SIAM J. COMPUT.
Vol. 29, No. 2, pp. 433–458
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PROXIMITY IN ARRANGEMENTS OF ALGEBRAIC SETS*

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Abstract. Let X be an arrangement of n algebraic sets X_i in d-space, where the X_i are either parametrized or zero-sets of dimension $0 \leq m_i \leq d-1$. We study a number of decompositions of d-space into connected regions in which the distance-squared function to X has certain invariances. Each region is contained in a single connected component of the complement of the bifurcation set $\mathcal{B}$ of the family of distance-squared functions or of certain subsets of $\mathcal{B}$. The decompositions can be used in the following proximity problems: given some point, find the k nearest sets X_i in the arrangement, find the nearest point in X, or (assuming that X is compact) find the farthest point in X and hence the smallest enclosing $(d-1)$-sphere. We give bounds on the complexity of the decompositions in terms of n, d, and the degrees and dimensions of the algebraic sets X_i.

Key words. k nearest points, Voronoi regions, bifurcation sets, critical points, symbolic computation

AMS subject classifications. 68U05, 68Q40, 51M20, 53A07, 57R45

PII. S0097539796300945

1. Introduction.

Let X be the union of n algebraic sets X_i of dimension $0 \leq m_i \leq d-1$ in d-space which are defined either by parametrizations or, more generally, as zero-sets. The dimension d of the ambient space is assumed to be arbitrary but fixed. Given a point $p \in \mathbb{R}^d$ with rational or, more generally, with algebraic number coordinates and a set of defining polynomials of X with rational coefficients, we would like to do the following:

1. find the k nearest sets X_i;
2. find the nearest point in X;
3. and, provided that X is compact, find the farthest point in X (and hence the smallest sphere with center p enclosing X).

For all of these proximity problems it is convenient to decompose d-space into certain connected regions, depending on X, in which the distance-squared function to X has certain invariances. A number of such decompositions are possible. Some decompositions have many invariants but also many regions, and it is of interest to bound the number of regions in terms of n, d, and the degrees and dimensions of the algebraic sets X_i. For example, the coarsest decomposition considered below consists of the first-order Voronoi regions, and the finest consists of the regions in the complement of the bifurcation set of the family of all distance-squared functions on X. However, all the decompositions studied here have the property that the proximity problems above can be solved in $O(\log n) \cdot P$ time (discarding the preprocessing time for constructing the decomposition), where P is a polynomial in the degrees and coefficient sizes of both the defining polynomials of X and the minimal polynomials of the algebraic number coordinates of p (from section 4 on we shall often concentrate on the combinatorial complexity, where the degrees and coefficient sizes of these polynomials are assumed to be bounded by some constant independent of n). Decompositions of

*Received by the editors March 20, 1996; accepted for publication (in revised form) May 24, 1998; published electronically October 5, 1999. Most of this work was done at the Max-Planck-Institut für Informatik in Saarbrücken, Germany, and has appeared in preliminary form as internal report MPI-96-1-003.

http://www.siam.org/journals/sicomp/29-2/30094.html

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d-space into regions made of points having certain proximity properties with respect to some collection of submanifolds of $\mathbb{R}^d$ have been studied both in computational geometry and in singularity theory, but there hasn't been much interaction between these fields.

Most of the works in computational geometry consider either the classical, first-order, Voronoi diagram of sets of isolated points or extensions to arrangements of linear subspaces of $\mathbb{R}^d$. The relation between higher-order Voronoi diagrams in $\mathbb{R}^d$ and arrangements in $\mathbb{R}^{d+1}$ is investigated by Edelsbrunner and Seidel [12]. A few works also consider Voronoi diagrams of arrangements of curved objects. First-order Voronoi diagrams of disjoint convex semialgebraic sites in d-space are studied in the book of Sharir and Agarwal [22]. Alt and Schwarzkopf [1] study first-order Voronoi diagrams of parametrized (semialgebraic) curve-segments and points in the plane. These authors are also interested in the local geometry of Voronoi edges: for example, they point out that end-points of self-Voronoi-edges (in the singularity theory literature known as symmetry sets) correspond to centers of osculating circles at curvature extrema of a planar curve and also to a cusp singularity of the evolute (or focal set). The local geometry of such symmetry sets and of evolutes has been studied in great detail in a number of singularity theory works.

