Y \leq O_2(P_1Y)$ and $O^2(P_1/O_2(P_1)) \cong L_2(r)$ in case of ${}^2F_4(r)$. Hence $O^2(P_1/O_2(P_1)) \cong L_2(r)$ or $U_3(r)$. By 15.7 we have that $C_{V_M/Y_M}(O_2(P_1Y)) \leq Y_P/Y_M$. Assume $[\omega, P_1] \not\leq P_1$. Then we have $K \cong \Omega_8^+(r)$ and ω induces a graph automorphism of order 3. But as $m_3(\Omega_8^+(r)) \geq 4$ all 3–elements are good, a contradiction. So $[\omega, P_1] \leq P_1$. Then as ω acts fixed point freely on Y_P/Y_M, we get that $O^2(P_1)$ centralizes Y_P and so it does not contain any good p–element.

As ω acts nontrivially on some root subgroup of K, we also get $r > 2$. This now implies that $K \cong L_n(r), n \leq 4$, $U_n(r), n \leq 5$, $Sp_{2n}(r), n \leq 3$, $G_2(r)$, ${}^3D_4(r)$, ${}^2F_4(r)$, $\Omega_8^-(r)$. □

bnot2

Proposition 15.20 *$b \neq 2$.*

Proof: Suppose false. Then we are in the situation of 15.19. In particular in all cases we have that $|V_M/Y_M| \leq q^2|Y/Y \cap O_2(\tilde{M})|^2$.

Let first $K \cong^3 D_4(r)$ or ${}^2F_4(r)$, then we have that $|Y : Y \cap O_2(\tilde{M})| \leq r^5$ by 1.5. Hence we have that $|V_M/Y_M| \leq q^2 r^{10} \leq r^{12}$, contradicting 3.45.

We have that $C_{V_M/Y_M}(K) = 1$ as otherwise by 15.7 $Y_P \cap C_{V_M}(K) > Y_M$ and so K is covered by M^g but $Y \leq O_2(M^g)$.

Let first $[[V_M/Y_M, K], C_S(K)] = 1$. This in fact happens if $[V_M/Y_M, K]$ is irreducible. By 15.7 we have that $C_{[V_M/Y_M,K]}(Y) = Y_P/Y_M$. As $[Y, \omega] = Y$, we get that Y projects onto a subgroup of $S \cap K$ and so $C_{[V_M/Y_M,K]}(S \cap K) = Y_P/Y_M$. Let K_1 be a preimage of K, then we get that $\tilde{M} = K_1(\tilde{M} \cap M^g)$. If $M \cap M^g$ does not contain a good p–element, we must have that $m_p(Aut_M(K)) \geq 3$. Hence by 5.5 we may assume that $p = 3$ and either $K \cong SL_3(4)$ or $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. In the former a good E is in $N(Y_P)$, a contradiction. So assume that $Z_3 \wr Z_3$

and $m_3(K) \leq 2$. But we have that $|K|$ is divisible by 3 and as the center of a Sylow 3–subgroup is of order 3, we get that a Sylow 3–subgroup of $C_{\tilde{M}/C_{\tilde{M}}(V_M/Y_M)}(K)$ is in K. Hence also $m_3(Aut_M(K)) = 3$. So in any case we have that $m_p(Aut_M(K)) \geq 3$ for $p \in \sigma(M)$.

As $M = K_1(\tilde{M} \cap M^g)$ and $Y \leq O_2(\tilde{M} \cap M^g)$, Y is normalized by S, we see that $K^S = K$.

Let first $m_p(K) \geq 3$ for some $p \in \sigma(M)$. Then we get $K \cong L_4(r)$, $p \Big| r-1$, $U_4(r)$ or $U_5(r)$, $p \Big| r+1$, or $Sp(6,r)$ or $\Omega^-(8,r)$ and $p \Big| r^2-1$. In any case a p–element in K centralizes an elementary abelian subgroup of order p^3 and so all p–elements in K are good. Let P_1 be as in 15.19, then we have that $P_1/O_2(P_1) \cong L_2(r)$, $L_2(r^2)$ or $U_3(r)$ and in any case p divides the order of $O^{(\infty)}(P_1)$, contradicting 15.19.

