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A Combinatorial Description of Knotted Surfaces and Their Isotopies

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We discuss the diagrammatic theory of knot isotopies in dimension 4. We project a knotted surface to a three-dimensional space and arrange the surface to have generic singularities upon further projection to a plane. We examine the singularities in this plane as an isotopy is performed, and give a finite set of local moves to the singular set that can be used to connect any two isotopic knottings. We show how the notion of projections of isotopies can be used to give a combinatoric description of knotted surfaces that is sufficient for categorical applications. In this description, knotted surfaces are presented as sequences of words in symbols, and there is a complete list of moves among such sequences that relate the symbolic representations of isotopic knotted surfaces.

1. INTRODUCTION

Algebraic and categorical descriptions of knot diagrams have played key roles [28] in classical knot theory since the discovery of the Jones polynomial [15]. In higher dimensions, diagrammatic descriptions of knotted surfaces that generalize classical knot diagrams, their Reidemeister moves, and braid theories have been made by several authors [17, 16, 4, 25, 31].

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The purpose of this paper is to give algebraic and categorical interpretations of knot diagrams in dimension 4.

We remind the reader of the papers by Fischer [11] and Kharlamov–Turaev [19]. In Fischer's thesis, the axioms of a certain type of 2-category are given. In these axioms some obvious relations are pre-supposed. In [19] the problem of composition is discussed in relation to these axioms. Meanwhile, Baez and Neuchl [1] have given a definition of a braided monoidal 2-category that serves as an alternative to the Kapranov–Voevodsky [18] axioms. Moreover, they have constructed a 2-categorical analogue of the quantum double [10]. For further studies of categorical structures of knotted surfaces, we need to have moves that explicitly include a height function in each still of movie descriptions of knotted surfaces.

Recall that categorical/algebraic descriptions for classical knots were obtained by fixing a height function on the plane into which a given knot is projected (this will be reviewed in Section 2). In this case, the three Reidemeister moves were augmented by two moves that take into consideration the height function of the plane.

For knotted surfaces a *movie description* is obtained when a height function is fixed on the 3-space into which a given knotted surface is projected. The height function is regarded as the time direction in the movie. We [3, 4] generalized the Reidemeister moves for knotted surfaces that were obtained by Roseman [25] to the case when there is a height function in 3-space. This generalization will be reviewed in Section 3.3.2.

To obtain categorical/algebraic descriptions of knotted surfaces, we will fix a height function on each cross section (called a *still* of a movie), and we will diagrammatically describe the interchange between distant critical levels of the height function. In this case the diagrammatic changes that occur between stills have more variety as do the diagrammatic moves that describe the isotopies.

1.1. *Organization*

The paper is organized as follows.

In Section 2, we discuss the classical Reidemeister theory of knot diagrams. The set of Reidemeister moves must be augmented when a height function is fixed on the plane into which a knot is projected. In the classical case, we have three types of moves to diagrams: (1) those that change the topology of the underlying graph (these are the Reidemeister moves); (2) those in which the topology of the underlying graph is unchanged but the local configuration of crossings and critical points of the height function changes; (3) those that involve interchanging distant critical points. Each of the diagrammatic moves can be interpreted cinematically as the local picture of a surface in 4-space.

In Section 3, we develop the known theory of knotted surface isotopies in analogue to the classical theory. The moves to diagrams are the Roseman moves; these affect the topology of the underlying diagram. Then we project a knotted surface diagram to a plane to obtain a chart description (in the sense of Kamada) of the surface. We list a sufficient set of moves to charts in Theorem 3.2.3. The moves to diagrams on which a height function is fixed are the movie moves of [3, 4]. The moves to diagrams on which a height function is fixed in each still form an augmentation to the set of movie moves. In Section 3.5, we give a combinatorial description of the knotted surfaces and their equivalences that should be suitable for categorical applications.

In Section 4, we show how to prove that each of the lists that have been compiled form a sufficient set of moves as the diagrams become more restricted. The idea of the proof is to interpret each of the moves as a codimension 1 singularity and then to use singularity theory to classify these.

In Section 5, we give an overview of the 2-categorical structure that will arise from the description given here. The axiomatization of this structure is being worked out by Baez and Langford.

2. THE CLASSICAL THEORY OF KNOT DIAGRAMS AND REIDEMEISTER MOVES

We discuss the Reidemeister moves, their algebraic interpretation, and their interpretation as local pictures of surfaces embedded in 4-dimensional space.

2.1. *Classical Knot Diagrams*

A *classical knot* is an embedded circle $K\colon S^1 \to \mathbf{R}^3$ in 3-space. The image $K(S^1)$ is projected generically into a plane Π^2. The projection is *generic* in the sense that a finite number of transverse intersections of arcs occur, and these intersections are isolated double points. The three elementary Reidemeister moves are exemplified in the top three pictures of Fig. 1 with one possible choice of crossing indicated in the figures. (We leave the reader to draw the other choices of crossings.) The Reidemeister moves are moves to *knot diagrams*—projections of knots into Π^2 that have crossing information indicated at the double points. These moves can be considered as surfaces properly mapped into $\Pi^2 \times I$ by regarding the strings to trace out a continuous surface as they move in $\Pi^2 \times I$. The boundaries of the surface at $\Pi^2 \times \{0\}$ and $\Pi^2 \times \{1\}$ are strings before/after the move respectively. The intersection of the surface with an interior plane, say

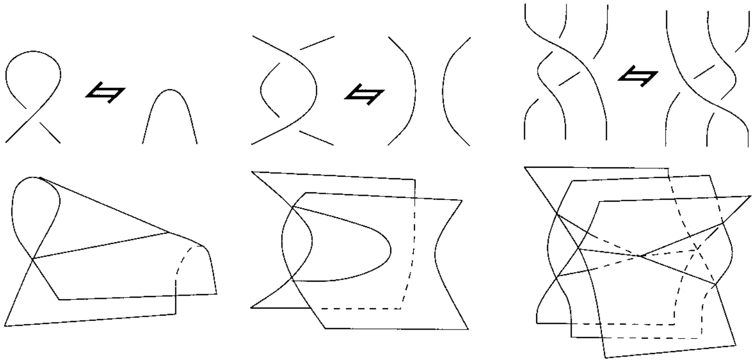

FIG. 1. Reidemeister moves and surfaces.

$\Pi^2 \times \{1/2\}$, contains a singularity or a Morse critical point of the self intersection set of the surface.

The singularity is a branch point if the corresponding Reidemeister move is of type I, it is a point of tangency of the double point curve if the move is of type II, and it is a triple point if the move is of type III. Figure 1 illustrates this relation.

2.2. *Reidemeister Moves with Height Functions*

In this section we review how height functions were used in classical knot theory to obtain categorical/algebraic descriptions.

Consider a classical knot, $K\colon S^1 \to \mathbf{R}^3$, a generic projection, $p_1\colon S^1 \to \Pi^2$, and the corresponding diagram D. In the plane Π^2 we choose a projection, $p_2\colon \Pi^2 \to L$, onto a line, L, such that the composition $g = p_2 \circ p_1 \circ K$ satisfies the following general position assumption:

1. The critical points of g are all Morse singularities and they each occur at distinct levels.

2. The crossing points of p_1 project to distinct levels and these levels are distinct from the critical levels of g.

In this case Morse critical points are maximal and minimal points. A knot diagram with such a projection p_2 is illustrated in Fig. 2. The following result is well known:

2.2.1. THEOREM. *Two knot diagrams with height function are isotopic if and only if one can be obtained from the other by a finite sequence of the Reidemeister moves that are illustrated in Fig.* 1, *the moves illustrated in* 3, *the variants of these figures obtained by other choices of crossing, their mirror images with respect to the horizontal and vertical axes, and moves in which the relative heights of distant critical points are interchanged.*

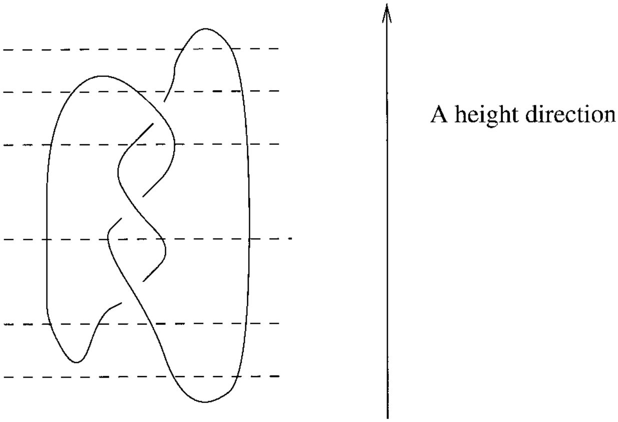

FIG. 2. A knot diagram with a height function.

Next we review how this theorem has categorical interpretations.

2.3. *Categorical Interpretations of Knot Diagrams*

A *strict braided monoidal category* is a category $\mathscr{C}$ that satisfies the following conditions:

1. There is an associative covariant functor

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}.$$

2. There is a distinguished object, **1**, such that

   (a) for any object $V \in \text{obj}\,\mathscr{C}$

$$V \otimes \mathbf{1} = V = \mathbf{1} \otimes V,$$

and

   (b) for any morphism $f: V \to W$

$$f \otimes \text{id}_{\mathbf{1}} = \text{id}_{\mathbf{1}} \otimes f = f.$$

3. There is a natural family of isomorphisms

$$R: V \otimes W \to W \otimes V$$

such that

$$R_{U,\, V \otimes W} = (\text{id}_V \otimes R_{U,\, W}) \circ (R_{U,\, V} \otimes \text{id}_W)$$

and

$$R_{U\otimes V,\,W}=(R_{U,\,W}\otimes \mathrm{id}_V)\circ(\mathrm{id}_U\otimes R_{V,\,W}).$$

One can show that the braiding, R, satisfies the Yang-Baxter condition:

$$\begin{aligned}(\mathrm{id}_W\otimes R_{U,\,V})\circ(R_{U,\,W}\otimes \mathrm{id}_V)\circ(\mathrm{id}_U\otimes R_{V,\,W})\\ =(R_{V,\,W}\otimes \mathrm{id}_U)\circ(\mathrm{id}_V\otimes R_{U,\,W})\circ(R_{U,\,V}\otimes \mathrm{id}_W).\end{aligned}$$

The category is said to be *pivotal* if it satisfies the following conditions: For each object V there is a dual object V^* and morphisms

$$b_V\colon \mathbf{1}\to V\otimes V^*$$

and

$$d_V\colon V^*\otimes V\to \mathbf{1}$$

such that

$$\begin{aligned}(\mathrm{id}_V\otimes d_V)\circ(b_V\otimes \mathrm{id}_V)&=\mathrm{id}_V,\\ (d_V\otimes \mathrm{id}_{V^*})\circ(\mathrm{id}_{V^*}\otimes b_V)&=\mathrm{id}_{V^*},\\ (\mathrm{id}_U\otimes d_V)\circ(R_{V^*,\,U}\otimes \mathrm{id}_V)&=(d_V\otimes \mathrm{id}_U)\circ(\mathrm{id}_{V^*}\otimes \bar{R}_{V,\,U})\end{aligned}$$

as maps $V^*\otimes U\otimes V\to U$. We assume that $V^{**}=V$, and the composition

$$(\mathrm{id}_V\otimes d_{V^*})\circ(R_{V^{**},\,V}\otimes \mathrm{id}_{V^*})\circ(\mathrm{id}_{V^{**}}\otimes b_V)$$

gives the identification. (The category is *rigid* if the last condition is dropped.)

As in [28], for example, the axioms for a braided monoidal category have graphical interpretations that correspond to the Reidemeister moves—or conglomerations thereof. The graphical calculus pictures in [28] indicate the interpretations that we summarize. The map R corresponds to a crossing, the map b corresponds to a minimum, and the map d corresponds to a maximum point of the diagram. Arcs in the diagram that have no critical points, correspond to identity mappings. The Yang–Baxter relation corresponds to the Reidemeister type III move. The invertibility of R corresponds to the type II move. The identification between V^{**} and V corresponds to the type I move. The identities that are satisfied by b and d correspond to the moves introduced in Fig. 3.

To complete our discussion of the relationship between categories and knot diagrams, we recall the following theorems of Freyd and Yetter [12] and Turaev [27].

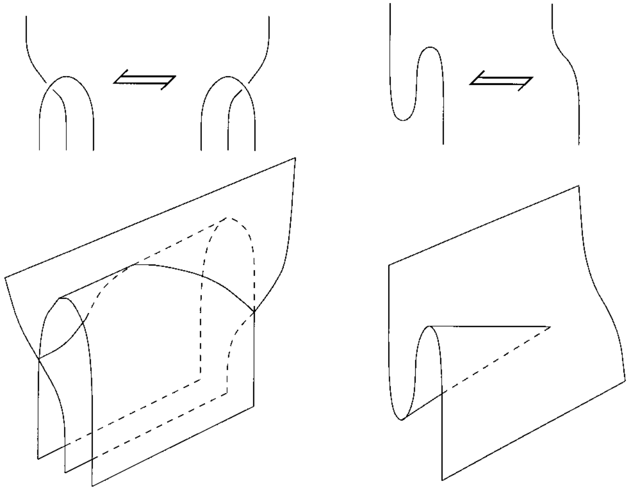

FIG. 3. Additional Reidemeister moves with height functions.

2.3.1. THEOREM [12]. *The category of regular isotopy classes of oriented tangles is the free braided (strict) rigid category on one object generator.*

2.3.2. THEOREM [27]. *The category of ambient isotopy classes of tangles is the free pivotal braided monoidal category on one self-dual object.*

2.4. *Singularities and Additional Moves*

In this section we review how we prove the sufficiency of two additional moves when a height function is present. We have already observed that Reidemeister moves are obtained by examining Morse critical points of crossing points of one dimensional higher knot diagrams. The additional moves are derived from other types of singularities. Here we give two figures indicating how these two moves are related to cusps and folds of mappings from 2-manifolds to the plane.

Just as the three Reidemeister moves have interpretations as surfaces embedded in 4-space, so do the moves that are introduced in Fig. 3. The figure indicates that moving an arc over a maximum point corresponds to the transverse intersection of a fold line and a double point arc. The cancellation of a local maximum and local minimum corresponds to a cusp singularity of the fold lines. We combine the height function on the plane Π onto which the knot is projected with the time direction of the isotopy to obtain a projection of the knot times an interval onto a plane. The singularities of this projection are the fold lines that are traced out by the maximal and minimal points and the cusps. The critical points of the

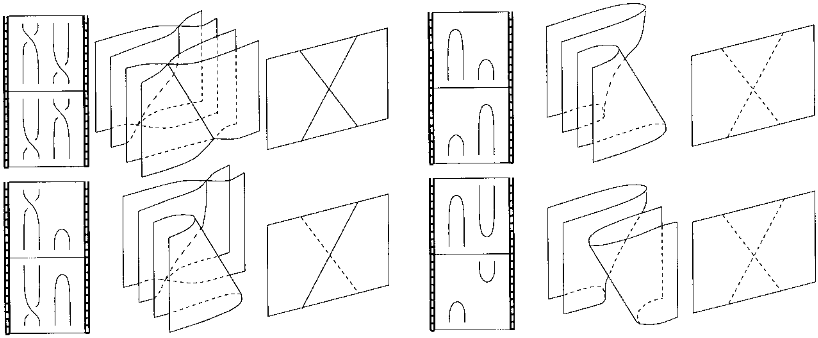

FIG. 4. Interchanging crossings and critical points.

multiple point set correspond to the Reidemeister moves—these are critical points in the time direction of the isotopy.

Thus in classical knot theory the moves that are used to isotope knots correspond to Morse critical points and singularities of surface projections.

2.5. *Exchanging Critical Points*

When a knot diagram is interpreted algebraically or categorically, the diagram represents a sequence of symbols. Similarly, braid theory gives the knot as the closure of a word in the braid group. In the latter case the knot is given as a sequence of braid generators, and a complete set of relations among the generators is known.

When we use a height function to describe the knot diagrams, there are explicit relations between crossings and critical points, distant crossings, and distant critical points. These relations are found by looking at the plane that has the interval factors (the interval onto which the diagram is projected) times (the time direction in the isotopy). Indeed, the distant critical points and crossing points trace out lines in this plane, and these lines cross as their height levels are exchanged. In Fig. 4 we have indicated these exchanges, and their interpretation as surfaces traced out during the isotopy. (In the figure we have not included intermediate arcs that may be present. For example, if $|i-j|>2$, the braid generators σ_i and σ_j in the top left hand side of the picture will be separated by a number of vertical strings, and the corresponding surfaces will be separated by as many walls.) The film strip icon will help us interpret these exchanges in the sequel.

3. DIAGRAMS AND SYMBOLIC REPRESENTATIONS OF KNOTTED SURFACES

In this section a complete symbolic representation of knotted surfaces will be given so that the category representing the surfaces can be defined.

The section is organized as follows. We recall the definitions of crossings and their lifts to the abstract surfaces. We review the Roseman Theorem and the Movie Move Theorem. We discuss putting a height function on the stills in a movie, and we present a list of moves that are sufficient for surface isotopies in that setting. Finally, we discuss the interchange of distant critical points, and we show how to interpret these interchanges graphically. The graphical interpretation gives rise to a notion of charts and chart moves that generalize those given by Kamada in the case of surface braids [17].