One of the main topics of singularity theory is the classification of stable and unstable singularities of functions and maps, and of the bifurcation sets in the parameter space of families of functions and maps. The bifurcation set of a family of functions $F : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d \times \mathbb{R}$, $(p, x) \mapsto (p, f(p, x))$ consists of all points p in parameter space $\mathbb{R}^d$ for which the function $x \mapsto f(p, x)$ has an unstable (degenerate) singularity. The family of distance-squared functions from any point $p \in \mathbb{R}^d$ to a parametrized m-dimensional surface X in d-space is a particular example of such a family, and the bifurcation set of this family is precisely the union of the evolute and the symmetry set of X. Porteous [16, 17] has used the classification of families of functions by Thom (from the early 1960s) to study the relation between the geometry of evolutes and the curvature of surfaces. Bruce, Giblin, and Gibson [6] have classified the singularities of symmetry sets of planar curves and of surfaces and space-curves in $\mathbb{R}^3$; see also the recent paper by Bruce [5]. Symbolic algorithms for computing bifurcation sets of families of projection maps have been studied by Rieger [19, 20] and these algorithms can also be used, with some minor modifications, to compute other bifurcation sets, such as evolutes and symmetry sets.

1.1. Assumptions and some notation. Let $X := \cup X_i \subset \mathbb{R}^d$ be a collection of n closed algebraic sets X_i and set $m_i := \dim X_i$ $(0 \le m_i \le d-1)$ and $m := \sup m_i$. For parametrized algebraic m_i-surfaces $x \mapsto X_i(x)$ we denote the maximal degree of the d component functions of $X_i(x)$ by δ_i, and set $\delta := \sup \delta_i$ $(1 \le i \le n)$. For the more general case of zero-sets $X_i = h_i^{-1}(0)$, where $h_i := (h_i^1, \ldots, h_i^{d-m_i}) : \mathbb{R}^d \to \mathbb{R}^{d-m_i}$, we assume that X_i, or rather its complexification, is a complete intersection (i.e., its codimension is equal to the number of defining equations) and we set $\Delta_i := \Pi_j \deg h_i^j = \deg X_i$ and $\Delta := \sup \Delta_i$ (geometrically, $\deg X_i$ is the number of real and complex intersection points, including those "at infinity," of X_i and a "generic" linear subspace of $\mathbb{R}^d$ of dimension $d - m_i$).

The following notation will be used in this paper: $Z(I)$ denotes the zero-set of an ideal I, $I(Z)$ the ideal of polynomials vanishing on Z, $I : J$ the ideal quotient, and $\mathrm{cl}Z$ denotes the closure of the set Z. Also, $\langle g_1, \ldots, g_s \rangle$ denotes the ideal generated by the g_i, $1 \le i \le s$. The components of a vector $x = (x^1, \ldots, x^d)$ are denoted by superscripts, so that subscripts can be used to enumerate elements of sets; and $(x^d)^3$

denotes the third power of the dth component.

Next, we need some notation from singularity theory. For most parts of this paper (sections 2 to 5), it is enough to remember the following notations: a nondegenerate critical point of a function, where the matrix of second derivatives has maximal rank, is of type A_1 (or of Morse type), a pair of A_1 points having the same critical values is denoted by A_1^2, and the least degenerate of the degenerate critical points is of type A_2. The bifurcation set $\mathcal{B} \subset \mathbb{R}^d$ of a family of distance-squared functions is a generally singular hypersurface whose regular components correspond to (codimension 1) singularities of type A_2 or A_1^2 of the distance-squared function. However, in section 6 we need to count the strata of the singular locus of $\mathcal{B}$ corresponding to more degenerate types of singularities (of codimension ≥ 2). The following notation for these singularities is more or less standard (for more details on the classification of functions up to $\mathcal{K}$- and $\mathcal{R}$-equivalence see chapter 8 of Dimca's book [10]). A function $f : \mathbb{R}^d \to \mathbb{R}$, $x := (x^1, \ldots, x^d) \mapsto f(x)$ has an A_k-singularity at $x = 0$ if there exists a smooth coordinate change $h : \mathbb{R}^d \to \mathbb{R}^d$, defined in the neighborhood of $x = 0$, such that $f \circ h(x) = c + (x^1)^{k+1} + \sum_{i=2}^d \epsilon_i (x^i)^2$, where c is some constant and $\epsilon_i = \pm 1$; see [2]. In other words, A_k denotes an equivalence class of function-germs (we consider pm;u f and h near $x = 0$), and the formula above describes a particular representative of this class. The equivalence classes A_k are orbits of the Mather groups $\mathcal{K}$ and $\mathcal{R}$. We shall abuse the notation $A_{\geq k}$ slightly: it will denote *all* classes of singularities in the closure of the A_k orbit, not just the orbits represented by $c + (x_1)^{\geq k+1} + \sum \epsilon_i (x^i)^2$. Finally, the function f has an $A^r_{\geq 1}$-singularity at a set of points $\{x_1, \ldots, x_r\}$, if it has an $A_{\geq 1}$-singularity at each $x_i \in \mathbb{R}^d$ and the r critical values $f(x_i)$ coincide.