Assume now $m_p(K) = 2$. Then as K is normal in $\tilde{M}/C_{\tilde{M}}(V_M/Y_M)$ we have an outer automorphism of order p. By 5.18 this is not a p–element besides $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M. Suppose we have an outer automorphism which is not a field automorphism, then we have $K \cong L_3(r)$, $p = 3$ or $U_5(r)$, $p = 5$. In the latter as $5 \Big| r+1$ we have that $m_5(K) > 2$. Hence we have that $K \cong L_3(r)$ and $3 \Big| r-1$. With 5.11 we get that all 3–elements in K are good, contradicting 15.19. So we have that $Z_3 \wr Z_3$ is a Sylow 3–subgroup of M and some field automorphism is induced. Now 3 divides r^2-1 and so by 5.11 we may assume that a Sylow 3–subgroup of K is extraspecial of order 27. Let u be the 3–element inducing the field automorphism. Then u centralizes in K an elementary abelian subgroup of order 9. Hence all 3–elements in $C_K(u)$ are good. So 3 does not divide $C_{P_1}(u)$, which is a contradiction.

So we have that $m_p(K) \leq 1$. Hence $Out(K)$ possesses a nonabelian Sylow p–subgroup. This shows that $p = 3$ and we have $K \cong L_3(r)$ or $U_3(r)$, or $p = 5$ and we have $K \cong U_5(r)$. As some p–element has to induce a diagonal automorphism, we have that p divides $r-1$ in the linear case and $r+1$ in the unitary case. But then in any case $m_p(K) \geq 2$.

So we have that $[[V_M, K], C_S(K)] \neq 1$. In particular $[V_M, K]$ involves at least two nontrivial irreducible K–modules.

Let $K \cong G_2(r)$, then $|Y/Y \cap O_2(\tilde{M})| \leq r^3$ by 1.5, so we have that $|V_M/Y_M| \leq r^8$, a contradiction to 3.45.

Let $K \cong \Omega^-(8,r)$, then by 1.5 $|Y/Y \cap O_2(\tilde{M})| \leq r^6$ and so $|V_M/Y_M| \leq r^{14}$, contradicting 3.45.

We are left with $K \cong Sp(6,r)$, then $|V_M/Y_M| \leq q^2r^{12}$, $K \cong L_4(r)$, $|V_M/Y_M| \leq q^2r^8$, $K \cong Sp(4,r)$, $|V_M/Y_M| \leq q^2r^6$ and $K \cong L_3(r)$ and $|V_M/Y_M| \leq q^2r^4$. In the last two cases we have equality, $q = r$ and $|Y/Y \cap O_2(\tilde{M})| = r^3$ or r^2. In all cases by 3.45 there are exactly two nontrivial irreducible modules involved. Now Y is a $2F$–module offender on these modules and so with 3.29 we get that they are the natural ones.

Let $W_1 = C_{V_M/Y_M}(C_S(K))$. Then we have that $C_{W_1}(Y) = Y_P/Y_M$ and so $C_{\tilde{M}}(C_{W_1}(S)) \leq M^g$. This now shows that $|Y/Y \cap O_2(\tilde{M})| \leq r^5$ in case of $K \cong Sp(6,r)$ and $|Y/Y \cap O_2(\tilde{M})| \leq r^3$ in case of $K \cong L_4(r)$. So we get that $|V_M/Y_M| \leq q^2r^{10}$, q^2r^6, respectively and so also in these cases we get that $r = q$ and $|Y/O_2(\tilde{M})| = q^5$, q^3 respectively. So in all cases we have $q = r$ and V_M/Y_M is an extension of a natural module by a natural module. As $O_2(C_{\tilde{M}/C_{\tilde{M}}(V_M/Y_M)}(K)) = 1$, we even get that V_M/Y_M is a direct sum of two natural modules for K. Let L be the point stabilizer in K on the natural module. Then we have that L is covered by M^g and as $Y \leq O_2(M^g)$ and normal in $M \cap M^g$, we see that the projection of Y onto L is in $O_2(L)$ and so $Y = O_2(L) \leq K$. But as V_M/Y_M is a direct sum of two modules, we now get $|C_{V_M/Y_M}(Y)| = q^2$, contradicting $C_{V_M/Y_M}(Y) = Y_P/Y_M$ by 15.7 and $|Y_P/Y_M| = q$. □

16 Proof of the Theorem

In this final chapter we collect the results of this paper to prove the main theorem.