In Section 3.5, we use the graphical interpretation to give a combinatorial description of knotted surfaces that should suffice for any categorical applications. Proofs of the sufficiency of the sets of moves that we propose in each setting will be postponed until Section 4.

3.1. *Diagrams and Their Isotopies*

We define the crossing points of a knot diagram, and review the Roseman Moves.

3.1.1. DEFINITIONS. Let F be a closed manifold of dimension 2, and let $K\colon F \to \mathbf{R}^4$ denote an embedding. Choose a projection $p\colon \mathbf{R}^4 \to \mathbf{R}^3$ such that the composition $p \circ K$ is a generic map of an 2-manifold into 3-space. The hyperplane, $\mathbf{R}^3$ is a chosen subspace of $\mathbf{R}^4$ so that $K(F) \subset \mathbf{R}^4 \backslash \mathbf{R}^3$. The multiple point manifolds are defined as follows. Let $f = p \circ K$. Let

$$\tilde{C}_r = \{(x_1, ..., x_r) \colon x_j \in F \quad x_s \neq x_t \quad \text{for } s \neq t \;\&\; f(x_1) = f(x_2) = \cdots = f(x_r)\};$$

this is a manifold of dimension $3 - r$. There is a free action of the permutation group Σ_r on $\tilde{C}_r$. The associated r-fold cover

$$D_r = \tilde{C}_r \times_{\Sigma_r} \{1, 2, ..., r\}$$

is called the *r-decker manifold* (double, triple, and quadruple decker manifolds when appropriate). The r-decker manifold is mapped into F via the map $[(x_1, ..., x_r); j] \mapsto x_j$. The quotient $C_r = \tilde{C}_r / \Sigma_r$ is called the r-tuple manifold, and this is mapped into $\mathbf{R}^3$ via the map $f_r\colon [x_1, ..., x_r] \mapsto f(x_1)$. Evidently, the r-to-1 covering space $D_r \to C_r$ factors through these maps. For convenience, we will include branch points in the double point manifold.

Recall that a generic map from a 2-manifold to 3-space has embedded points, double point curves, isolated triple points, and branch points. *A knotted surface diagram* consists of a generic projection of the surface into 3-space together with *crossing information* (defined in the next two sentences) included along the image of the double and triple point manifolds. The sheet of the diagram that is further from the hyper-plane onto which the surface is projected is broken; that is, a small tubular neighborhood of the image of one of the sheets of the double decker manifold is removed from the surface F. At a triple point, this will mean that there is an indication of a top, middle, and bottom sheet. Knotted surface diagrams of surfaces are also called *broken surface diagrams*. See [7] for more details. The local pictures of knotted surface diagrams are depicted in Fig. 5. We may abuse notation and not make the distinction

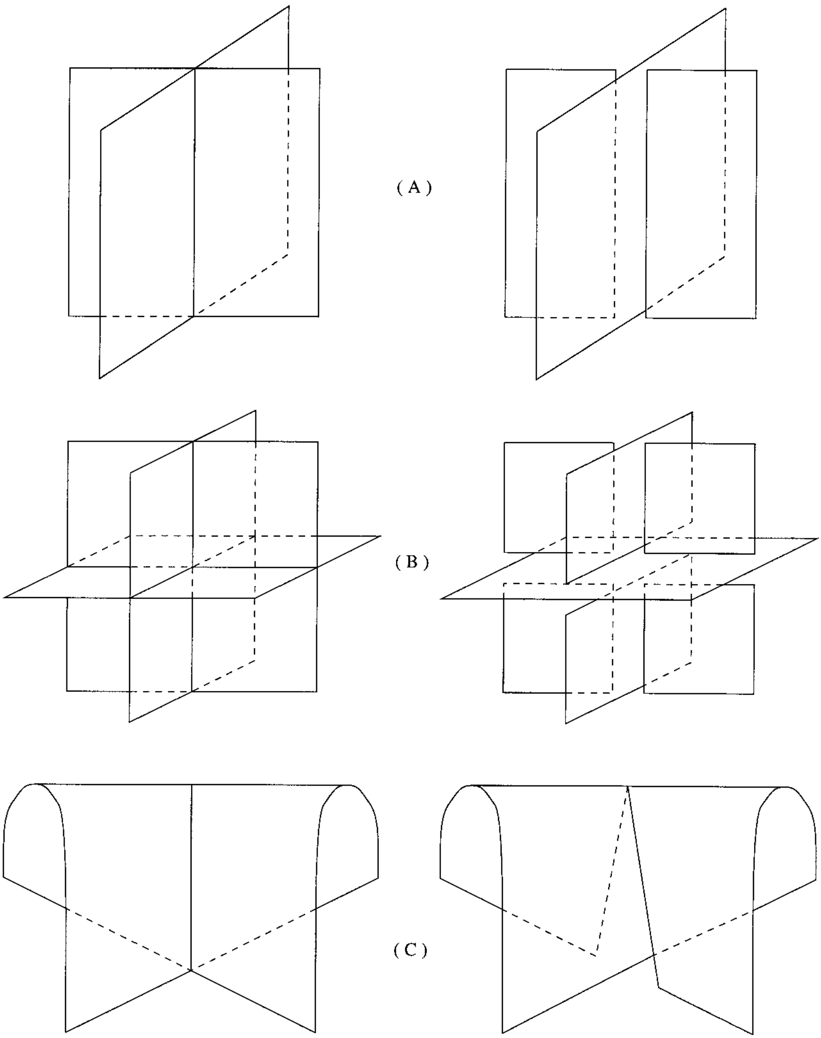

FIG. 5. Projections and broken diagrams of knotted surfaces.

between the diagram and the projection of the knotted surface. In particular, the moves to diagrams will be drawn as moves to projections.

3.1.2. Roseman Moves. For such diagrams of knotted surfaces, Roseman obtained a complete set of moves generalizing the Reidemeister moves. Thus two diagrams represent isotopic knottings if and only if they are related to each other by a finite sequence of moves taken from the Roseman moves that are depicted in Fig. 6.

One proves that the Roseman moves are a sufficient set of moves for knot isotopies, by showing how each move corresponds to a Morse critical

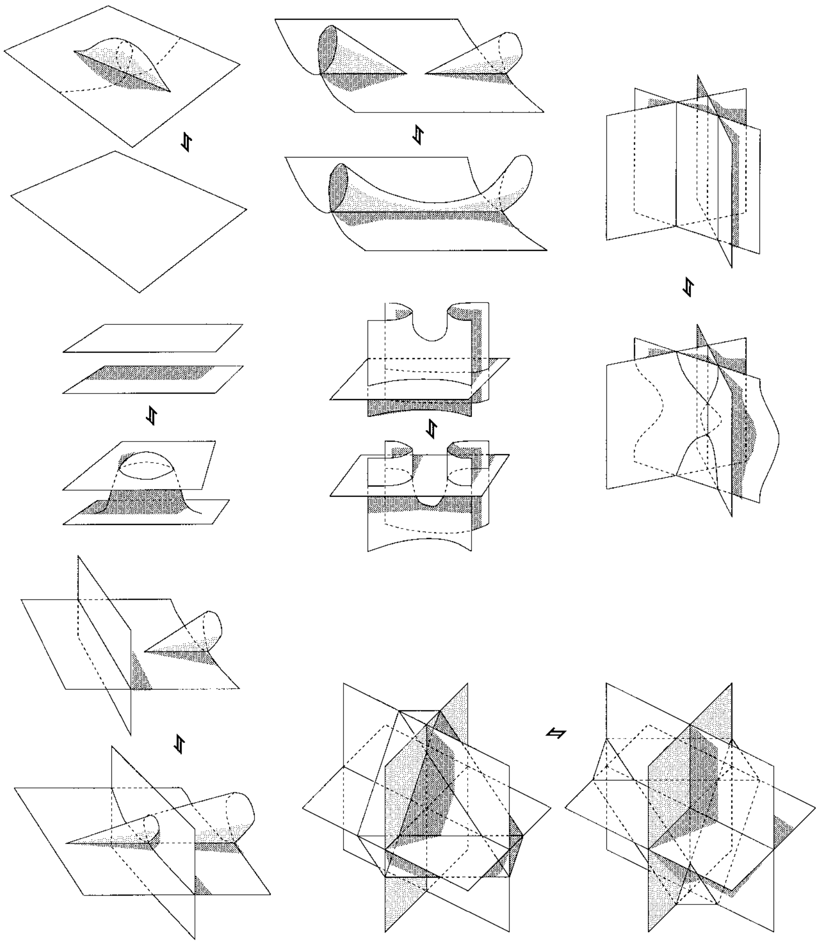

Fig. 6. Roseman moves of knotted surfaces.

point on one of the multiple point sets where the isotopy direction provides a height function. Alternatively, Goryunov [14] has classified the codimension one singularities of stable maps from $\mathbf{C}^2$ to $\mathbf{C}^3$, and the real pictures of the versal unfoldings of these singularities correspond to the Roseman moves.

3.2. *Charts of Knotted Surfaces and Their Moves*

Charts for surface braids were defined by Kamada [17, 16]. Here we define charts for any generic projections of knotted surfaces.

3.2.1. Definition. Consider a surface embedded in $\mathbf{R}^4$, and choose a projection $p\colon \mathbf{R}^4 \to \mathbf{R}^3$ that is generic with respect to the knotting $K\colon F \to \mathbf{R}^4$. We define a *retinal plane* to be a plane, P, in $\mathbf{R}^3$ with a projection $\pi\colon \mathbf{R}^3 \to P$ such that $p \circ K(F) \subset \mathbf{R}^3 \backslash P$.

3.2.2. Definition. Consider the image $\mathcal{I} = \pi \circ p \circ K(F)$ of a generic projection of a given knotted surface in the retinal plane. Let D denote the projections of the double points, triple points, and branch points considered as subsets of $\mathcal{I}$. Assume without loss of generality that the map $\pi \circ p \circ K$ is generic. Let S denote the image of the fold lines and cusps of the generic map $\pi \circ p \circ K$ in $\mathcal{I}$. Without loss of generality assume that D and S are in general position.

Let the *chart*, $C = C(K, p, \pi)$, *of K with respect to p and π*, be the planar graph $D \cup S$ considered as a subset of $\mathcal{I}$ which is further contained in the retinal plane. We label the the chart C according to the following rules.

The image D is depicted by a collection of solid arcs while the image S is depicted by a collection of dotted arcs in our figures. In the figures a thick dotted arc can be either an arc in D or an arc in S.

There are seven types of vertices in the chart C; these vertices correspond to isolated stable singularities of codimension 0.

(1) The projection of a triple point gives rise to a 6-valent vertex. Every edge among the six coming into the vertex is colored solidly.

(2) Each branch point in the projection of the knotted surface $K(F)$ corresponds to a 3-valent vertex. Two of the edges at the vertex are colored as dotted arcs (the fold lines); the other edge is solidly colored (the double arc that ends at the branch point).

(3) Each cusp of the projection π gives rise to a 2-valent vertex in which both edges are colored as dotted arcs.

(4) The projection of a point at which an arc of double point crosses a fold is a 4-valent vertex. Two of the edges at this vertex are solid; the

other two are dotted. A circle in the retinal plane that encompasses such a vertex encounters the edges in the cyclic order (solid, solid, dotted, dotted).

(5) The points of the retinal plane at which the double points cross are 4-valent vertices at which all of the incoming edges are solid.

(6) The points of the retinal plane at which the fold lines cross are 4-valent vertices at which all of the incoming edges are dotted.

(7) The points of the retinal plane at which an arc of D crosses an arc of S are 4-valent vertices at which there are two solid edges and two dotted edges. A circle encompassing the vertex encounters the edges in cycle order (dotted, solid, dotted, solid).

We use the projection of the knotted surface K in 3-space to label the edges of the chart as follows (Fig. 7). Consider a ray R that is perpendicular to the retinal plane. Assume that R is in general position with $p(K(F))$, and assume that the end of the ray lies on an edge E. The edge E is the image of the double point arc or a fold line of $p(K(F))$. Let E' be the preimage (either the double point arc or a fold line). Let m (resp. n) be the number of sheets of $p(K(F))$ that are farther away (resp. closer to) from the retinal plane than E' along the ray. Then the pair of the integers (m, n)

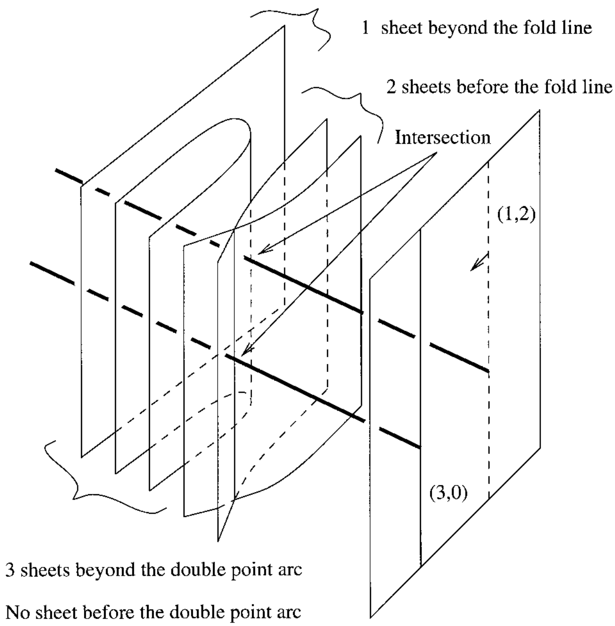

FIG. 7. Labeling of charts.

is assigned to the edge E as a label. The label does not depend on the choice of point along the edge near which the ray R starts.

Furthermore, we indicate a *normal* to fold lines. A fold line is formed by two sheets coming into it. In the retinal plane, one side of the fold line is the image of these sheets. We indicate this side by a normal vector to the edge of the chart that are the images of fold lines.

Next we consider the moves to charts for isotopic knotted surfaces. Specifically, we will prove

3.2.3. THEOREM. *Two charts of the isotopic knotted surfaces are related by the moves depicted in Fig.* 8 *through* 10.

The moves that are depicted in Fig. 11 will be discussed in Section 3.5. In these figures labels and normals are not specified for simplicity.

3.3. KNOTTED SURFACE DIAGRAMS WITH HEIGHT FUNCTIONS. We define height functions for knotted surface diagrams and give the list of movie moves.

3.3.1. DEFINITION. A projection $p_2 \colon \mathbf{R}^3 \to \mathbf{R}$ is *a generic height function for the knotting* if

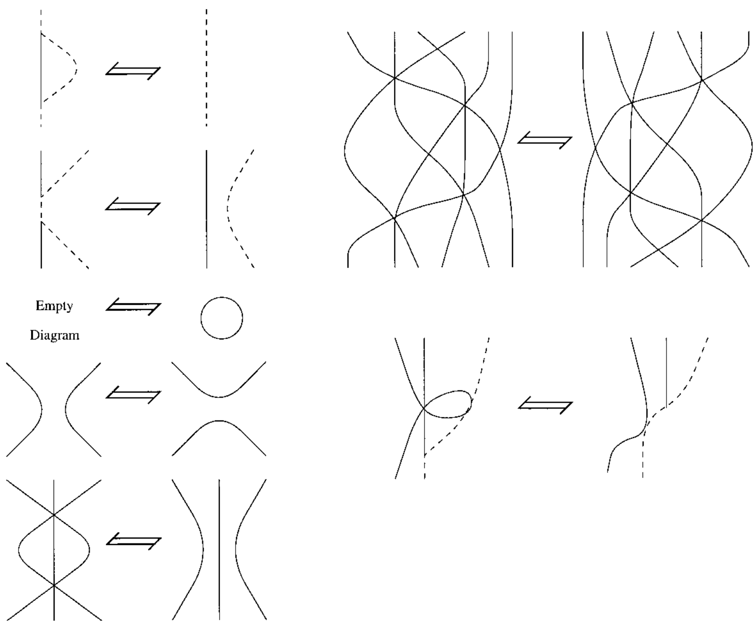

FIG. 8. Chart moves, part I.

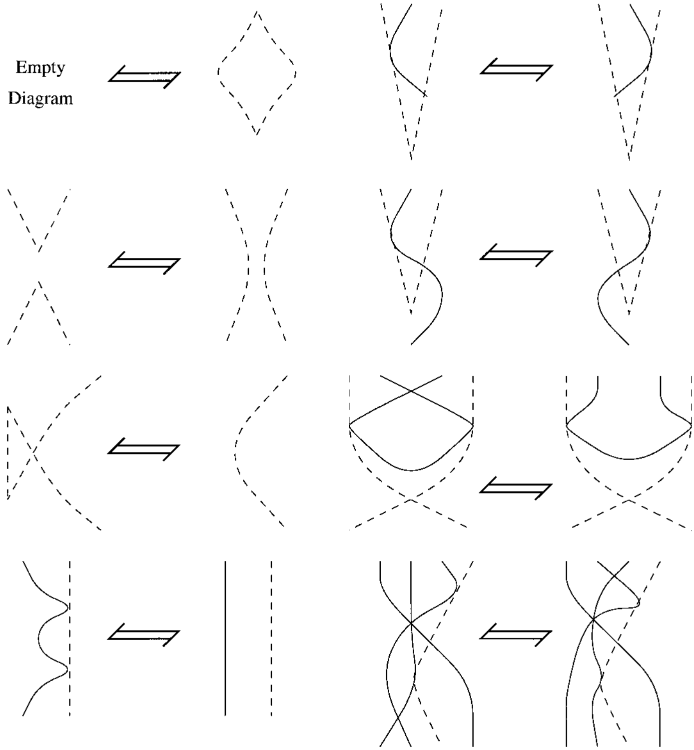

FIG. 9. Chart moves, part II.