Throughout this paper, $d^k f(p, x)[v_1, \ldots, v_l]$ will always denote the kth differential of a function f with respect to the variables $x \in \mathbb{R}^m$ and not with respect to the parameter $p \in \mathbb{R}^d$, and this k-linear form $d^k f$ should be multiplied with the vectors $v_i \in \mathbb{R}^m$, $1 \leq i \leq l$. Occasionally, we shall omit the parameters and variables (p, x).

Given a hypersurface $M \subset \mathbb{R}^d$, we denote the arrangement cut-out by M by $\mathcal{A}(M)$. We denote by $|\mathcal{A}(M)|$ the size of this arrangement, that is, the number of i-cells, $0 \leq i \leq d$, in $\mathcal{A}(M)$. Note that the connected regions of $\mathbb{R}^d \setminus M$ are the d-cells in $\mathcal{A}(M)$.

The Voronoi diagram of order k of a set $S := \{X_1, \ldots, X_n\}$ of algebraic sets $X_i \subset \mathbb{R}^d$ is defined as follows. Set $\mu_p(X_i) := \inf_{q \in X_i} \|q - p\|^2$ and let $\tilde{S} \subset S$ be a subset with k elements, $1 \leq k \leq n - 1$. Then

$$V_k(\tilde{S}) := \{p \in \mathbb{R}^d : \mu_p(X_i) < \mu_p(X_j), \text{for all } (X_i, X_j) \in \tilde{S} \times (S \setminus \tilde{S})\}$$

is the kth-order Voronoi cell of $\tilde{S}$ (which, in general, is not connected). The kth-order Voronoi surface V_k of S is the union of the boundaries of such Voronoi cells, i.e., $V_k := \cup_{\tilde{S} \subset S} \partial V_k(\tilde{S})$, and the kth-order Voronoi diagram is the arrangement $\mathcal{A}(V_k)$.

1.2. Contents of following sections. In section 2 we study the bifurcation set $\mathcal{B}$ of the family of distance-squared functions on an arrangement of m_i-surfaces X_i in $\mathbb{R}^d$ which are parametrized by polynomial maps. In particular, we give bounds for the number of regions in the complement of $\mathcal{B}$ (and, in fact, for $|\mathcal{A}(\mathcal{B})|$) and describe certain invariants which characterize these regions. We also obtain a result on the local topology of $\mathcal{B}$ that yields a priori information on how the semialgebraic components of $\mathcal{B}$ are glued together. This result is valid both for parametrized surfaces X_i and for zero-sets and does not assume that the family of distance-squared functions is versal (this is a common assumption in singularity theory works on this subject that does not necessarily hold for "almost all" algebraic surfaces X_i of some bounded degree).

In section 3 we consider the more general case of arrangements of algebraic zero-sets X_i (note that most zero-sets do not have a global parametrization given as the image of some polynomial map). For zero-sets we exploit the geometric characterization of the singularities of the distance-squared function in terms of the contact order (or intersection multiplicity) of X_i with certain $(d-m_i)$-spheres, where $m_i = \dim X_i$. This avoids the problem of finding local parametrizations of the X_i given by analytic maps (working with polynomials is much more convenient). The more classical case of contact between hypersurfaces X_i and osculating circles is treated in subsection 3.1; the more complicated case of contact between algebraic sets X_i of codimension $d - m_i \geq 2$ and $(d - m_i)$-spheres is studied in subsection 3.2.

In section 4 we describe an algorithm for determining the regions in the complement of the bifurcation set $\mathcal{B}$. This algorithm is similar, in its overall structure, to the algorithms in [19, 20] and uses standard techniques from computational algebra. We also describe solutions to the proximity problems 1 to 3 stated at the beginning of this introduction, which are based on the decomposition of d-space into regions in the complement of $\mathcal{B}$ or of certain subsets of $\mathcal{B}$.

In section 5 we present a few examples of these decompositions for curves and points in the plane, which have been computed with the methods described in section 4.