We have a uniqueness group M and a Sylow 2–subgroup S of M which is also a Sylow 2–subgroup of G. We assume that there is at least one further maximal 2–local subgroup in G containing S. By 6.17 we get that $F^*(M) = O_2(M)$. Further by 7.3 also $N_G(S) \leq M$. By 8.14 for any 2–local subgroup H containing S we get $F^*(H) = O_2(H)$. Then by 9.1 there is a unique uniqueness group containing S. Now we define $M_0 = N_M(S \cap C_M(Y_M))$. Then 10.5 shows that there are at least two maximal 2-locals containing M_0. Starting with such a H such that

(1) $H \not\leq M$

(2) $C_H(O_2(H)) \leq O_2(H)$

(3) Y_H is maximal with respect to (1) and (2)

(4) $M \cap H$ is maximal with respect to (3)

(5) H is maximal with respect to (1) - (4)

we define in 11.4, 11.5 and 11.6 certain groups P relative to H which are minimal with respect containing M_0 but not be contained in M. These groups are called nice. In 12.28 we show the existence of such a nice P if $Y_H \leq O_2(M)$ and in 13.8 if $Y_H \not\leq O_2(M)$. A nice P is one of the following groups

P contains S but $P \not\leq M$ and one of the following holds

(1) $E(P/C_P) \cong L_2(q^2)$ and Y_P is the orthogonal module.

(2) $E(P/C_P) \cong L_2(q) \times L_2(q)$ and Y_P is the $\Omega^+(4, q)$–module.

(3) $E(P/C_P) \cong L_2(q)$ or $P/C_P \cong \Sigma_3$ and Y_P is a sum of natural modules.

(4) $E(P/C_P) = K_1 \times K_2$, $K_1 \cong K_2 \cong A_5$, $Y_P = V_1 \times V_2$, where $[K_i, Y_P] = V_i$ and $[K_{3-i}, V_i] = 1$. Further V_i is the orthogonal K_i–module and K_1 is not normal in P/C_P.

(5) $P/O_2(P) \cong \Sigma_3 \wr Z_2$ or $\Sigma_3 \times \Sigma_3$ and Y_P involves just orthogonal modules and at most three of them.

(6) P/C_P is an extension of a cyclic group of order $q^2 - 1$ by Galois automorphisms and P acts semiregularly on Y_P, with an element of order $q - 1$ in M.

(7) P/C_P is an extension of a cyclic group of prime order greater than three, which acts semiregularly on Y_P, Further $Y_P = Y_M \times Y_M^t$ for some $t \in P$.

In (1) - (5),(7) the group P is minimal with respect not to be in M.

Then we make the following definition.

We define a group $\tilde{Y_P}$. In the cases (3),(6) and (7) we just set $\tilde{Y_P} = Y_P$. If we are in (1) or (2) then let $\tilde{Y_P}$ be the preimage of $C_{Y_P/Y_M}(S \cap E(P/C_P))$. In case (5) let $\tilde{Y_P}$ the group generated by the commutators of the transvections in S. In case (4) let $\tilde{Y_P} = C_{Y_P}(S \cap E(P/C_P))$.

Now set

$$V_M = \langle \tilde{Y_P}^M \rangle.$$

Suppose that $C_M(V_M)$ contains a good E. As $N_G(Y_P) \not\leq M$, we get that P is not as in (3), (6) or (7). Set $\tilde{P} = \langle C_P(x) \mid 1 \neq x \in \tilde{Y_P} \rangle$. In case of (4) or (5) we have $P = \tilde{P}S$, a contradiction. In case (1) and (2) we have always some element $y \in \tilde{Y_P} \setminus Y_M$ whose centralizer in P/C_P involves $L_2(q)$. Hence $\langle \tilde{P}, S \rangle = P$ by minimality. So by 5.11 we have $m_p(C_M(V_M)) \leq 1$. Let $T \leq S$ such that $S \cap C_M(V_M) \leq T$ and $TC_M(V_M)/C_M(V_M) = O_2(M/C_M(V_M))$. Set $\hat{M} = N_M(T)$. Then we have with 2.5 that $\hat{M}$ contains some good E. So

$$O_2(\langle \hat{M}, P \rangle) = 1.$$

We have $C_M(V_M)T \leq C_M$. Hence we get that $Y_M = Y_{\hat{M}}$.

Then we study the amalgam $\Gamma(\hat{M}, P)$ and show in 14.36 that $b = b_\Gamma = 2$. Then 15.20 gives the final contradiction.

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