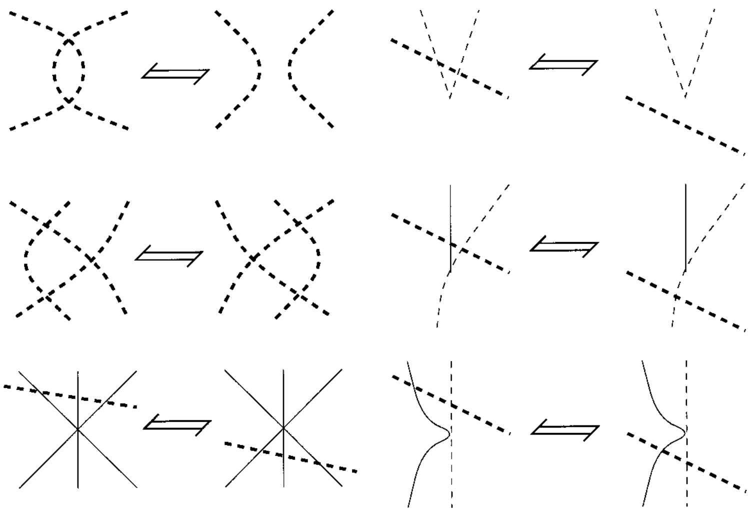

FIG. 10. Chart moves, part III.

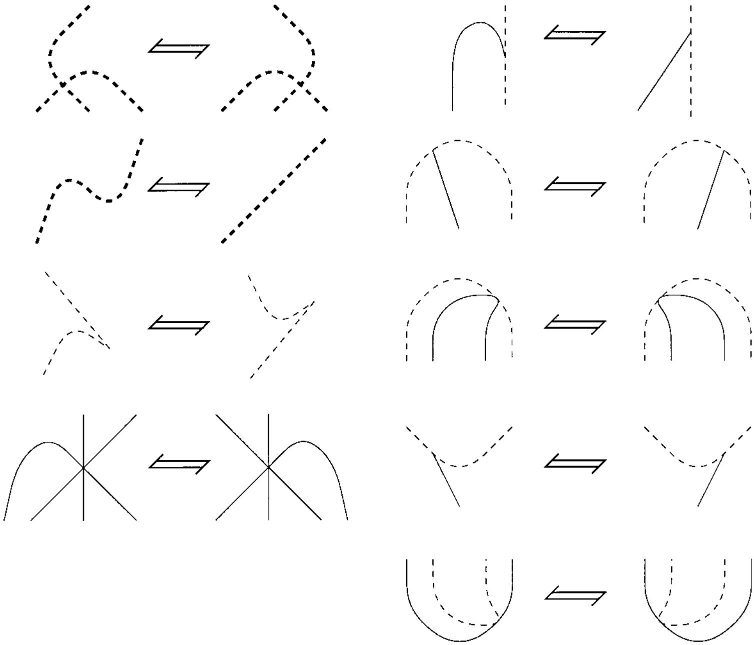

FIG. 11. Chart moves, part IV.

1. $p_2 \circ f_r$ has only non-degenerate critical points for all $r = 1, ..., n+1$, and
2. each critical point is at a distinct critical level of p_2.

Condition (1) for $r = 1$ states that $p_2 \circ f$ has non-degenerate critical points. Here we define critical points to include branch points and triple points. So in condition (2), we have that each critical point of either the manifold or its multiple point set is at a different critical level.

3.3.2. KNOTTED SURFACE MOVIES. *A knotted surface movie* consists of a knotted surface diagram together with a choice of height function for the diagram. The main theorem in [3, 4] is the following:

3.3.3. THEOREM [3, 4]. *Two knotted surfaces movies represent isotopic knottings if and only if they are related by a finite sequence of moves to movies depicted in Figs.* 12, 13, 14 *or interchanging the levels of distant critical points.*

In the illustration of moves to movies we have shown local pictures where the surface is cut between critical levels by a plane and the crossing

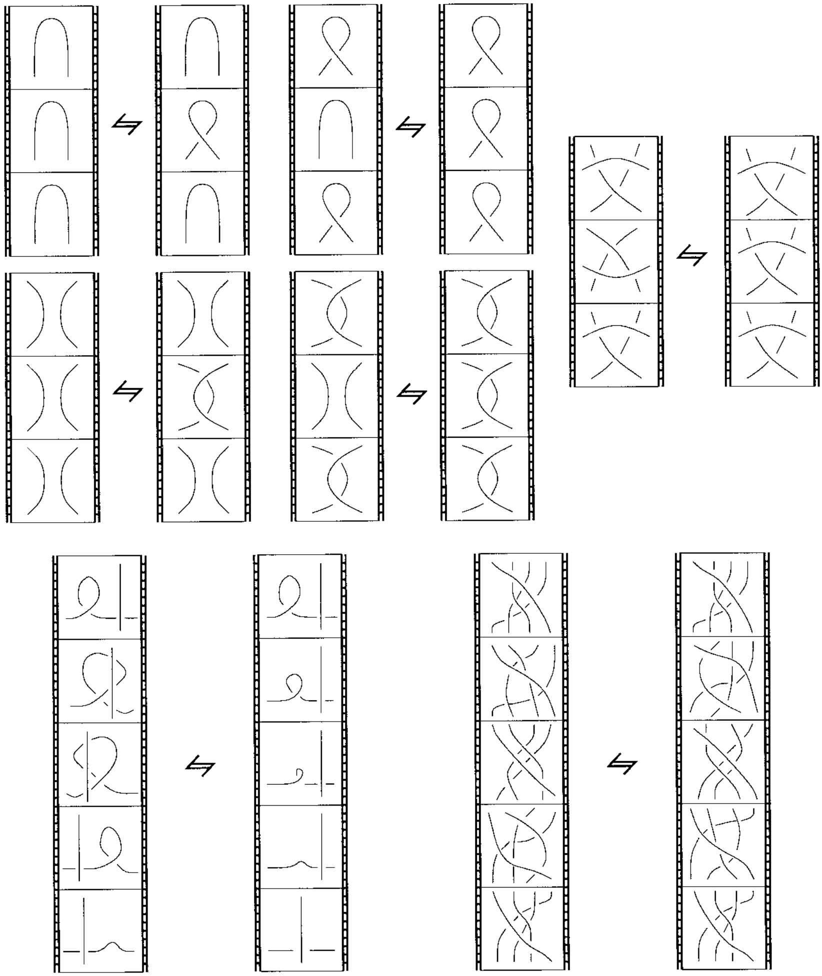

FIG. 12. Movie versions of the Roseman moves.

information is indicated. Thus the stills represent the level sets of the height function. We also remind the reader that only one possible choice of crossing information is indicated as with the classical Reidemeister moves. In the next section, we discuss the need to fix a height function in each of the stills.

3.4. MOVIES FOR WHICH A HEIGHT FUNCTION IS FIXED IN EACH STILL. We begin this section with an example that indicates the geometric need to fix height functions in the stills. Subsequently, we give an overview of the categorical justification of these height functions. We analyze the

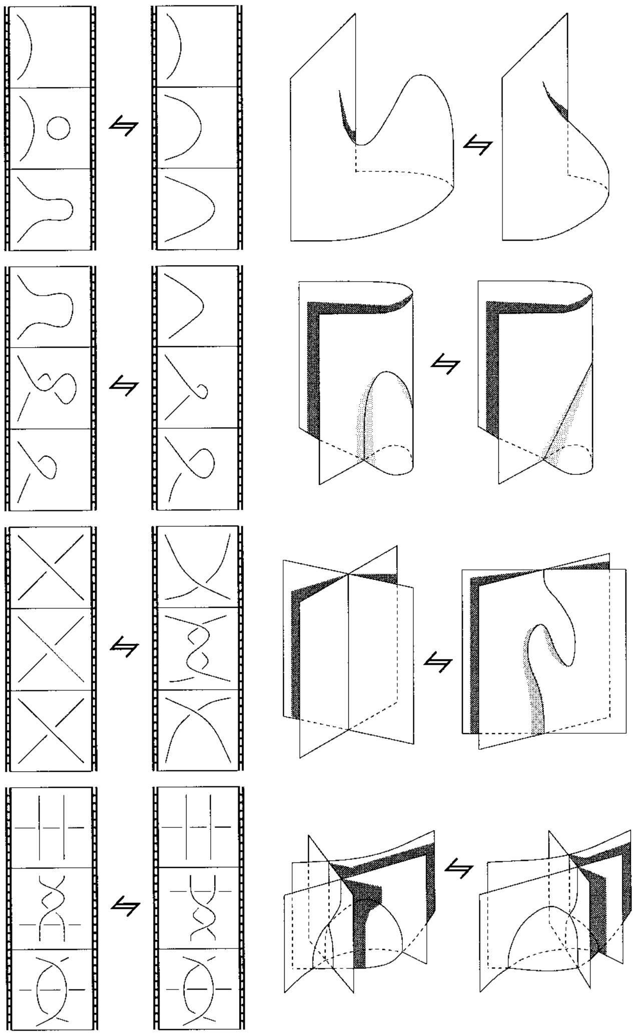

FIG. 13. Movie moves in which the topology of the image is unchanged (1).

projections of the surfaces when a height function is fixed in 3-space and in the stills, and we list a sufficient set of moves (that are an annotation of the movie moves) in the new more restricted setting.

3.4.1. EXAMPLE. Consider the isotopy of the trefoil knot diagram that is depicted in Fig. 15. A diagram with 3-fold symmetry is rotated clockwise through an angle of $2\pi/3$. Of the 3 Reidemeister moves illustrated in the

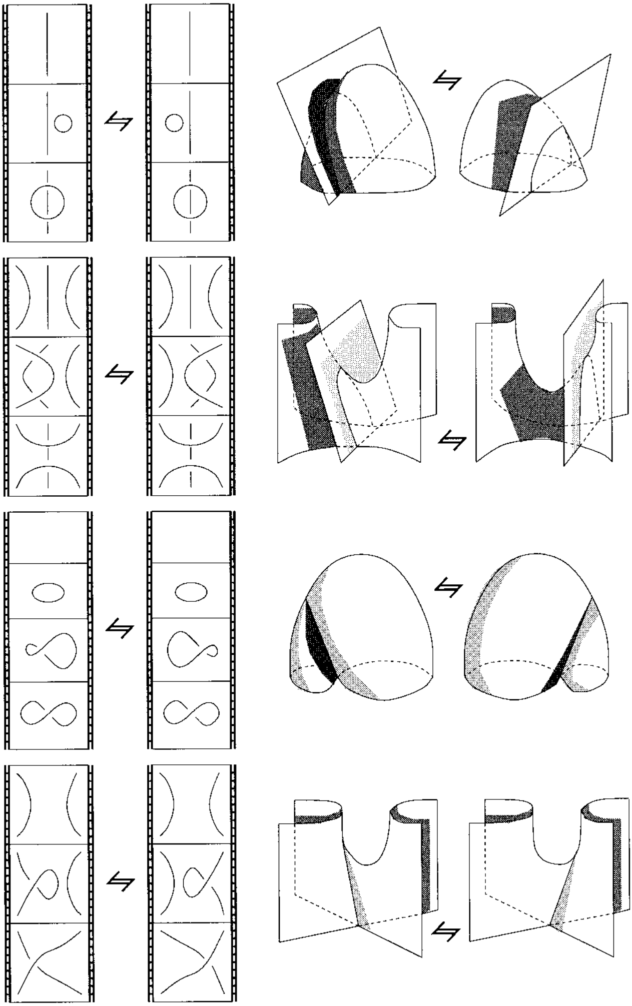

FIG. 14. Movie moves in which the topology of the image is unchanged (2).

introduction, none is employed in this isotopy, but the isotopy clearly changes the diagram so exactly what is happening here?

The diagram of the trefoil was rotated, or equivalently the position of the top of the diagram changed. In Fig. 15 we indicate how the height function is changed by the non-Reidemeister moves depicted in Figs. 16 and 4.

The significance of such a rotation in knotted surface movies is that when this rotation occurs in a movie, it may give a surface which is not

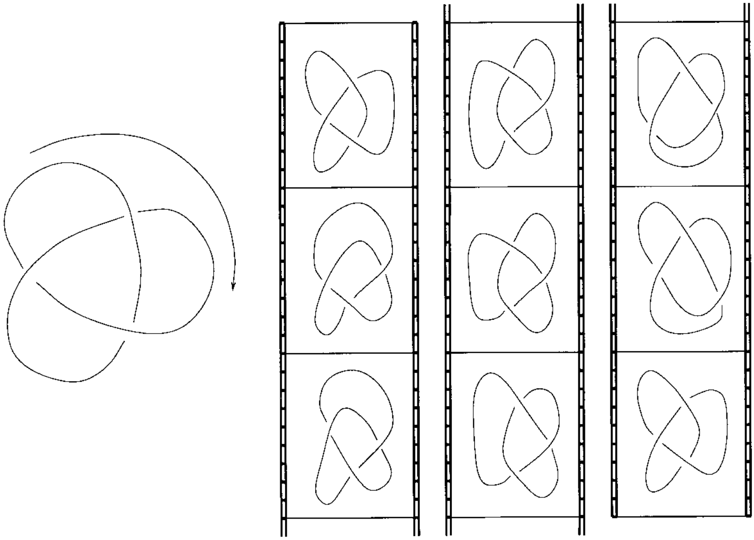

FIG. 15. Rotation of trefoil.

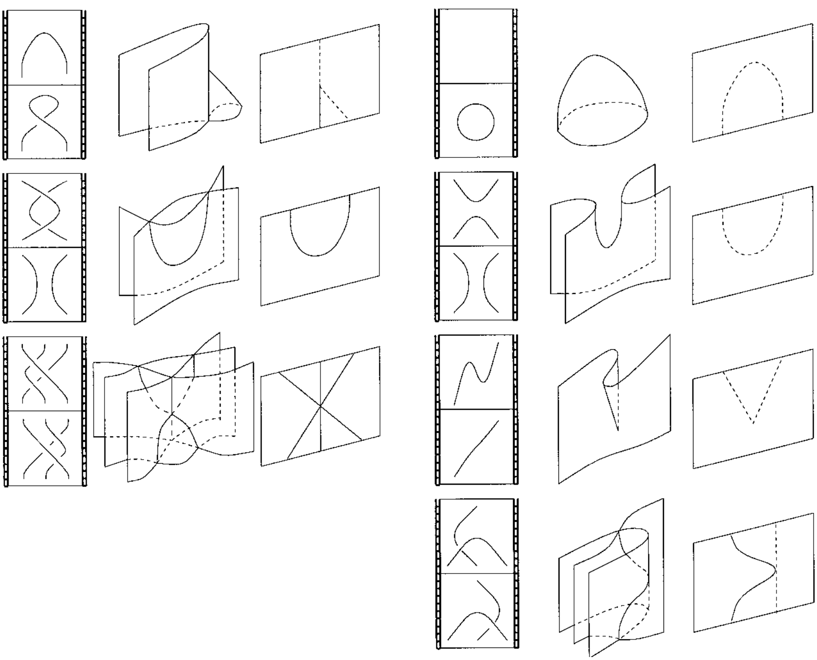

FIG. 16. Elementary string interactions.

isotopic to the surface without the rotation in the corresponding stills. Thus we need to be able to describe such changes in diagrams.

Recall that Fig. 4 included those moves to knot diagrams that involved interchanging distant critical points and crossings. The non-Reidemeister moves in Fig. 16 are local changes to the knot diagrams with height functions. Again the film strip icon is used because we are thinking of the diagrams as non-critical cross sections of a surface.

The scene consisting of a twisted trefoil can occur as a scene in a larger movie. In fact, Roseman uses this scene in his video, "Twisting and Turning in 4-Dimensions" [26] (See also [9]).

3.4.2. Categorical Motives. In Section 5, we will generalize the categorical structure of a braided monoidal category to apply a similar structure to embedded surfaces in 4-space. Here we give an overview of the notion and motivation for a 2-category.

In a 2-category there are objects, morphisms, and 2-morphisms. Objects are symbolized as dots, morphisms are symbolized as arrows that start and end at a specific pair of dots, and 2-morphisms are symbolized as polygons whose vertices are the dots, and whose edges are the arrows. Strictly speaking, a 2-morphism is a 2-gon between two given 1-morphisms, but composition and pasting allows the more general case to be well-defined.

The philosophy of 2-categories is that an equality between a pair of 1-morphisms (or even a similarity or equivalence) should be replaced by a 2-morphism that expresses that equality, similarity, or equivalence.