Finally, in section 6, we compare the combinatorial complexities of the arrangements $\mathcal{A}(\mathcal{B})$ and of the kth-order Voronoi diagrams $\mathcal{A}(V_k)$. Note that the boundaries V_k of the Voronoi regions of order k are subsets of $\mathcal{B}$, and the bounds in sections 2 and 3 are therefore upper bounds for the complexity of the kth-order Voronoi diagram of arrangements of algebraic sets in terms of n, d, and the degrees and dimensions of these sets. A comparison of the combinatorial complexities of the bifurcation set $\mathcal{B}$ and of the Voronoi boundaries V_k for the more special arrangements studied in [1] and [22] shows that there is a considerable gap, which can be partially understood by studying the combinatorial complexity of certain intermediate sets $V_k \subset R_k \subset \mathcal{B}$. Even so, for general arrangements of algebraic sets, an asymptotically tight bound (at least in terms of combinatorial complexity) for the number of regions of $\mathbb{R}^d \setminus V_k$ remains a widely open problem (for n intersecting hypersurfaces in $\mathbb{R}^d$ we show, for example, that $|\mathcal{A}(V_1)| \sim O(n^{d+1})$ and $|\mathcal{A}(V_1)| \sim \Omega(n^d)$).

2. The complement of the bifurcation set $\mathcal{B}$ of a family of distance-squared functions. In this section the algebraic m_i-surfaces X_i of the arrangement are parametrized by polynomial maps $x \mapsto X_i(x)$, where $x = (x^1, \ldots, x^{m_i}) \in \mathbb{R}^{m_i}$. The necessary modifications in the (more general) case of zero-sets will be described in section 3. The family of distance-squared functions on X_i is defined by

$$F_i : \mathbb{R}^d \times \mathbb{R}^{m_i} \to \mathbb{R}^d \times \mathbb{R}, \quad (p, x) \mapsto (p, f_i(p, x) := \|X_i(x) - p\|^2).$$

Recall that an element of this d-parameter family of functions in m_i variables is a Morse function if its critical points are nondegenerate (i.e., the corresponding matrix of second derivatives has maximal rank) and have distinct critical values. The bifurcation set $\mathcal{B}_i \subset \mathbb{R}^d$ of the family F_i is the set of "bad" parameters p for which $x \mapsto f_i(p, x)$ fails to be a Morse function. The set $\mathcal{B}_i$ is the union of the local bifurcation set

$$\mathcal{E}_i := \{p \in \mathbb{R}^d : \exists x : df_i(p, x) = 0, \text{rank } d^2 f_i(p, x) < m_i\},$$

consisting of $A_{\geq 2}$-singularities, and the level bifurcation set

$$\mathcal{S}_i := \mathrm{cl}\{p \in \mathbb{R}^d : \exists x \neq \bar{x} : df_i(p,x) = df_i(p,\bar{x}) = 0, f_i(p,x) = f_i(p,\bar{x})\},$$

consisting of $A_{\geq 1}^{\geq 2}$-singularities. (The notation $\mathcal{E}_i$ and $\mathcal{S}_i$ indicates that, from a classical differential geometry point of view, the local and level bifurcation sets are evolutes and symmetry sets, respectively; see section 3. Also, recall that in all functions depending on the parameters $p \in \mathbb{R}^d$, the differentials are with respect to the remaining variables.)

The bifurcation set $\mathcal{B}$ of the arrangement associated to $X = \cup X_i$ is the union of the bifurcation sets $\mathcal{B}_i$ of the X_i and the following intersurface level bifurcation sets:

$$\mathcal{S}_{i,j} := \{p \in \mathbb{R}^d : \exists x, \bar{x} : df_i(p,x) = df_j(p,\bar{x}) = 0, f_i(p,x) = f_j(p,\bar{x})\},$$

that is,

$$\mathcal{B} := \bigcup_{1\leq i\leq n} \mathcal{E}_i \cup \bigcup_{1\leq i\leq n} \mathcal{S}_i \cup \bigcup_{1\leq i<j\leq n} \mathcal{S}_{i,j}.$$

This definition of $\mathcal{B}$ assumes that $1 \leq \dim X_i \leq d-1$, but it can be extended easily to include isolated points $X_i = \{q_i\}$. For a point q_i the sets $\mathcal{E}_i$ and $\mathcal{S}_i$ are defined to be empty, for a point pair q_i, q_j the set $\mathcal{S}_{i,j}$ is defined to be the hyperplane perpendicular to $q_j - q_i$ through $(q_i + q_j)/2$, and for surface-point pairs X_i ($\dim X_i \geq 1$), $X_j = \{q_j\}$ we define

$$\mathcal{S}_{i,j} := \{p \in \mathbb{R}^d : \exists x : df_i(p,x) = 0, f_i(p,x) = \|q_j - p\|^2\}.$$