In the classical case, a knot diagram represents a morphism. (In the representation of the braided monoidal category, it represents a map from $\mathbf{C}$ to $\mathbf{C}$.) In dimension 4, we have a movie description of a knotted surface which we will give as a sequence of classical knot diagrams where there is a height function fixed in each diagram, and a pair of subsequent diagrams differ at most by one of the moves depicted in Figs. 16 or 4. Each still of the movie (one element in the sequence of diagrams) will represent a 1-morphism. Thus we regard the knotted surface as a composition of 2-morphisms in a 2-category. To get this description, then, we will need to fix a height function on each still so that we can associate a 1-morphism to the diagram as in the classical case. For this purpose, we need a complete set of Reidemeister moves when height functions are fixed for each still.

3.4.3. Definition. Let $\pi\colon \mathbf{R}^3 \to \mathbf{R}^2$ be the projection to a retinal plane. The *vertical axis* is defined by a projection $v\colon \mathbf{R}^2 \to \mathbf{R}$ (the image is the up/down axis) and we require that the composition $v \circ \pi$ is a generic height function for the knotting in the sense defined in Section 3.3.1. The *horizontal axis* is defined by a projection $h\colon \mathbf{R}^2 \to \mathbf{R}$ (the image is the left/right axis). The horizontal axis of the retinal plane will be used to define a height function in each of the stills of the knotted surface movie.

Let us examine the singularities of the projection of $K(M)$ onto the retinal plane. First there are the cusp and fold singularities of a surface as classified by Whitney. Second, the double point manifold has singularities, critical points, and crossing points. Since branch points can occur, the double point manifold is a manifold with boundary, and the branch points lie along the fold lines of the projection onto the retinal plane. Other maximal and minimal points of the double point manifold can also occur along the fold lines. Finally, the triple points of the projection are isolated, and these are the three fold intersections of arcs of the double point manifold. Each of these situations is a local and stable phenomenon, and they are illustrated by the drawings in the Fig. 16.

3.4.4. Definitions. Consider the singular levels of the projection of the knotted surface on the retinal plane. Suppose $t = 1/2$ is a singular value on the vertical axis and no other singularities occur for $t \in [-1, 2]$, then we say that the inverse images of the $t = 0$ and the $t = 1$ levels differ by an *elementary string interaction* or *ESI with respect to the movie description with a still height function.*

There are seven basic types of ESIs. They are depicted in Fig. 16. We describe the singularities.

1. When a branch point occurs, it will occur at a fold line, and this is called a *type I Reidemeister move*. The double point arc ends at the fold line.

2. When a maximal point or minimal point occurs on the interior of a double point arc, this is a *type II Reidemeister move*. The pair of strings involved has no fold lines.

3. When an isolated triple point occurs among three double point arcs and there are three sheets of surface intersecting pairwise along these arcs, this is called a *type III Reidemeister move*. Three strings involved have no fold lines.

4. A Morse critical point of the surface F of index 0 or 2 with respect to projection onto the vertical axis is *a birth or death* of an unknotted circle. Small circles at each maximal/minimal point have one maximal and one minimal point with respect to the height function in the still (given by projection onto the horizontal axis).

5. A Morse critical point of index 1 on the surface is a *saddle*. At a saddle point, a single pair of optimal point (one maximum and one minimum) either is introduced or cancelled.

6. A cusp on a fold line is called a *switch back move*.

7. When a double point arc crosses a fold line so that in the projection onto the retinal plane the double point arc crosses the fold line this is called a *camel-back move* or a *ψ-move.*

Of course we include the following variations for the ESIs. Each film can run backward, crossing information can vary, and in cases 1, 6, and 7 we turn both of the stills upside down by reflecting through a central horizontal axis. (The remaining ESIs are symmetric under such a reflection.)

The first five of the above are called *ESIs with respect to the movie descriptions*. They are used in the movie moves. The remaining two ESIs do not change the topological type of the knot diagram (considered as a graph in the plane), but they give local changes to the diagram when the height function is changed. At this point we are not including the interchange of distant critical points to be among the ESIs, but will include these later to obtain a combinatorial description of the knotted surface.

3.4.5. Singularities of Knotted Surface Isotopies. We will examine singularities in the retinal plane as an isotopy of a knotted surface is performed.

Consider an isotopy K_t between knottings $K_0, K_1: F \to \mathbf{R}^4$ for $t \in [0, 1]$. For each t, K_t is an embedding. Recall that $p: \mathbf{R}^4 \to \mathbf{R}^3$ (resp. $\pi: \mathbf{R}^3 \to \mathbf{R}^2$) denotes the projection onto a hyperplane (resp. the retinal plane). The moves that are used to decompose the knotted surface isotopy are codimension 1 singularities. We will "watch" the projection of the isotopy on the retinal plane. If $\mathbf{R}^2$ is the retinal plane, then the isotopy provides a map $\mathscr{K}$ from $F \times [0, 1]$ onto $\mathbf{R}^2 \times [0, 1]$. Here is a list of the local types of changes we see in the fold lines and multiple point sets:

1. Changes in the fold lines;

2. Changes in the positions of the double points and triple points in relation to the fold lines where the changes do not affect the topological type of the double point set and the fold lines remain fixed;

3. Changes in the positions of the double points and triple points in relation to the optimal point of the multiple point sets;

4. Critical points of the multiple point sets in the direction of the isotopy.

Let us describe these more concretely.

(1) There are 5 types of changes in the fold line set that can occur. Elliptical and hyperbolic confluence of cusps are two of these. A fold line can undergo cusp singularity because the vertical direction of the retinal plane provides a height function for the fold lines. In the presence of a nearby saddle point, a cusp can change from pointing downward to

pointing upward. The singular point that serves as the intermediate point is called a horizontal cusp [20]. Finally, there can be a swallow-tail singularity in the fold set. These moves are illustrated in Figs. 17, 18, 19, 20, and 21, respectively.

(2) There are 9 situations in which the double points and triple points change their position in relation to the fold lines. A branch point may pass through an optimum of the surface (Fig. 22) or a saddle point of the surface (Fig. 23); in either case the fold line has a local optimum. A branch point may pass through a cusp of the fold line set (Fig. 24). Similar to these three moves, we have a double point curve passing over a fold-line near a maximum point (Fig. 25), saddle point (Fig. 26), or cusp (Fig. 27); the changes are realized when the point at which the double line passes over the fold intersects the optimal points on the fold line set. A double point curve can pass back and forth over a fold line and this situation is replaced with the double point curve not passing at all over the folds (Fig. 28). There may be a triple point in a neighborhood of a fold line, and the move in this case passes the triple point over the fold (Fig. 29).

Finally, a pair of fold lines may cross, and a pair of double points may pass over the pair of fold lines. By interchanging the relative height of the double points it is possible to interchange the fold-line over which a given arc passes. The two arcs then are connected by a type II move. The singularity that one sees in the retinal plane that connects these two moves occurs when the double arc becomes tangent to the direction of projection; in following the projection of the double arcs, one sees them undergo a type I Reidemeister move in the retinal plane. Fig. 30 contains an illustration.

(3) The optima of the double points can change in a cusp-like fashion (Fig. 31), or a maximum point can be pushed through a branch point (Fig. 32). A triple point can be pushed over a maximum point of the double point set (Fig. 33).

(4) The remaining changes are movie parametrizations of the Roseman moves. Their description as Morse critical points on the multiple point set appears in [4].

In relation to these changes we observe that the changes described in (1) affect only the fold lines. Those changes in (2) affect the relative position of multiple points and fold lines. Those changes in (3) affect the relative height of the multiple points. Those changes in (4) affect the topology of the projection of the diagram.

3.4.6. Theorem. *Let two knotted surface diagrams be given, each as a sequence of ESIs. Then one is obtained from the other by a finite sequence*

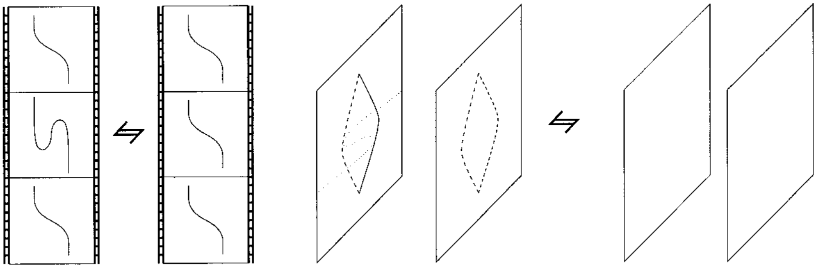

FIG. 17. An elliptic confluence of cusps.

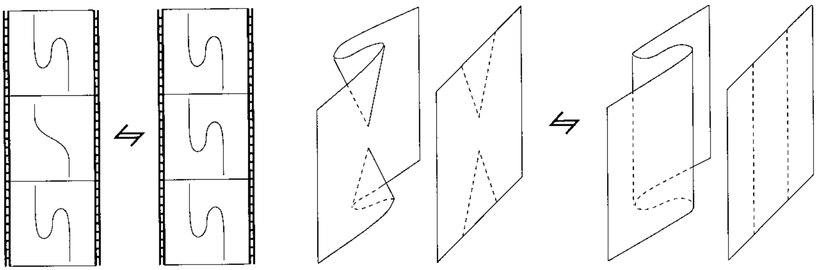

FIG. 18. A hyperbolic confluence of cusps.

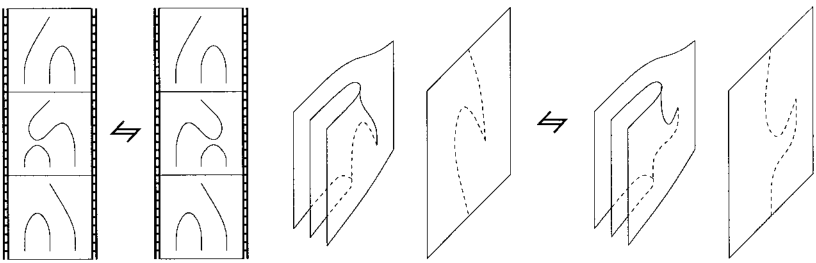

FIG. 19. A cusp on the set of fold-lines.

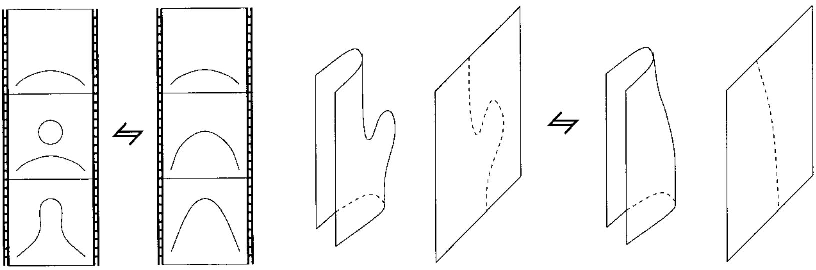

FIG. 20. A horizontal cusp.

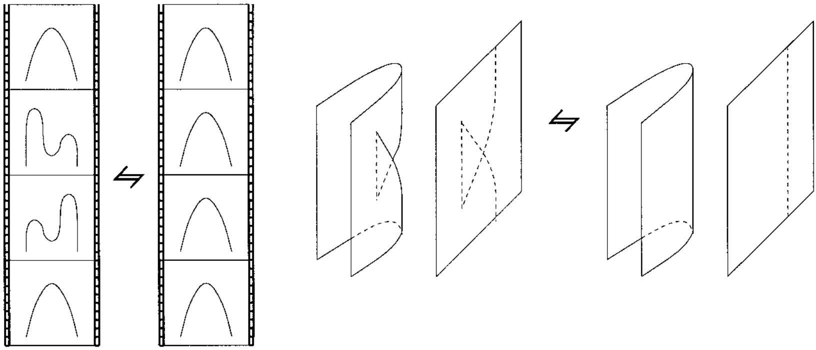

FIG. 21. A swallow-tail on the fold lines.

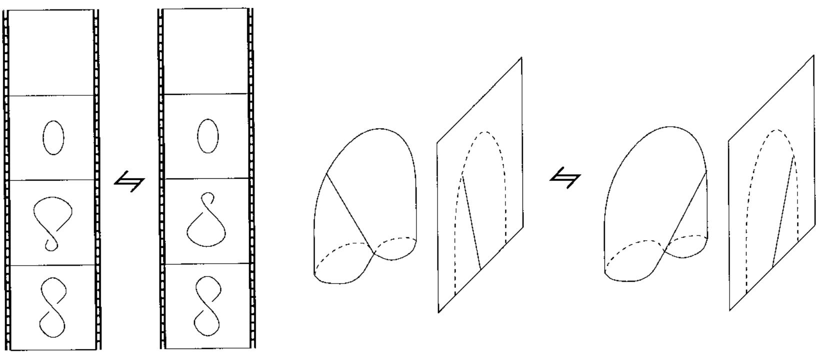

FIG. 22. A branch point passes over a maximum point of the surface.

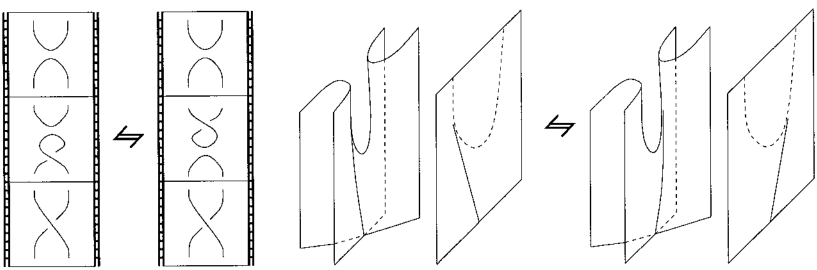

FIG. 23. A branch point passes over a saddle point of the surface.

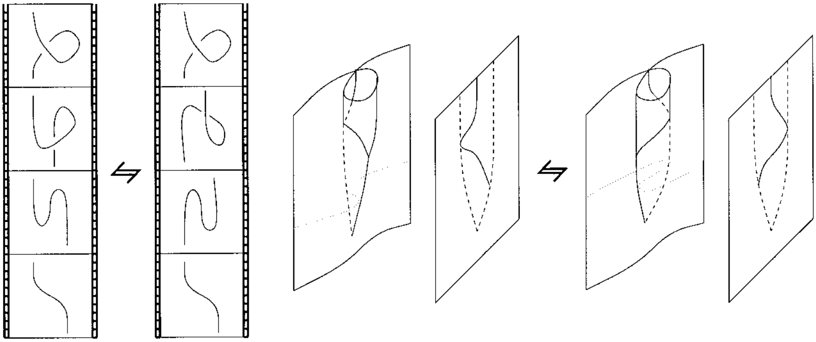

FIG. 24. A branch point passes through a cusp.

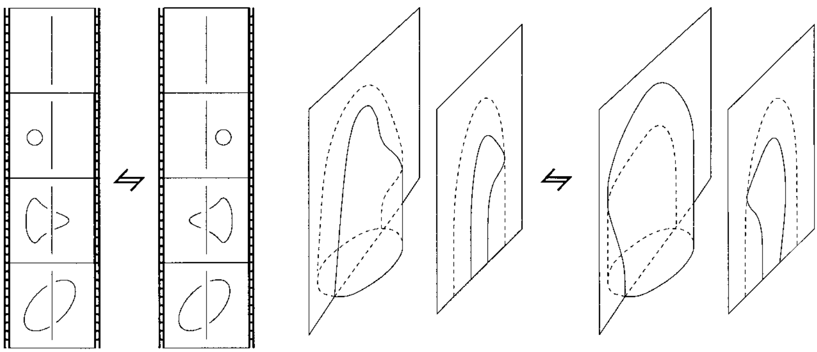

FIG. 25. A double point arc passes over a fold line near a maximum point.

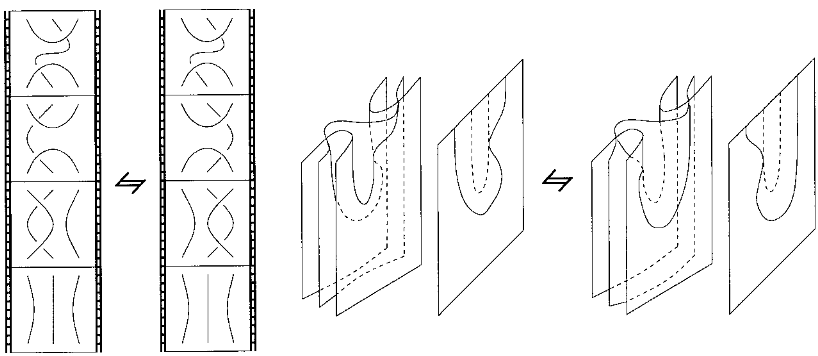

FIG. 26. A double point arc passes over a fold line near a saddle point.