The definitions of the local bifurcation sets $\mathcal{E}_i$ and of the intersurface level bifurcation sets $\mathcal{S}_{i,j}$ are fairly straightforward from a computational point of view. The definition of the intrasurface level bifurcation sets $\mathcal{S}_i$ is less straightforward: the inequalities $x \neq \bar{x}$, together with the defining equations appearing in the definition, yield semialgebraic sets $\mathcal{S}_i' \subset \mathbb{R}^d \times \mathbb{R}^{2m_i}$ which are not closed. It is, however, possible to close up the sets $\mathcal{S}_i'$ by adding a set of boundary points $\partial\mathcal{S}_i'$ on the diagonal $\{x = \bar{x}\} \subset \mathbb{R}^{2m_i}$ (see below). Furthermore, the closed sets $\tilde{\mathcal{S}}_i := \mathcal{S}_i' \cup \partial\mathcal{S}_i'$ can be defined by polynomial equations (inequations are not required), which is a big advantage from a computational algebra point of view.

Let $\bar{\mathcal{S}}_i \subset \mathbb{R}^d \times \mathbb{R}^{2m_i}$ denote the set defined by the defining equations of $\mathcal{S}_i'$ (omitting the inequalities $x \neq \bar{x}$); this is a closed set which coincides with $\mathcal{S}_i'$ away from the diagonal, but has too high dimension on the diagonal. Then the vanishing ideal of the closure of $\mathcal{S}_i'$ is given by the ideal quotient

$$I(\bar{\mathcal{S}}_i) : (I(\bar{\mathcal{S}}_i) + \langle x^1 - \bar{x}^1, \ldots, x^{m_i} - \bar{x}^{m_i}\rangle),$$

whose generators can be determined using Gröbner basis methods (see, for example, [3, Chapter 6.2]). For zero-sets X_i of codimension ≥ 2 we actually close up the sets $\mathcal{S}_i'$ in this way (see section 3.2). However, for arrangements of parametrized sets (and of zero-sets of codimension 1; see section 3.1) the defining equations of the closure of $\mathcal{S}_i'$ can be constructed in a more direct way that avoids the expensive computation of Gröbner bases and also yields some useful information about the topology of the

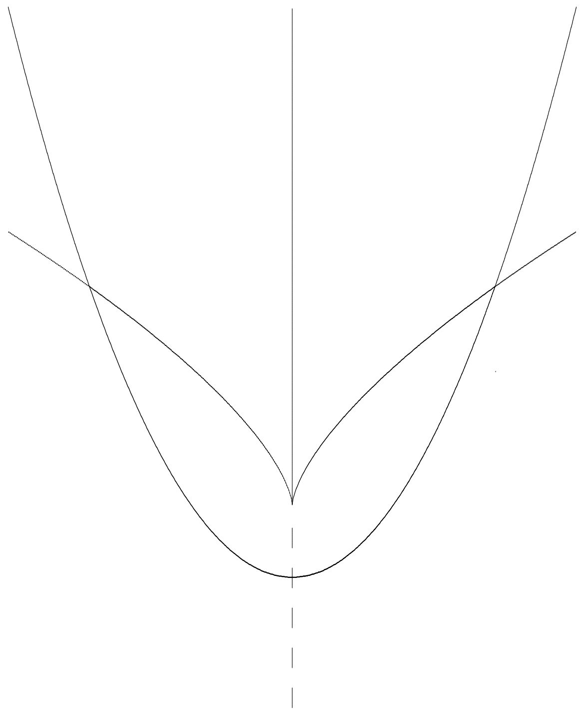

Fig. 1. *The bifurcation set $\mathcal{B}_i$ of a single parabola X_i: the local bifurcation set $\mathcal{E}_i$ is the cusp-shaped curve and the level bifurcation set $\mathcal{S}_i$ is the solid half-line whose boundary $\partial\mathcal{S}_i$ in the union $\hat{\mathcal{S}}_i$ of the solid and dashed vertical line coincides with the cusp point. The distance-squared function to X_i from points of $\mathcal{B}_i$ has the following singularities: A_3 at the cusp, A_2 for all other points of $\mathcal{E}_i$, and A_1^2 for all other points of $\mathcal{S}_i$.*

bifurcation set $\mathcal{B}_i$. We first give an outline of this construction and its topological consequences; more detailed statements follow in Proposition 2.1 and its proof.