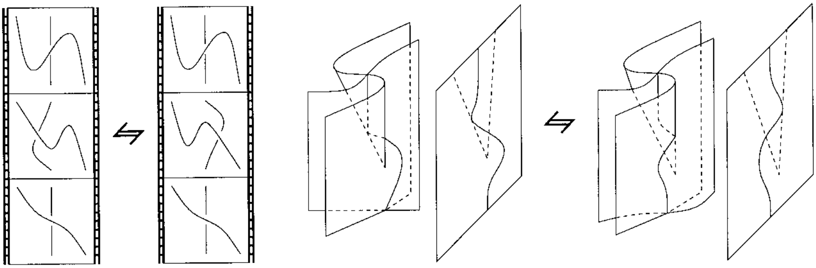

FIG. 27. A double arc passes over a fold line near a cusp.

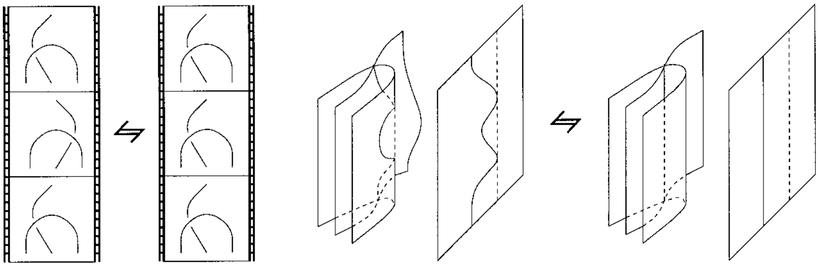

FIG. 28. Removing redundant double points crossing the fold lines.

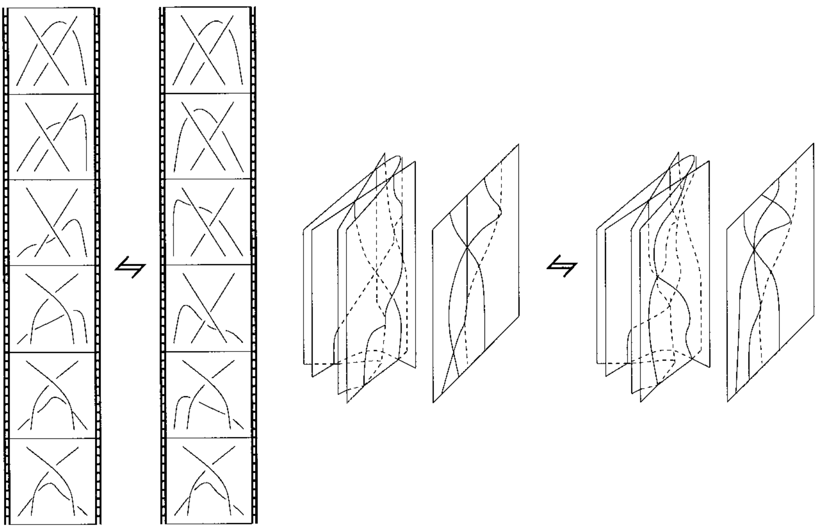

FIG. 29. A triple point near a fold line.

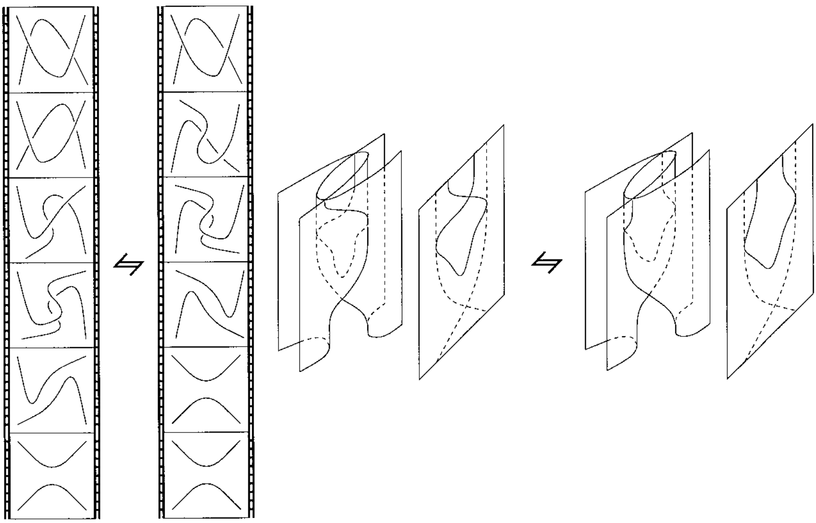

FIG. 30. A double point arc becomes tangent to the line of projection.

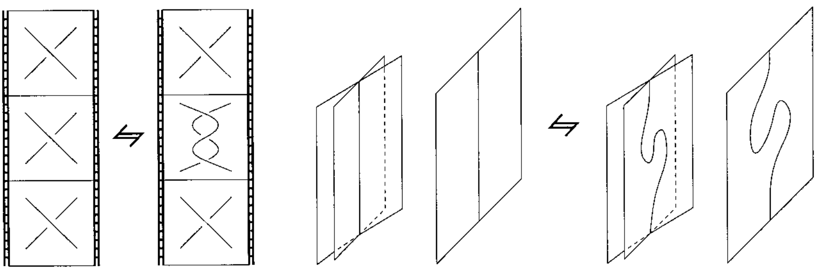

FIG. 31. A cusp on the double point set.

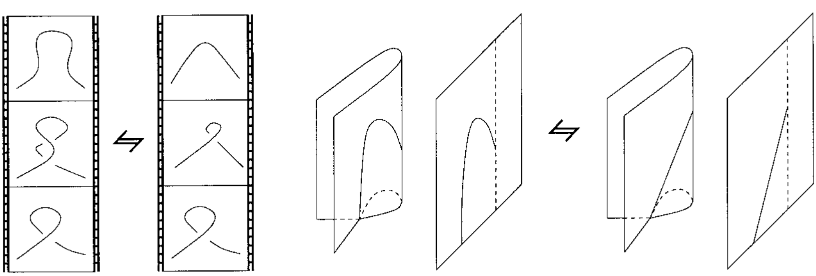

FIG. 32. A maximum point of the double point set being pushed through a branch point.

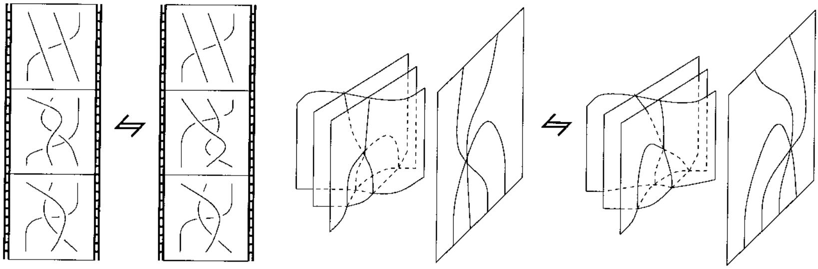

FIG. 33. A triple point passing through a maximum on the double point set.

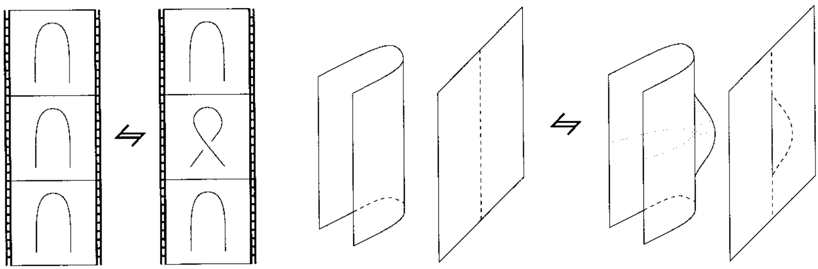

FIG. 34. An elliptic confluence of branch points.

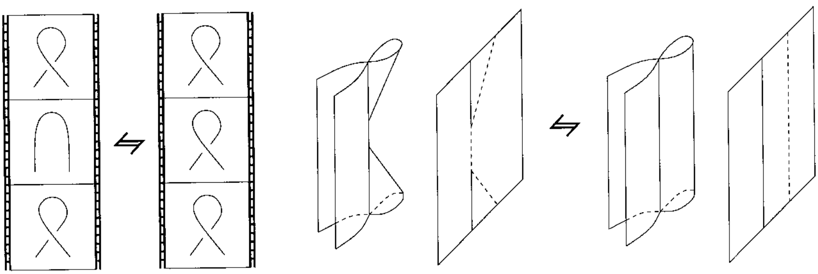

FIG. 35. A hyperbolic confluence of branch points.

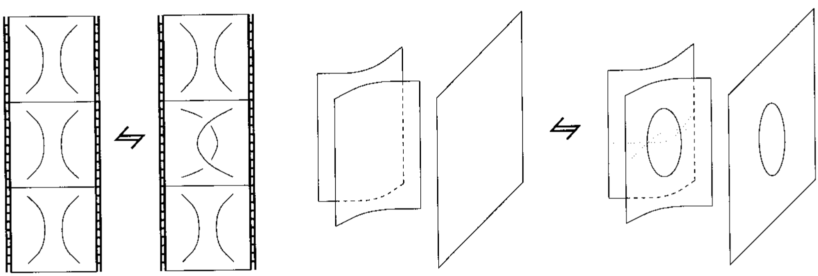

FIG. 36. An elliptic confluence of double points.

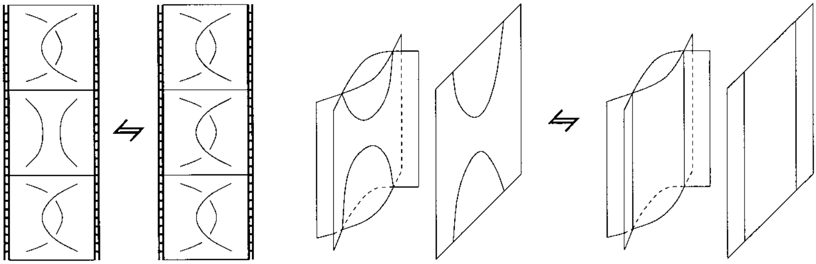

FIG. 37. A hyperbolic confluence of double points.

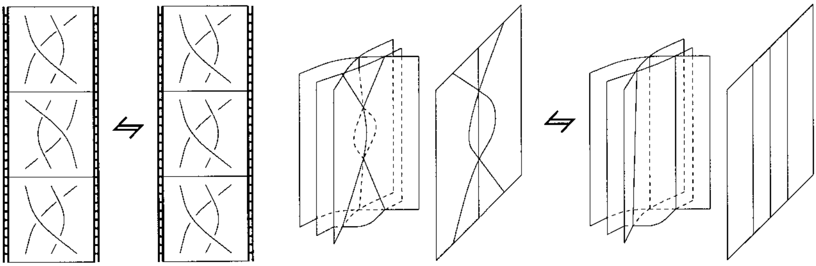

FIG. 38. Canceling triple points.

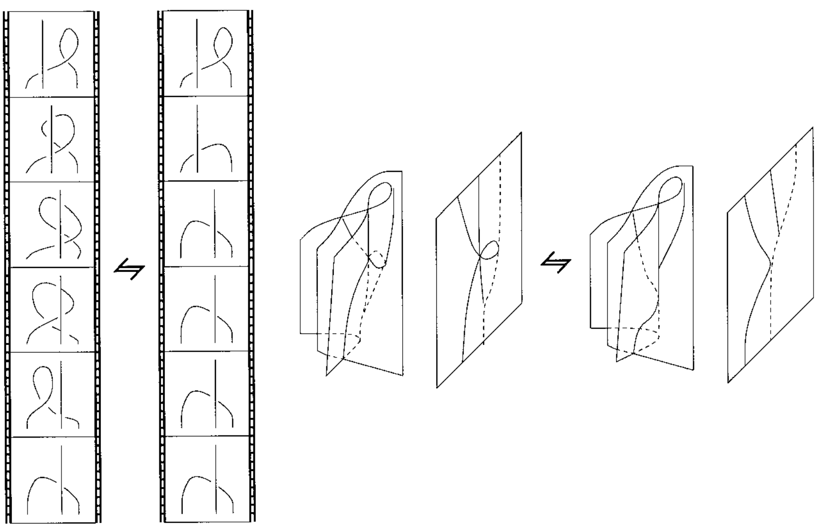

FIG. 39. A branch point moving through a triple point.

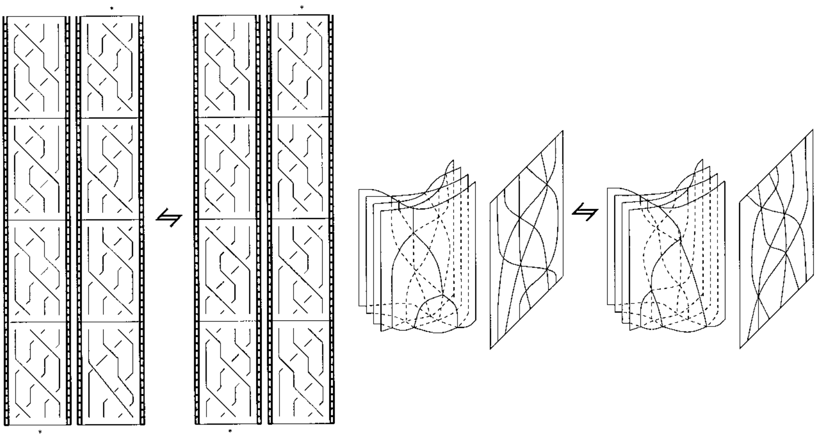

FIG. 40. A quadruple point in the isotopy.

of local moves taken from those depicted in Figs. 17, *through* 40, *or by exchanging the order in which ESIs occur when they occur in disjoint neighborhoods.*

Here we did not strictly specify what we mean by exchanging the order of ESIs. In Section 3.5, we will give a concrete description of moves addressing this point. Let us motivate the sequel.

3.4.7. MOTIVATION. We want to give a complete combinatoric or algebraic description to the set of knotted surface diagrams. To this end, we must explicitly describe the set of moves to the classical knot diagrams that occur at the critical levels of the surface with respect to the height function in the retinal plane. In particular, we must include among the critical data the crossings of double point arcs, the crossings of fold lines, and the crossings between double points and fold lines. Once we establish that the set of moves to classical diagrams include these interchanges, we find that the set of moves to knotted surface diagrams must take into account these subtleties.

In categorical language, we have identified new 2-morphisms that are natural equivalences which used to be considered to be equalities. Thus these equivalences must satisfy some further equalities. This language does not cast aspersions on the previous results—for example, the Reidemeister Theorem, the movie move theorem, or even Theorem 3.4.6. Each of these theorems provides a valid technique for moving knots (or knotted surfaces) around in space (or 4-space). But as we specify the diagrams as certain combinatorial data, the moves can affect those data. And we have to take into consideration those changes.

In the next section, we will examine all of the folds and double lines as they are projected to the retinal plane. In this way we will take into account the folds and double points that are not necessarily close (on the diagram of the surface) but that have projections that are close. In the language of singularity theory, we are examining the *multi-local* situation.

3.5. *Complete Symbolic Representations*

In Section 3.2, we illustrated the changes that occur in the retinal plane among the double point lines, triple points, branch points, and fold lines in the retinal plane. In Section 3.3.2, we illustrated the movie moves. And in Section 3.4, we illustrated the local changes in the movie description that can occur when a height function is included in the retinal plane. Here we amalgamate these results to give a complete list of moves to charts when a height function is fixed in the retinal plane.

3.5.1. DEFINITION. *The full set of elementary string interactions* (*FESIs*) are those illustrated in Figs. 16 and 4. These include the 3 classical Reidemeister moves, the two moves (also found in [23]) that involve changing a height function, and the four (multi-local) moves that involve interchanging the height of crossings and critical points.

3.5.2. EXAMPLE. Consider the diagram of the trefoil that is illustrated in Fig. 2. At any critical level, one can read across the diagram (from left to right) a sequence of symbols taken from the set X, $\bar{X}$, $\bigcup$, and $\bigcap$. The symbols can be adorned with double subscripts—the left subscript will indicate the number of straight strings to the left of the critical point, the right subscript will indicate the number of straight strings to the right of the critical point. In this way the diagram illustrated gives rise to the sequence

$$\bigcap_{0,0}\bigcap_{0,2}X_{1,1}X_{1,1}X_{1,1}\bigcup_{0,2}\bigcup_{0,0}.$$

Clearly any such classical knot diagram that has a height function can be described in similar manner. We turn to give the combinatorial description in the following.

3.5.3. DEFINITION. Let a set of symbols $X_{m,n}$, $\bar{X}_{m,n}$, $\bigcap_{m,n}$ and $\bigcup_{m,n}$ be given. Define *the initial number of a symbol*, $\iota(Y_{m,n})$, and *the terminal number of a symbol*, $\tau(Y_{m,n})$, (where $Y_{m,n}$ is one of the above symbols) as follows: $\iota(X_{m,n})=\iota(\bar{X}_{m,n})=\tau(X_{m,n})=\tau(\bar{X}_{m,n})=m+n+2$, $\iota(\bigcap_{m,n})=m+n$, $\tau(\bigcap_{m,n})=m+n+2$, $\iota(\bigcup_{m,n})=m+n+2$, $\tau(\bigcup_{m,n})=m+n$.

A *word* is a sequence $Y_0\cdots Y_k$ in symbols $Y_j=X_{m,n}$, $\bar{X}_{m,n}$, $\bigcap_{m,n}$ or $\bigcup_{m,n}$ where m and n are non-negative integers such that $\tau(Y_j)=\iota(Y_{j+1})$.

For a word $W = Y_0 \cdots Y_k$ with Y_0 and Y_k non-empty, $\tau(W)$ is defined by $\tau(Y_k)$ and $\iota(W)$ is defined by $\iota(Y_0)$.

The empty word is allowed as a word, and any given word need not involve all of the symbols.

A *sentence* is a sequence $(W_0, W_1, ..., W_f)$ of words such that W_0 and W_f are the empty words, and for any $i = 0, ..., f-1$, W_{i+1} is obtained from W_i by performing one of the following changes.

1. Cancellation or creation of a pair of adjacent symbols $\bigcap_{m,n} \bigcup_{m,n}$ in the word. More specifically, if $W_i = U \bigcap_{m,n} \bigcup_{m,n} V$ (resp. $W_i = UV$) where U and V are words such that $\tau(U) = m+n = \iota(V)$, then $W_{i+1} = UV$ (resp. $W_{i+1} = U \bigcap_{m,n} \bigcup_{m,n} V$). (Similar explicit expressions for W_i and W_{i+1} are omitted in the following.)

2. Cancellation or creation of a pair of adjacent symbols $\bigcup_{m,n} \bigcap_{m,n}$ in the word.

3. A replacement of $\bigcap_{m,n} X_{m,n}$ by $\bigcap_{m,n}$, or vice versa; a replacement of $\bigcap_{m,n} \bar{X}_{m,n}$ by $\bigcap_{m,n}$, or vice versa; a replacement of $X_{m,n} \bigcup_{m,n}$ by $\bigcup_{m,n}$, or vice versa; or a replacement of $\bar{X}_{m,n} \bigcup_{m,n}$ by $\bigcup_{m,n}$, or vice versa.

4. Cancellation or creation of a pair $X_{m,n} \bar{X}_{m,n}$ or $\bar{X}_{m,n} X_{m,n}$.

5. A replacement of one of the following:
$X_{m,n} X_{m+1,n-1} X_{m,n}$ by $X_{m+1,n-1} X_{m,n} X_{m+1,n-1}$ or vice versa,
$X_{m,n} X_{m+1,n-1} \bar{X}_{m,n}$ by $\bar{X}_{m+1,n-1} X_{m,n} X_{m+1,n-1}$ or vice versa,
$X_{m,n} \bar{X}_{m+1,n-1} \bar{X}_{m,n}$ by $\bar{X}_{m+1,n-1} \bar{X}_{m,n} X_{m+1,n-1}$ or vice versa,
$\bar{X}_{m,n} X_{m+1,n-1} X_{m,n}$ by $X_{m+1,n-1} X_{m,n} \bar{X}_{m+1,n-1}$ or vice versa,
$\bar{X}_{m,n} \bar{X}_{m+1,n-1} X_{m,n}$ by $X_{m+1,n-1} \bar{X}_{m,n} \bar{X}_{m+1,n-1}$ or vice versa, or
$\bar{X}_{m,n} \bar{X}_{m+1,n-1} \bar{X}_{m,n}$ by $\bar{X}_{m+1,n-1} \bar{X}_{m,n} \bar{X}_{m+1,n-1}$.

Note that these correspond to various crossing types of Reidemeister type III move.

6. A replacement of $\bigcap_{m,n} X_{m+1,n-1}$ by $\bigcap_{m+1,n-1} \bar{X}_{m,n}$, or vice versa; a replacement of $\bigcap_{m,n} \bar{X}_{m+1,n-1}$ by $\bigcap_{m+1,n-1} X_{m,n}$, or vice versa; a replacement of $X_{m,n} \bigcup_{m-1,n+1}$ by $\bar{X}_{m-1,n+1} \bigcup_{m,n}$, or vice versa; or a replacement of $\bar{X}_{m,n} \bigcup_{m-1,n+1}$ by $X_{m-1,n+1} \bigcup_{m,n}$, or vice versa.

7. Cancellation or creation of a pair $\bigcap_{m,n} \bigcup_{m+1,n-1}$ or $\bigcap_{m+1,n-1} \bigcup_{m,n}$.

8. A replacement of $Y_{m,n} Y'_{i,j}$ by $Y'_{i',j'} Y_{m',n'}$ where Y and Y' denote either X, $\bar{X}$, $\bigcap$ or $\bigcup$ and $|m-i| > 1$, $m+n = i+j$. The values of the subscripts i', j', m', n' depend on the value of the Y and Y' in the replacement. For example, if both Y and Y' take values from X or $\bar{X}$, then the primed subscripts have the same values as the unprimed subscripts. If one of Y and Y' (say Y) is X or $\bar{X}$ and the other (say Y') is $\bigcap$ or $\bigcup$, then one of the subscripts of Y changes by ± 2, and the subscripts of Y' do not change. If both of Y and Y' are $\bigcap$ or $\bigcup$, then two of the four subscripts change by ± 2—The signs are the same (different) if Y and Y' are different (the same).

Since the letters X, $\bar{X}\bigcap$, and $\bigcup$ correspond to crossings, maxima, and minima in a cross sectional knot diagram, we leave the reader to work out the values of the subscripts in the various cases by examining Fig. 4. In the following (especially in Theorem 3.5.5) we abuse notation when this phenomena happen, and use the notation $Y'_{i,j} Y_{m,n}$ for the replacement of $Y_{m,n} Y'_{i,j}$ instead of $Y'_{i',j'} Y_{m',n'}$.

Thus when the symbol Y appears for $X_{m,n}$, $\bar{X}_{m,n}$, $\bigcap_{m,n}$ or $\bigcup_{m,n}$, the same subscripts of Ys are kept for consecutive words to simplify the notation. We thank J. Baez and L. Langford for pointing out this phenomena.

Notice that successive words in a sentence differ by a certain changes, but in some circumstances the place where the change takes place is crucial information. For example, the sentence fragment $(..., \bigcap_{0,0}, \bigcap_{0,0}\bigcup_{0,0}\bigcap_{0,0}, ...)$ is ambiguous. It could mean $(..., \bigcap_{0,0}\bigstar, \bigcap_{0,0}\bigcup_{0,0}\bigcap_{0,0}, ...)$ or $(..., \bigstar\bigcap_{0,0}, \bigcap_{0,0}\bigcup_{0,0}\bigcap_{0,0}, ...)$ where the $\bigstar$ indicates the point at which the insertion takes place. In the former case the operation is an insertion of $\bigcup\bigcap$ and corrresponds to a saddle point. In the latter, the operation is an insertion of $\bigcap\bigcup$ and corresponds to a birth of a simple closed curve. Thus the information carried in a sentence must include the point of change between words. Precisely speaking we include the location at which an FESI is performed, and which FESI is performed. However to simplify notation we only indicate sequences of words in the following. We will specify the point of change as this ambiguity occurs.

In the following Theorem 3.5.5 we discuss equivalences among sentences. We remark here that when we change a sentence by a local replacement, it may happen that the result is a sentence such that $W_i = W_{i+1}$ for some i. This violates the definition of a sentence, so we delete W_{i+1} in this case. The opposite case may also happen (we may have to first introduce W_{i+1} which is equal to W_i before we make a replacement). Thus, strictly speaking, we allow such repetitions of words in sentences and define an equivalence relation, and work on equivalence classes. Note that this phenomenon corresponds to taking repetitive slices in the movie description, to see slow motion pictures.

Let F be a knotted surface in 4-space, and let $p(F)$ be its generic projection onto a vertical axis in the retinal plane. For each non-critical value $y \in \mathbf{R}$, the inverse image of y in $\mathbf{R}^3$ consists of a classical knot diagram. The horizontal axis in the retinal plane provides a height function for this diagram. We can use the height function to express such a diagram as a sequence of symbols defined in Section 2.5. The critical values correspond to the changes in stills that are expressed by one of the FESIs that are depicted Figs. 16, 4; these interactions correspond to the operations that connect any two words in a sentence. Thus any knotted surface diagram (with projection onto the vertical axis in the retinal plane) gives rise to a sentence. Conversely, given a sentence we can construct a knotted surface diagram: Each word gives a knot diagram, and each successive pair of words gives rise to a FESI. In summary we have proved

3.5.4. THEOREM. *To any knotted surface diagram a sentence is assigned. For any sentence there is a knotted surface whose corresponding sentence is the given one.*

A proof of the following Theorem will be given in the next section. It combines three results we have presented in this section:

(1) moves on charts,

(2) movie moves,

(3) movie moves with height functions on each still.

3.5.5. THEOREM. *Two sentences represent isotopic knotted surfaces if and only if one can be obtained from the other by a finite sequence of moves where the moves are taken from the list that follows.*

In the following list, parts of sentences are given. If the left hand side of the relation is found as a part of a sentence, then the part is replaced by the right hand side, or vice versa.

In the following Y, Y', Y'' denote either X, $\bar{X}$, $\cap$, or $\cup$. In this case we abuse notation and use the same subscripts for consecutive words even though those values can change from word to word depending on the value of Y. We also assume for any consecutive word PQ of P and Q that $\tau(P)=\iota(Q)$. The symbols W and V represent any words satisfying this condition.

1. $(W\cap_{m,n}V,\ W\cap_{m,n}X_{m,n}V,\ W\cap_{m,n}V)\leftrightarrow(W\cap_{m,n}V)$.
2. $(W\cap_{m,n}X_{m,n}V, W\cap_{m,n}V,\ W\cap_{m,n}X_{m,n}V)\leftrightarrow(W\cap_{m,n}X_{m,n}V)$.
3. $(WV)\leftrightarrow(WV,\ WX_{m,n}\bar{X}_{m,n}V,\ WV)$.
4. $(WX_{m,n}\bar{X}_{m,n}V,\ WV,\ WX_{m,n}\bar{X}_{m,n}V)\leftrightarrow(WX_{m,n}\bar{X}_{m,n}V)$.
5. $(WX_{m,n}X_{m+1,n-1}X_{m,n}V,\ WX_{m+1,n-1}X_{m,n}X_{m+1,n-1}V,$
$WX_{m,n}X_{m+1,n-1}X_{m,n}V)\leftrightarrow(WX_{m,n}X_{m+1,n-1}X_{m,n}V)$.
6. $(WX_{m,n+2}X_{m+1,n+1}X_{m,n+2}X_{m+2,n}X_{m+1,n+1}X_{m,n+2}V,$
$WX_{m+1,n+1}X_{m,n+2}X_{m+1,n+1}X_{m+2,n}X_{m+1,n+1}X_{m,n+2}V,$
$WX_{m+1,n+1}X_{m,n+2}X_{m+2,n}X_{m+1,n+1}X_{m+2,n}X_{m,n+2}V,$
$WX_{m+1,n+1}X_{m+2,n}X_{m,n+2}X_{m+1,n+1}X_{m+2,n}X_{m,n+2}V,$
$WX_{m+1,n+1}X_{m+2,n}X_{m,n+2}X_{m+1,n+1}X_{m,n+2}X_{m+2,n}V,$
$WX_{m+1,n+1}X_{m+2,n}X_{m+1,n+1}X_{m,n+2}X_{m+1,n+1}X_{m+2,n}V,$
$WX_{m+2,n}X_{m+1,n+1}X_{m+2,n}X_{m,n+2}X_{m+1,n+1}X_{m+2,n}V,$
$WX_{m+2,n}X_{m+1,n+1}X_{m,n+2}X_{m+2,n}X_{m+1,n+1}X_{m+2,n}V)$
$\leftrightarrow(WX_{m,n+2}X_{m+1,n+1}X_{m,n+2}X_{m+2,n}X_{m+1,n+1}X_{m,n+2}V,$

$WX_{m,n+2}X_{m+1,n+1}X_{m+2,n}X_{m,n+2}X_{m+1,n+1}X_{m,n+2}V,$
$WX_{m,n+2}X_{m+1,n+1}X_{m+2,n}X_{m+1,n+1}X_{m,n+2}X_{m+1,n+1}V,$
$WX_{m,n+2}X_{m+2,n}X_{m+1,n+1}X_{m+2,n}X_{m,n+2}X_{m+1,n+1}V,$
$WX_{m+2,n}X_{m,n+2}X_{m+1,n+1}X_{m+2,n}X_{m,n+2}X_{m+1,n+1}V,$
$WX_{m+2,n}X_{m,n+2}X_{m+1,n+1}X_{m,n+2}X_{m+2,n}X_{m+1,n+1}V,$
$WX_{m+2,n}X_{m+1,n+1}X_{m,n+2}X_{m+1,n+1}X_{m+2,n}X_{m+1,n+1}V,$
$WX_{m+2,n}X_{m+1,n+1}X_{m,n+2}X_{m+2,n}X_{m+1,n+1}X_{m+2,n}V)$

7. $(W\bigcap_{m+1,n-1}\bar{X}_{m+1,n-1}X_{m,n}V,$
$W\bigcap_{m+1,n-1}X_{m,n}\bar{X}_{m,n}\bar{X}_{m+1,n-1}X_{m,n}V,$
$W\bigcap_{m,n}\bar{X}_{m+1,n-1}\bar{X}_{m,n}\bar{X}_{m+1,n-1}X_{m,n}V,$
$W\bigcap_{m,n}\bar{X}_{m+1,n-1}X_{m+1,n-1}\bar{X}_{m,n}\bar{X}_{m+1,n-1}V,$
$W\bigcap_{m,n}\bar{X}_{m,n}\bar{X}_{m+1,n-1}V,\ W\bigcap_{m,n}\bar{X}_{m+1,n-1}V)$
$\leftrightarrow(W\bigcap_{m+1,n-1}\bar{X}_{m+1,n-1}X_{m,n}V,\ W\bigcap_{m+1,n-1}X_{m,n}V,$
$W\bigcap_{m,n}\bar{X}_{m+1,n-1}V).$

8. $(WV,\ W\bigcap_{m,n}\bigcup_{m-1,n+1}V,\ WV)\leftrightarrow(WV).$

9. $(W\bigcap_{m,n}\bigcup_{m-1,n+1}V,\ WV,\ W\bigcap_{m,n}\bigcup_{m-1,n+1}V)$
$\leftrightarrow(W\bigcap_{m,n}\bigcup_{m-1,n+1}V).$

10. $(W\bigcap_{m,n}V,\ W\bigcap_{m,n}\bigcap_{m+2,n}\bigcup_{m+1,n+1}V,$
$W\bigcap_{m,n}\bigcap_{m,n+2}\bigcup_{m+1,n+1}V,\ W\bigcap_{m,n}V)\leftrightarrow(W\bigcap_{m,n}V).$

11. $(W\bigcap_{m+1,n-1}\bar{X}_{m,n}V,\ W\bigcap_{m,n}X_{m+1,n-1}V,\ W\bigcap_{m+1,n-1}\bar{X}_{m,n}V)$
$\leftrightarrow(W\bigcap_{m+1,n-1}\bar{X}_{m,n}V).$

12. $(W\bigcap_{m+1,n-1}X_{m,n}\bigcup_{m+1,n-1}V,\ W\bigcap_{m+1,n-1}\bar{X}_{m+1,n-1}\bigcup_{m,n}V,$
$W\bigcap_{m+1,n-1}\bigcup_{m,n}V,\ WV)\leftrightarrow(W\bigcap_{m+1,n-1}X_{m,n}\bigcup_{m+1,n-1}V,$
$W\bigcap_{m,n}\bar{X}_{m+1,n-1}\bigcup_{m+1,n-1}V,\ W\bigcap_{m,n}\bigcup_{m+1,n-1}V,\ WV).$

13. $(W\bigcap_{m+2,n-2}\bar{X}_{m+1,n-1}\bigcup_{m,n}V,\ W\bigcap_{m+2,n-2}X_{m,n}\bigcup_{m+1,n-1}V,$
$WX_{m,n-2}\bigcap_{m+2,n-2}\bigcup_{m+1,n-1}V,\ WX_{m,n-2}V)$
$\leftrightarrow(W\bigcap_{m+2,n-2}\bar{X}_{m+1,n-1}\bigcup_{m,n}V,$
$W\bigcap_{m+1,n-1}X_{m+2,n-2}\bigcup_{m,n}V,\ W\bigcap_{m+1,n-1}\bigcup_{m,n}X_{m,n-2}V,$
$WX_{m,n-2}V).$

14. $(W\bigcap_{m,n}\bar{X}_{m+1,n-1}X_{m-1,n+1}X_{m,n}V,$
$W\bigcap_{m+1,n-1}X_{m,n}X_{m-1,n+1}X_{m,n}V,$
$W\bigcap_{m+1,n-1}X_{m-1,n+1}X_{m,n}X_{m-1,n+1}V,$
$WX_{m-1,n-1}\bigcap_{m+1,n-1}X_{m,n}X_{m-1,n+1}V,$
$WX_{m-1,n-1}\bigcap_{m,n}\bar{X}_{m+1,n-1}X_{m-1,n+1}V,$
$WX_{m-1,n-1}\bigcap_{m,n}X_{m-1,n+1}\bar{X}_{m+1,n-1}V)$
$\leftrightarrow(W\bigcap_{m,n}\bar{X}_{m+1,n-1}X_{m-1,n+1}X_{m,n}V,$
$W\bigcap_{m,n}X_{m-1,n+1}\bar{X}_{m+1,n-1}X_{m,n}V,$
$W\bigcap_{m-1,n+1}\bar{X}_{m,n}\bar{X}_{m+1,n-1}X_{m,n}V,$
$W\bigcap_{m-1,n+1}X_{m+1,n-1}\bar{X}_{m,n}\bar{X}_{m+1,n-1}V,$
$WX_{m-1,n-1}\bigcap_{m-1,n+1}\bar{X}_{m,n}\bar{X}_{m+1,n-1}V,$
$WX_{m-1,n-1}\bigcap_{m,n}X_{m-1,n+1}\bar{X}_{m+1,n-1}V).$

15. $(WY_{i,j}Y'_{m,n}V,\ WY'_{m,n}Y_{i,j}V,\ WY_{i,j}Y'_{m,n}V) \leftrightarrow (WY_{i,j}Y'_{m,n}V)$ where $|i-m|>1$, $i+j=n+m$, and the subscripts change from word to word depending on the values of Y and Y'.