To find the defining equations of the closure of $\mathcal{S}_i'$, we first "blow up" the diagonal $\{x = \bar{x}\} \subset \mathbb{R}^{2m_i}$ by a change of coordinates β that replaces the pair of points $(x, \bar{x})$ by $(x, x + \lambda \cdot \omega)$, where $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{P}^{m_i - 1}$ (i.e., we represent the point $\bar{x}$ by moving it some distance λ along a ray through x and direction ω). The map β is an isomorphism for $\lambda \neq 0$ "blowing down" the hyperplane $\{\lambda = 0\}$ to the diagonal $\{x = \bar{x}\}$, which is a linear subspace of $\mathbb{R}^{2m_i}$ of codimension m_i. Next, we choose a certain set of generators of $I(\bar{\mathcal{S}}_i)$ and divide them by suitable powers of λ. The modified generators define a closed algebraic set $\tilde{\mathcal{S}}_i$ that coincides with the sets $\bar{\mathcal{S}}_i$ and $\mathcal{S}_i'$ in the complement of the diagonal $\{\lambda = 0\}$. The set of boundary points of $\mathcal{S}_i'$ on the diagonal is given by $\partial\mathcal{S}_i' = \tilde{\mathcal{S}}_i \cap \{\lambda = 0\}$. Using the defining equations of the sets $\tilde{\mathcal{S}}_i$, $\mathcal{S}_i'$, and $\bar{\mathcal{S}}_i$ and excluding a Zariski closed subset of X_i in the space of all algebraic sets (where m_i, δ_i, and d are fixed) one easily checks the following properties. The set $\mathcal{S}_i'$ has dimension $d-1$, and "almost all" of its points $(p, x, \bar{x})$ correspond to pairs of A_1-singularities of the distance-squared function having the same critical values $f_i(p, x) = f_i(p, \bar{x})$. The set $\partial\mathcal{S}_i'$ has dimension $d-2$ and almost all of its points correspond to A_3-singularities. (By contrast, most points of $\bar{\mathcal{S}}_i \cap \{\lambda = 0\}$ merely correspond to A_1-singularities and form a $d+m_i$-dimensional set.) It therefore follows that the projection of $\partial\mathcal{S}_i'$ into the parameter space $\mathbb{R}^d$ is contained in the local bifurcation set $\mathcal{E}_i$ (which corresponds to

$A_{\geq 2}$-singularities). The projection of $\partial\mathcal{S}'_i$, denoted by $\partial\mathcal{S}_i$, also forms the boundary of the semialgebraic level bifurcation set $\mathcal{S}_i \subset \mathbb{R}^d$ in the smallest real algebraic set $\hat{\mathcal{S}}_i$ containing $\mathcal{S}_i$: moving along a generic path from $\mathcal{S}_i$ to $\hat{\mathcal{S}}_i \setminus \mathcal{S}_i$ we first get a pair of real A_1-singularities having the same critical value, which coalesce in an A_3-singularity as we cross the boundary $\partial\mathcal{S}_i$ and then become complex. Figure 1 illustrates this situation in the simple case of a single parabola $X_i(x) = (x, x^2)$ in the plane. We can now give a more precise description of this construction.

PROPOSITION 2.1.

(i) *For all* $(p, x, x) \in \partial\mathcal{S}'_i$*, the distance-squared function* $x \mapsto f_i(p, x)$ *has an* $A_{\geq 3}$*-singularity at* x*. This implies that* $\pi(\partial\mathcal{S}'_i) \subset \mathcal{E}_i$*, where* $\pi : \mathbb{R}^d \times \mathbb{R}^{2m_i} \to \mathbb{R}^d$ *denotes the projection onto the first factor.*

(ii) *The degree of* $\partial\mathcal{S}'_i$ *is of order* $\delta_i^{2m_i+1}$ *and that of* $\cup\partial\mathcal{S}'_i$ *of order* $n \cdot \delta^{2m+1}$.

Proof. (i) The proof of the first part of the proposition follows the construction outlined above. Set $A := (a^1, \ldots, a^{m_i-1}, 1)$, then the map given by

$$\beta : \mathbb{R}^{2m_i} \to \mathbb{R}^{2m_i}, \quad (x, \lambda, A) \mapsto (x, x + \lambda \cdot A)$$

"blows down" the hyperplane $\{\lambda = 0\}$ to the diagonal $\{x = \bar{x}\}$ and has maximal rank for $\lambda \neq 0$. Note that we have replaced the space of directions $\omega \in \mathbb{P}^{m_i-1}$ by the affine chart of vectors A in $\mathbb{R}^{m_i}$ whose last component is equal to 1. To cover all of $\mathbb{P}^{m_i-1}$, m_i such charts are required, but it is easy to check that the arguments below do not depend on the choice of chart. We then claim that the set $\tilde{\mathcal{S}}_i$ can be defined by the following three equations (omitting the inequality $\lambda \neq 0$): by $df_i(p, x) = 0$ (as before), and by

$$U_i(p, x, \lambda, A) := \lambda^{-1}\big(df_i(p, x + \lambda \cdot A) - df_i(p, x)\big) = 0,$$

and by

$$V_i(p, x, \lambda, A) := \lambda^{-3}\Big(f_i(p, x + \lambda \cdot A) - f_i(p, x) - \lambda df_i(p, x)[A] - \frac{\lambda^2}{2}\langle U_i, A\rangle\Big) = 0.$$