16. $(WY_{i,j}Y'_{k,\ell}Y''_{m,n}V,\ WY'_{k,\ell}Y_{i,j}Y''_{m,n}V,$ $WY'_{k,\ell}Y''_{m,n}Y_{i,j}V,\ WY''_{m,n}Y'_{k,\ell}Y_{i,j}V)$ $\leftrightarrow (WY_{i,j}Y'_{k,\ell}Y''_{m,n}V,\ WY_{i,j}Y''_{m,n}Y'_{k,\ell}V,$ $WY''_{m,n}Y_{i,j}Y'_{k,\ell}V,\ WY''_{n,m}Y'_{k,\ell}Y_{i,j}V)$ where $|i-k|>1$, $|i-m|>1$, $|k-m|>1$, $i+j=k+\ell=n+m$, and the subscripts change from word to word depending on the values of Y, Y', and Y''.

17. $(WY_{i,j}X_{m,n-1}X_{m+1,n-2}X_{m,n-1}V,$ $WX_{m,n-1}Y_{i,j}X_{m+1,n-2}X_{m,n-1}V,$ $WX_{m,n-1}X_{m+1,n-2}Y_{i,j}X_{m,n-1}V,$ $WX_{m,n-1}X_{m+1,n-2}X_{m,n-1}Y_{i,j}V,$ $WX_{m+1,n-2}X_{m,n-1}X_{m+1,n-2}Y_{i,j}V)$ $\leftrightarrow (WY_{i,j}X_{m,n-1}X_{m+1,n-2}X_{m,n-1}V,$ $WY_{i,j}X_{m+1,n-2}X_{m,n-1}X_{m+1,n-2}V,$ $WX_{m+1,n-2}Y_{i,j}X_{m,n-1}X_{m+1,n-2}V,$ $WX_{m+1,n-2}X_{m,n-1}Y_{i,j}X_{m+1,n-2}V,$ $WX_{m+1,n-2}X_{m,n-1}X_{m+1,n-2}Y_{i,j}V)$ where $i<m-1$ or $i>m+2$, $i+j=m+n$, and the subscripts change from word to word depending on the value of Y.

18. $(WY_{i,j}\bigcap_{m+1,n-1}\bigcup_{m+2,n-2}V,\ W\bigcap_{m+1,n-1}Y_{i,j}\bigcup_{m+2,n-2}V,$ $W\bigcap_{m+1,n-1}\bigcup_{m+2,n-2}Y_{i,j}V,\ WY_{i,j}V)$ $\leftrightarrow (WY_{i,j}\bigcap_{m+1,n-1}\bigcup_{m+2,n-2}V,\ WY_{i,j}V)$ where $|i-m|>1$, $i+j+2=m+n$, and the subscripts change from word to word depending on the value of Y. Note here also that in the second word of the first sentence, $Y_{i,j}$ should be replaced by $Y_{i,j+2}$ (resp. $Y_{i+2,j}$) if $i<m-1$ (resp. $i>m+1$).

19. $(WY_{i,j}X_{m,n}\bigcup_{m,n}V,\ WX_{m,n}Y_{i,j}\bigcup_{m,n}V,\ WX_{m,n}\bigcup_{m,n}Y_{i,j}V,$ $W\bigcup_{m,n}Y_{i,j}V)$ $\leftrightarrow (WY_{i,j}X_{m,n}\bigcup_{m,n}V,\ WY_{i,j}\bigcup_{m,n}V,\ W\bigcup_{m,n}Y_{i,j}V)$ where $|i-m|>1$, $i+j=m+n$, and the subscripts change from word to word depending on the value of Y.

20. $(WY_{i,j}\bigcap_{m+1,n-1}\bar{X}_{m,n}V,\ W\bigcap_{m+1,n-1}Y_{i,j}\bar{X}_{m,n}V,$ $W\bigcap_{m+1,n-1}\bar{X}_{m,n}Y_{i,j}V,\ W\bigcap_{m,n}X_{m+1,n-1}Y_{i,j}V)$ $\leftrightarrow (WY_{i,j}\bigcap_{m+1,n-1}\bar{X}_{m,n}V,\ WY_{i,j}\bigcap_{m,n}X_{m+1,n-1}V,$ $W\bigcap_{m,n}Y_{i,j}X_{m+1,n-1}V,\ W\bigcap_{m,n}X_{m+1,n-1}Y_{i,j}V)$ where $|i-m|>1$, $i+j+2=m+n$, and the values of the subscripts change from word to word depending on the value of Y.

21. $(W\bigcap_{m+1,n+1}X_{m,n+2}\bar{X}_{m+2,n}\bigcup_{m+1,n+1}V,$ $W\bigcap_{m+1,n+1}\bar{X}_{m+2,n}X_{m,n+2}\bigcup_{m+1,n+1}V,$

$W\bigcap_{m+1,n+1}\bar{X}_{m+2,n}\bar{X}_{m+1,n+1}\bigcup_{m,n+2}V$,
$W\bigcap_{m+2,n}X_{m+1,n+1}\bar{X}_{m+1,n+1}\bigcup_{m,n+2}V$,
$W\bigcap_{m+2,n}\bigcup_{m,n+2}V$, $W\bigcup_{m,n}\bigcap_{m,n}V)$
$\leftrightarrow(W\bigcap_{m+1,n+1}X_{m,n+2}\bar{X}_{m+2,n}\bigcup_{m+1,n+1}V$,
$W\bigcap_{m,n+2}\bar{X}_{m+1,n+1}\bar{X}_{m+2,n}\bigcup_{m+1,n+1}V$,
$W\bigcap_{m,n+2}\bar{X}_{m+1,n+1}X_{m+1,n+1}\bigcup_{m+2,n}V$, $W\bigcap_{m,n+2}\bigcup_{m+2,n}V$,
$W\bigcup_{m,n}\bigcap_{m,n}V)$.

22. $(WY_{i,j}V,\ WY_{i,j}Z_{m,n}\bar{Z}_{m,n}V,\ WZ_{m,n}Y_{i,j}\bar{Z}_{m,n}V)$
$\leftrightarrow(WY_{i,j}V,\ WZ_{m,n}\bar{Z}_{m,n}Y_{i,j}V,\ WZ_{m,n}Y_{i,j}\bar{Z}_{m,n}V)$
where $(Z,\bar{Z})$ is either of $(X,\bar{X})$, $(\bar{X},X)$, $(\bigcap,\bigcup)$, or $(\bigcup,\bigcap)$, $|i-m|>1$, $i+j=m+n$, and the values on the subscripts change from word to word depending on the value of Y.

23. $(WZ_{m,n}V,\ WZ_{m,n}\bar{Z}_{m,n}Z_{m,n}V,\ WZ_{m,n}V)\leftrightarrow(WZ_{m,n}V)$
where the pair $(\bar{Z},Z)$ was introduced and the pair $(Z,\bar{Z})$ was cancelled in the left hand side, and $(Z,\bar{Z})$ is either of $(X,\bar{X})$, $(\bar{X},X)$, $(\bigcap,\bigcup)$, or $(\bigcup,\bigcap)$.

24. $(W\bigcap_{m+1,n}V,\ W\bigcap_{m+1,n}\bigcup_{m,n+1}\bigcap_{m,n+1}V,\ W\bigcap_{m,n+1}V)$
$\leftrightarrow(W\bigcap_{m+1,n}V,\ W\bigcap_{m,n+1}\bigcup_{m+1,n}\bigcap_{m+1,n}V,\ W\bigcap_{m,n+1}V)$.

25. $(WX_{m+1,n-1}X_{m,n}V,\ WX_{m,n}\bar{X}_{m,n}X_{m+1,n-1}X_{m,n}V$,
$WX_{m,n}X_{m+1,n-1}X_{m,n}\bar{X}_{m+1,n-1}V)$
$\leftrightarrow(WX_{m+1,n-1}X_{m,n}V,\ WX_{m+1,n-1}X_{m,n}X_{m+1,n-1}\bar{X}_{m+1,n-1}V$,
$WX_{m,n}X_{m+1,n-1}X_{m,n}\bar{X}_{m+1,n-1}V)$.

26. $(W\bigcap_{m,n}V,\ W\bigcap_{m,n}X_{m,n}\bar{X}_{m,n}V,\ W\bigcap_{m,n}\bar{X}_{m,n}V)$
$\leftrightarrow(W\bigcap_{m,n}V,\ W\bigcap_{m,n}\bar{X}_{m,n}V)$.

27. $(WV, W\bigcap_{m,n}\bigcup_{m,n}V,\ W\bigcap_{m,n}X_{m,n}\bigcup_{m,n}V)$
$\leftrightarrow(WV,\ W\bigcap_{m,n}\bigcup_{m,n}V,\ W\bigcap_{m,n}X_{m,n}\bigcup_{m,n}V)$
where in the left hand side $\bigcap_{m,n}X_{m,n}$ was introduced while in the right hand side $X_{m,n}\bigcup_{m,n}$ was introduced.

28. $(WV,\ W\bigcap_{m-1,n+1}\bigcup_{m-1,n+1}V$,
$W\bigcap_{m-1,n+1}\bar{X}_{m,n}X_{m,n}\bigcup_{m-1,n+1}V$,
$W\bigcap_{m,n}X_{m-1,n+1}X_{m,n}\bigcup_{m-1,n+1}V)$
$\leftrightarrow(WV,\ W\bigcap_{m,n}\bigcup_{m,n}V,\ W\bigcap_{m,n}X_{m-1,n+1}\bar{X}_{m-1,n+1}\bigcup_{m,n}V$,
$W\bigcap_{m,n}X_{m-1,n+1}X_{m,n}\bigcup_{m-1,n+1}V)$.

29. $(W\bigcup_{m,n}\bigcap_{m,n}V,\ W\bigcup_{m,n}\bigcap_{m,n}X_{m,n}V,\ WX_{m,n}V)$
$\leftrightarrow(W\bigcup_{m,n}\bigcap_{m,n}V,\ WX_{m,n}\bigcup_{m,n}\bigcap_{m,n}V,\ WX_{m,n}V)$.

30. $(W\bar{X}_{m+1,n-1}\bigcup_{m,n}\bigcap_{m+1,n-1}\bar{X}_{m,n}V$,
$WX_{m,n}\bigcup_{m+1,n-1}\bigcap_{m+1,n-1},\ \bar{X}_{m,n}V,\ WX_{m,n}\bar{X}_{m,n}V,\ WV)$
$\leftrightarrow(W\bar{X}_{m+1,n-1}\bigcup_{m,n}\bigcap_{m+1,n-1}\bar{X}_{m,n}V$,
$W\bar{X}_{m+1,n-1}\bigcup_{m,n}\bigcap_{m,n}X_{m+1,n-1}V$,
$W\bar{X}_{m+1,n-1}X_{m+1,n-1}V,\ WV)$.

31. $(WZ_1Z_2V,\ WZ'_1Z_2V,\ WZ'_1Z'_2V) \leftrightarrow (WZ_1Z_2V,\ WZ_1Z'_2V,\ WZ'_1Z'_2V)$ where the changes Z_i to Z'_i for $i=1, 2$ are FESIs.

Furthermore, we include the following variations to the list.

1. If $(S_1, ..., S_f) \leftrightarrow (S'_1, ..., S'_{f'})$ is in the list, then $(S_f, ..., S_1) \leftrightarrow (S'_1, ..., S'_{f'})$ is also a relation. This replacement corresponds to running a movie backwards.

2. If $(S_1, ..., S_f) \leftrightarrow (S'_1, ..., S'_{f'})$ is in the list, then $(T_1, ..., T_f) \leftrightarrow (T'_1, ..., T'_{f'})$ is also a relation where T_j (resp. T'_j) is obtained from S_j (resp. S'_j) as follows. If $S_j = Y^j_1 \cdots Y^j_k$ where Y^j_h are generators, then $T_j = Z^j_1 \cdots Z^j_k$ where $Z^j_h = X_{m,n}$ (resp. $\bar{X}_{m,n}$) if $Y^j_{k-h+1} = \bar{X}_{m,n}$ (resp. $X_{m,n}$) and $Z^j_h = \bigcap_{m,n}$ (resp. $\bigcup_{m,n}$) if $Y^j_{k-h+1} = \bigcup_{m,n}$ (resp. $\bigcap_{m,n}$), for all $j = 1, ..., f$, $h = 1, ..., k$. There is a similar replacement for T'_j (just put in primes).

3. If $(S_1, ..., S_f) \leftrightarrow (S'_1, ..., S'_{f'})$ is in the list, then $(T_1, ..., T_f) \leftrightarrow (T'_1, ..., T'_{f'})$ is also a relation where T_j (resp. T'_j) is obtained from S_j (resp. S'_j) as follows. If $S_j = Y^{j,1}_{m_1,n_1}, ..., Y^{j,k}_{m_k,n_k}$ where $Y^{j,h}_{m_h,n_h}$ are generators, then $T_j = (Y^{j,1}_{n_1,m_1})', ..., (Y^{j,k}_{n_k,m_k})'$, where $(Y^{j,h})' = Y^{j,h}$ if $Y^{j,h} = \bigcap$ or $\bigcup$, $(Y^{j,h})' = X$ (resp. $\bar{X}$) if $Y^{j,h} = \bar{X}$ (resp. X) for all $j = 1, ..., f$, $h = 1, ..., k$. This corresponds to reflecting the stills in their vertical axis.

4. Change X to $\bar{X}$ and vice versa in the relations consistently whenever possible. Recall that we had six variations for the Reidemeister type III move (listed as one ESI). Thus a given sentence may also be valid with such a replacement, and there is a move on sentences when these (and similar) replacements are valid. Add such variations to the list; they correspond to reflecting the stils from front to back in their plane of projection.

We remark here that the number of moves in Figs. 8 through 11, that of moves in the above theorem, and that of singularities in the proof of the theorem are different. This is because thick dotted lines in the figures represent either solid or dotted lines, and the symbol Z in the above theorem represents different generators. Thus a single move in one description can represent two or more moves in another. Furthermore, we count the codimension 1 singularities over the complex numbers, and some of the complex singularities split into two orbits over the reals.

3.5.6. Example. We demonstrate how a sequence from the above relations unties a knotted surface diagram. The first sequence represents an embedded 2-sphere with two critical points, two cusps, and two simple closed curves in the fold set. The subsequent sequences represent the result of applying various moves to the sequence until a standard unknot results.

$$
\begin{aligned}
(&\emptyset \\
&, \bigcap_{0,0} \bigcup_{0,0} \\
&, \bigcap_{0,0} X_{0,0} \bigcup_{0,0} \\
&, \bigcap_{0,0} X_{0,0} \bigcap_{2,0} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{2,0} X_{0,2} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{2,0} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} \bar{X}_{1,1} \bigcup_{2,0} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{1,1} X_{0,2} \bigcup_{2,0} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{1,1} \bigcup_{2,0} X_{0,0} \bigcup_{0,0} \\
&, \bigcap_{0,0} X_{0,0} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcup_{0,0} \\
&, \emptyset)
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&, \bigcap_{0,0} \bigcap_{2,0} X_{0,2} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{2,0} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
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&, \emptyset)
\end{aligned}
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&, \bigcap_{0,0} \bigcap_{2,0} X_{0,2} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} X_{0,2} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{1,1} X_{0,2} \bigcup_{0,2} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{1,1} X_{0,2} \bigcup_{2,0} \bigcup_{0,0} \\
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&, \emptyset)
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&, \bigcap_{0,0} \bigcap_{0,2} X_{0,2} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
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&, \bigcap_{0,0} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} X_{0,2} \bigcup_{1,1} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{0,2} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
&, \bigcap_{0,0} \bigcap_{1,1} X_{0,2} \bigcup_{0,2} \bigcup_{0,0} \\
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&, \bigcap_{0,0} \bigcap_{0,2} \bar{X}_{1,1} \bigcup_{0,2} \bigcup_{0,0} \\
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&, \bigcap_{0,0}\bigcap_{0,2}\bar{X}_{1,1}\bigcup_{0,2}\bigcup_{0,0} \\
&, \bigcap_{0,0}\bigcap_{0,2}X_{0,2}\bigcup_{1,1}\bigcup_{0,0} \\
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&, \bigcap_{0,0}\bigcup_{0,0} \\
&, \varnothing)
\end{aligned}
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4. SINGULARITIES AND KNOTTED SURFACE ISOTOPIES

In this section we provide proofs of Theorems stated in the preceding section.