It is easy to see that, away from the diagonal $\{\lambda = 0\}$,

$$df_i(p, x) = U_i(p, x, \lambda, A) = V_i(p, x, \lambda, A) = 0$$

and the original system

$$df_i(p, x) = df_i(p, x + \lambda \cdot A) = f_i(p, x) - f_i(p, x + \lambda \cdot A) = 0$$

define the same zero-sets $\mathcal{S}'_i \subset \mathbb{R}^d \times \mathbb{R}^{2m_i} \setminus \{\lambda = 0\}$. Furthermore, the right-hand sides of U_i and V_i are divisible by λ and λ^3 (by Taylor's theorem); hence $df_i = U_i = V_i = 0$ defines a closed algebraic variety $\tilde{\mathcal{S}}_i := \mathcal{S}'_i \cup \partial\mathcal{S}'_i \subset \mathbb{R}^d \times \mathbb{R}^{2m_i}$.

In fact, $\tilde{\mathcal{S}}_i$ is the smallest closed set containing $\mathcal{S}'_i$, and the boundary $\partial\mathcal{S}'_i := \tilde{\mathcal{S}}_i \cap \{\lambda = 0\}$ of $\mathcal{S}'_i$ in $\tilde{\mathcal{S}}_i$ corresponds to $A_{\geq 3}$-singularities of the distance-squared function $x \mapsto f_i(p, x)$. The boundary $\partial\mathcal{S}'_i$ is defined by the following equations:

$$\begin{aligned} df_i(p, x) &= 0, \\ U_i(p, x, 0, A) = d^2 f_i(p, x)[A] &= 0, \\ V_i(p, x, 0, A) = -\frac{1}{12} d^3 f_i(p, x)[A^3] &= 0. \end{aligned}$$

This system "recognizes" an $A_{\geq 3}$-singularity of $x \mapsto f_i(p,x)$ at x—the condition for an $A_{\geq 3}$-singularity is precisely that $df_i = 0$ and $d^2 f_i[v] = 0$, $d^3 f_i[v^3] = 0$ for some nonzero vector v (see, for example, Porteous [18, p. 397]).

(ii) The degree of the variety $\partial \mathcal{S}'_i$ defined by the above system of equations is at most of order $\delta_i^{2m_i+1}$ (by Bezout's theorem), so that the degree of the union of n such varieties is of order $\sum_{i=1}^n \delta_i^{2m_i+1} \leq n \cdot \delta^{2m+1}$. $\square$

Remarks. 1. Patching together the $\tilde{\mathcal{S}}_i$ in the m_i affine charts in the proof of part i yields a variety $V \subset \mathbb{R}^d \times \mathbb{R}^{m_i+1} \times \mathbb{P}^{m_i-1}$ whose projection π onto $\mathbb{R}^d$ is the intrasurface level bifurcation set $\mathcal{S}_i$. To compute the defining equations of $\mathcal{S}_i$ it is sufficient to use a single "good" affine chart for $\mathbb{P}^{m_i-1}$: for example, $(a^1, \ldots, a^{m_i-1}, 1)$ is good if $\dim V > \dim(V \cap \{a^{m_i} = 0\})$. In this case, the missing component of V "at infinity" will be closed up by the projection π.

2. Part i of the proposition also holds locally for germs of C^∞-submanifolds X_i of dimension m_i (note that the proof merely depends on the Taylor expansion of f_i at $(p,x) = (p_0, x_0)$). In particular, it also holds for algebraic zero-sets of dimension m_i (which will be studied in section 3), because these sets have a parametrization which is even analytic.

3. For $m = 1$ and $d = 2$, the set $\cup \partial \mathcal{S}'_i$ consists of isolated points $(p_l, x_l) \in \mathbb{R}^2 \times \mathbb{R}$ (this can be checked by a simple dimensional argument), and there are at most $O(n \cdot \delta^3)$ such endpoints by the proposition above. The projections p_l of these points into the plane are possible endpoints of the level-bifurcation set $\mathcal{S} = \cup \mathcal{S}_i$. However, their projections x_l onto $\mathbb{R}$ correspond to curvature extrema of X_i and each curvature extremum corresponds to one endpoint. But $X = \cup X_i$ has at most $O(n \cdot \delta)$ curvature extrema.