Observe that Theorem 3.5.5 is a combinatorial restatement of the moves to charts that are depicted in Figs. 8, 9, 10, 11, and moves in which distant critical points of the vertical direction are interchanged. Similarly, Theorem 3.4.6 can be restated in terms of the moves depicted in those figures. In particular, the Theorem states explicitly which of the moves in these four figures involve only local changes in the diagrams.

The local moves in the retinal plane listed in Theorem 3.5.5 are generic singularities of isotopies $\mathbf{R}^3 \times I \supset F \times I \to \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. This means that without loss of generality we can assume that the isotopy has only these types of singularities. These singularities in turn give rise to codimension 1 singularities of maps $\mathbf{R}^3 \supset F \to \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and vice versa. Hence the results follow once we prove that the list of local singularities of the surface isotopies described in Section 3.4.5 and the multi-local singularities depicted in Fig. 10 exhausts the codimension 1 singularities of surface maps $\mathbf{R}^3 \supset F \to \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ where the first map is the projection onto the retinal plane and the second map is the projection onto the vertical direction of the retinal plane.

In Section 4.1, and in particular, the table in Section 4.1.2, we give a correspondence between codimension 1 singularities and the moves depicted in Figures 8 through 11.

The figures illustrate the relations between these codimension 1 singularities and knotted surface isotopies.

Thus Theorems 3.5.5 and 3.4.6 will follow once we have given a complete classification of the codimension 1 singularities that occur when a surface is projected from $\mathbf{R}^4$ onto a plane in which a height function is given. Also observe that Theorem 3.2.3 follows by combining the classifications given by Goryunov [14], Rieger [24], and West [30]. Now we turn to a discussion of the singularities. For a description of the techniques for classifying smooth map-germs we refer the reader to the survey by Wall [29].

4.1. Codimension 1 Projections of Generic Surfaces. Let $V \subset \mathbf{R}^3$ denote the image of a *generic* map $f = p \circ K$ from a surface into $\mathbf{R}^3$. So, locally, (V, q) is a germ of either an embedded surface, of a pair of surfaces intersecting transversely, of a triple point, or of a branch point (cross-cap). Below we classify the codimension 1 germs and multi-germs of simultaneous projections of V into a plane and a line contained in this plane. More precisely, we classify the following s-germs of diagrams of maps (where $S = \{q_1, ..., q_s\}$ is a finite set of source points)

$$\pi_2 \circ g \colon (\mathbf{R}^3, S) \supset (V, S) \to (\mathbf{R} \times \mathbf{R}, 0) \to (\mathbf{R}, 0)$$

$$\{q_i = (x_i, y_i, z_i)\}_{1 \leqslant i \leqslant s} \mapsto (X, Y) \qquad \mapsto Y$$

up to germs of diffeomorphisms $h_i \in \mathrm{Diff}(\mathbf{R}^3, q_i)$, $k \in \mathrm{Diff}(\mathbf{R} \times \mathbf{R}, 0)$ and $l \in \mathrm{Diff}(\mathbf{R}, 0)$ such that:

$$(h_i(V), q_i) = (V, q_i), \qquad 1 \leqslant i \leqslant s$$

and

$$d\pi_2(k) + l \circ \pi_2 = 0.$$

It turns out that, considering complex multi-germs, there are 33 codimension 1 orbits under this equivalence relation (in the cases where moduli are present, the codimension of the entire modular stratum is equal to one). Some of these orbits split into distinct orbits of real multi-germs (as indicated by the $\pm$-signs in some normal forms below). So, considering simultaneous projections of *generic* (complex) surfaces in 3-space onto planes and lines (fixing height functions in the projection planes), there are 33 possible codimension 1 singularities—for ordinary projections onto planes (without considering height functions) there are 22 codimension 1 singularities.

Following the terminology in Mancini and Ruas [20] we call a projection germ at q_i into the plane *tangent* if q_i is a critical point of the height function, and *transverse* otherwise. The list of codimension 1 projections under the above equivalence then consists of the following parts.

(i) Tangent and transverse projections of embedded surfaces are classified in Propositions 3.1 and 3.2 of Mancini and Ruas [20]. Actually, these authors assume that $\pi_2 \circ g$ is a Morse function. One easily checks that there is one more such codimension 1 projection for which $\pi_2 \circ g$ is not Morse, namely

$$(x, y) \mapsto (x, x^3 + y^2),$$

whose versal deformation is $(x, x^3 + y^2 + tx)$.

(ii) Local and multi-local projections of surfaces with double curves and triple points are classified in Proposition 4.1 of Rieger [24], and local projections of branch points are classified in Theorem 8.6.1 of West [30]. Furthermore, there are two bi-local codimension 1 projections involving branch points. The first is given by

$$\begin{aligned}\{(x_1, y_1) &\mapsto (x_1, x_1 y_1, y_1^2) \mapsto (x_1, y_1^2),\\ (x_2, y_2) &\mapsto (x_2, y_2, x_2 + y_2^2) \mapsto (x_2, x_2 + y_2^2)\}\end{aligned}$$

and the second by the restriction of the pair of maps from $\mathbf{R}^3$ to $\mathbf{R}^2$

$$\{g_1=(x_1, z_1), g_2=(z_2, z_2 \pm x_2 + y_2)\}$$

to $y_1^2 - x_1^2 z_1 = 0$ and $x_2 y_2 = 0$, respectively. Note that the first component of both bi-germs is a fold of the projection of a branch point (which is given by a parametrization in the first case and as a zero-set in the second case); the second component is an ordinary fold (first case) and a projected double curve (second case). A versal deformation of both bi-germs can be obtained by adding the term $\{(0, 0), (0, t)\}$.

The above normal forms for codimension 1 projections, and the ones in [24] and [30], do not take into account a height function in the projection plane. In order to construct the corresponding normal forms of transverse codimension 1 projections, one has to change the projection $\pi_2\colon \mathbf{R}^2 \to \mathbf{R}$ to $\pi(X, Y) = aX + bY$, $h=(a, b) \in S^1$, such that $\langle h, l_i \rangle \neq 0$ for all limiting tangent lines l_i at 0 of projected double-curves and folds. (Also, for the tri-germs in [24] corresponding to triple-crossings of folds and projected double-curves there will be a modulus—as for the transverse triple-fold in [20]—given by the cross-ratio of the slopes of the three tangent lines and the direction of the height function.)

(iii) Finally, the following codimension 1 tangent projections of non-embedded points complete our list:

4.1.1. Proposition. *The tangent projections of a double-, triple- or branch point*

$$\pi_2 \circ g\colon (\mathbf{R}^3, S) \supset (V, S) \to (\mathbf{R} \times \mathbf{R}, 0) \to (\mathbf{R}, 0)$$

of codimension 1 *are equivalent to one of the following mono-germs g or bi-germs $g=\{g_1, g_2\}$ below. Note that the g_i marked with a $*$ denote the composition of a parametrization of (V, q_i) with a projection into the plane and hence are, locally, maps from $\mathbf{R}^2$ to $\mathbf{R}^2$. The terms u of a versal deformation $g + t \cdot u$ of g and either the defining equations r_i of $r_i^{-1}(0) = (V, q_i)$ or, in the cases $*$, a parametrization of (V, q_i) are given for each g (also, we set ε, $\varepsilon_i = \pm 1$).*

1. $g=(x+z, y+z+\varepsilon x^2)$, $u=(0, x)$, $r=xyz$
2. $g=(z, x+\varepsilon_1 z^2 + \varepsilon_2 y^2 + yz)$, $u=(0, z)$, $r=xy$
3. $g=(x, y^2+\varepsilon x^2)\ *$, $u=(0, x)$, $(x, y) \mapsto (xy, g(x, y))$
4. $g=(y^2, x+y^4)\ *$, $u=(0, y^2)$, $(x, y) \mapsto (xy, g(x, y))$
5. $g=(z, \varepsilon x + y + z^3)$, $u=(0, z)$, $r=xy$

6. $g=\{(z_1, \varepsilon x_1+y_1+z_1^2),\ (y_2^2, x_2)*\}$, $u=\{(0,0),(1,0)\}$, $r_1=x_1y_1$, $(x_2, y_2)\mapsto(y_2, g_2(x_2, y_2))$

7. $g=\{(z_1, \varepsilon_1 x_1+y_1+z_1^2),\ (\varepsilon_2 x_2+y_2, z_2)\}$, $u=\{(0,0),(1,0)\}$, $r_1=x_1y_1$, $r_2=x_2y_2$

8. $g=\{(x_1, y_1^2+\varepsilon_1 x_1^2)*,\ (\varepsilon_2 x_2+y_2, z_2)\}$, $u=\{(0,0),(1,0)\}$, $(x_1, y_1)\mapsto(y_1, g_1(x_1, y_1))$, $r_2=x_2y_2$.

Proof. The derivation of this classification combines the methods of [20] and [24] and involves fairly routine calculations—we omit the details. Roughly speaking, one determines the orbits of the appropriate group of equivalences inductively modulo increasingly higher powers k of the maximal ideal until some orbit either has codimension >1 or is $(k-1)$-sufficient (*i.e.* any representative g of this orbit is $(k-1)$-determined in the sense that $j^{k-1}f\sim j^{k-1}g$ implies $f\sim g$). The codimension and the sufficiency of an orbit can be determined from its tangent space at g. Let us briefly illustrate this for the first example in our list.

Let C_{xyz} denote the local ring of smooth function germs in the source variables and $\underline{m}_{xyz}$ its maximal ideal. Likewise, the C_{XY} and C_Y denote the rings of function germs in the target variables of g and of π_2, respectively. Let θ_g denote the C_{xyz}-module of vector fields over g (that is, sections of $g^*T\mathbf{R}^3$).

For the triple point $V=\{xyz=0\}$ one checks that the tangent space to the orbit of the corresponding group of equivalences at g is given by

$$T(g)=C_{xyz}\{x\,\partial g/\partial x,\ y\,\partial g/\partial y,\ z\,\partial g/\partial z\}+C_{XY}\{\partial/\partial X\}+C_Y\{\partial/\partial Y\}.$$

Note that $T(g)$ differs from the usual right-left tangent space in the following respects: the usual right tangent space is restricted to C_{xyz}-modules of vector fields tangent to $V=\{xyz=0\}$ and the usual left tangent space is restricted to C_{XY}-modules of vector fields tangent to the level set $Y=0$ at the origin (which preserve the "height" of the critical point). The codimension of g is defined to be $\dim_{\mathbf{R}}\theta_g/T(g)$.

For $g=(x+z,\ y+z+\varepsilon x^2)$ one calculates that $\underline{m}_{xyz}^3\cdot\theta_g\subset T(g)$ which means that g is 2-determined – in fact

$$T(g)=(C_{xyz},\ C_{xyz}\backslash\{x\}).$$

So g has codimension 1, and $G=g+t\cdot(0,x)$ is a versal deformation of g.

4.1.2. The Correspondence Between Chart Moves and Singularities. In this section we explicitly state which singularities in the various lists correspond to the moves on charts that we have depicted. In either Fig. 10 or Fig. 11, the illustration (i, j) refers to the move that is depicted in the

TABLE I

Figure Number	Reference
8(1, 1)	[21] 1 : 1 S_1^+; [14] I
8(1, 2)	[21] 1 : 1 S_1^-; [14] I
8(2, 1)	[21] 7 : 2; [14] II
8(3, 1)	[21] 7 : 2; [14] II
8(4, 1)	[14] III
8(2, 2)	[21] 7 : 5; [14] IV
8(5, 1)	[14] V
9(1, 1)	[20], 4_2^+ (Table 1); [24] (Table 1) item 1
9(1, 2)	[30] 8.6.1 (b)
9(2, 1)	[20] (Table 1) 4_2^-; [24] (Table 1) item 1
9(2, 2)	[24] (Table 1) item 7
9(3, 1)	[20] (Table 1) 5; [24] (Table 1) item 2
9(3, 2)	[24] (Table 1) item 8
9(4, 1)	[24] (Table 1) item 6
9(4, 2)	[24] (Table 1) item 19
10(1, 1)	[20] (Table 2) item 3; [24] (Table 1) 5, 10, or 12
10(1, 2)	[20] (Table 2) item 4 [24] (Table 1) item 4 or 11
10(2, 1)	[20] (Table 2) item 5 [24] (Table 1) items 3, 9, 13, 14
10(2, 2)	4.1 item (ii)
10(3, 1)	[24] (Table 1) items 17 or 18
10(3, 2)	[24] (Table 1) items 15 or 16
11(1, 1)	[20] (Table 2) item 2; 4.1.1 above, items 6, 7 or 8
11(1, 2)	4.1.1 above, item 4
11(2, 1)	4.1.1 above, item 5; or Sect. 4.1 (i)
11(2, 2)	4.1.1 above, item 3^+
11(3, 1)	[20] (Table 1), item 3′
11(3, 2)	4.1.1 above, item 2 ($\varepsilon_1 = \varepsilon_2$)
11(4, 1)	4.1.1 above, item 1
11(4, 2)	4.1.1 above, item 3^-
11(5, 2)	4.1.1 above, item 2 ($\varepsilon_1 \neq \varepsilon_2$)

ith row and jth column of the figure. Thus in Table I, 8 (1, 2) refers to the illustration in the first row second column of Fig. 8. In the second column we list the reference in square brackets, the theorem or table number, and the item in that list. Thus [30], 8.6.1, (b) refers to the second item in West's Theorem 8.6.1. In the case where no numbering is given in the table, we give either a brief description, or the number of the singularity if the table had been numbered.

We note that in the table the correspondences are sometimes many-to-one or one-to-many, for in the figures we have used thick dotted lines to indicate either fold lines or double point curves, and in the various lists of singularities plus and minus signs are included in some of the cases.

Also Goryunov's classification [14] is over the complexes, and so, for example, the confluences of branch points splits into two real cases. In any case, the correspondences between the figures and the singularities are not difficult to work out when the table is ambiguous.

4.1.3. PROOF OF THEOREM 3.5.5. Proposition 4.1.1, the results of [20], [24], and [30], give complete lists of the appropriate codimension 1 singularities in case a generic surface is given in 3-space. The table indicates that these singularities correspond exactly to the cases depicted in Figs. 9, 10, and 11. The illustrations in Fig. 8 correspond to the codimension 1 singularities classified in [21] and [22], or equivalently, these are chart depictions of the Roseman moves [25]. Thus any codimension 1 singularity is either found in one of these lists, or occurs when the height of distant critical points in the chart are interchanged. Theorem 3.5.5 contains a combinatorial description of each of the cases found in the charts, and thus it gives a complete list of changes to sequences of FESIs. This completes the proof.

5. THE 2-CATEGORY OF KNOTTED SURFACES

In this section we give an outline of the definition of the 2-category of knotted surfaces.

A (small) 2-category consists of the following data: (1) a set of objects **Obj**, (2) a set of 1-morphisms **1-Mor**, whose elements have source and target objects, (3) a set of 2-morphisms **2-Mor**, whose elements have source and target 1-morphisms. There are compositions of these morphisms defined, and we refer to [18] for more details since their definition takes 3 pages.

The set of objects in the 2-category is the non-negative integers. There is a tensor product **1-Mor** $\times$ **1-Mor** $\to$ **1-Mor** given as the sum of integers, $m \otimes n = m + n$. We assume that the tensor product is strictly associative $((a \otimes b) \otimes c = a \otimes (b \otimes c))$. The set of 1-morphisms is generated by the 1-morphisms $\bigcap_{m,n}$ $\bigcup_{m,n}$, $X_{m,n}$ and $\bar{X}_{m,n}$, where m, n are non-negative integers. The 1-morphism $X_{m,n}$ and $\bar{X}_{m,n}$ have source and target the integer $m+n+2$; the 1-morphism $\bigcap_{m,n}$ has target the integer $m+n$ and source $m+n+2$; the 1-morphism $\bigcup_{m,n}$ has target the integer $m+n+2$ and source $m+n$. Thus the set of 1-morphisms is the set of compositions of $\bigcap_{m,n}$ $\bigcup_{m,n}$, $X_{m,n}$ and $\bar{X}_{m,n}$, where compositions are made when the source of one coincides the target of the next. We associate a composition of 1-morphisms to a tangle diagram that has a fixed height function, and the composition of 1-morphisms is read from the bottom to top of the diagram. Thus a knotted surface is represented by a sequence of 1-morphisms.

The 2-morphisms are the moves that connect words in a sentence as described in Section 3.5. The relations among the 2-morphisms are those that are described in Theorem 3.5.5.

In the diagrammatical situation the 2-morphisms are represented as a sequence of tangle diagrams where a successive pair of diagrams in the sequence differs by at most an FESI. Furthermore, the moves to sentences are represented as the moves to movies as depicted in Theorem 3.4.6.

It is reasonable to conjecture a generalization of the Freyd–Yetter Theorem to this 2-category. A systematic and categorical way of describing these relations will be needed.

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