Proposition 2.2. *For all points p in a single connected region of $\mathbb{R}^d \setminus \cup \mathcal{E}_i$ the collection of distance-squared functions $\{x \mapsto f_i(p,x) : 1 \leq i \leq n\}$ has a constant number, c, of critical points, where*

$$n \leq c \leq \sum_{i=1}^{n} (2\delta_i - 1)^{m_i} \sim O(n \cdot \delta^m).$$

Proof. From the definition of the local bifurcation sets $\mathcal{E}_i$ we see that the distance-squared functions $x \mapsto f_i(p,x)$ have isolated critical points (of multiplicity 1) for all $p \in \mathbb{R}^d \setminus \cup \mathcal{E}_i$. Each f_i is nonnegative and has degree $2\delta_i$. Hence, each f_i has at least one local minimum and at most $(2\delta_i - 1)^{m_i}$ critical points; this yields the desired bounds for c. $\square$

Remark. For arrangements of hypersurfaces X_i (i.e., $m_i = d - 1$) the number of critical points c has the following geometrical interpretation: it is equal to the number of normal lines of $X = \cup X_i$ passing through the point p.

Proposition 2.3. *The number of connected regions of $\mathbb{R}^d \setminus \mathcal{B}$ (and, in fact, the size of $\mathcal{A}(\mathcal{B})$) are at most of order $n^{2d} \cdot \delta^{(2m+1)d}$. Furthermore, let $p \in \mathbb{R}^d \setminus \mathcal{B}$, and let*

$$\xi_{1,\nu_1}(p), \xi_{2,\nu_2}(p), \ldots, \xi_{c,\nu_c}(p)$$

denote the critical points $\xi_{l,\nu_l}(p)$ of the collection of distance-squared functions $x \mapsto f_{\nu_l}(p,x)$, where $\nu_l \in \{1, \ldots, n\}$, ordered by increasing distance. That is, $f_{\nu_l}(p, \xi_{l,\nu_l}(p)) < f_{\nu_{l+1}}(p, \xi_{l+1,\nu_{l+1}}(p))$. For all points p in a single connected region of $\mathbb{R}^d \setminus \mathcal{B}$ we have the following: (i) *the numbers $\nu_1, \ldots, \nu_c$ are invariant, and* (ii) *the maps $p \mapsto \xi_{l,\nu_l}(p)$ are continuous for $1 \leq l \leq n$.*

Proof. The bifurcation set $\mathcal{B}$ is a semialgebraic subset of a closed real algebraic set $\hat{\mathcal{B}} \subset \mathbb{R}^d$, and the number of connected regions cut out by $\mathcal{B}$ is less than or equal to the number of regions cut out by $\hat{\mathcal{B}}$. The number of connected regions of $\mathbb{R}^d \setminus \hat{\mathcal{B}}$ is equal to the $(d-1)$st Betti number of $\hat{\mathcal{B}}$ plus 1 (see below), and the desired upper bound follows at once from a result of Milnor [15] (which says that the sum of the Betti numbers of $\hat{\mathcal{B}}$ is of order $(\deg \hat{\mathcal{B}})^d$) and the bound for the degree of $\hat{\mathcal{B}}$ derived below. (On the other hand, the singular stratification of $\hat{\mathcal{B}}$ has $O((\deg \hat{\mathcal{B}})^d)$ strata and $|\mathcal{A}(\mathcal{B})| \le |\mathcal{A}(\hat{\mathcal{B}})|$, which yields the bound for $|\mathcal{A}(\mathcal{B})|$.)

The (linear) formula for the number of connected components of $\mathbb{R}^d \setminus \hat{\mathcal{B}}$ in terms of the $(d-1)$st Betti number of $\hat{\mathcal{B}}$ follows from standard duality results in algebraic topology: roughly speaking, either from Lefschetz duality (which yields an isomorphism between the zeroth homology group H_0 of $\mathbb{R}^d \setminus \hat{\mathcal{B}}$ and the dth cohomology group H^d of the pair $(\mathbb{R}^d, \hat{\mathcal{B}})$) and the isomorphism $H_d(\mathbb{R}^d, \hat{\mathcal{B}}) \cong H_{d-1}(\hat{\mathcal{B}})$ (coming from the standard exact homology sequence of the pair $(\mathbb{R}^d, \hat{\mathcal{B}})$); or, more directly, one can use Alexander duality to get an isomorphism between H_0 of $\mathbb{R}^d \setminus \hat{\mathcal{B}}$ and H^{d-1} of $\hat{\mathcal{B}}$. The more precise argument (included in parentheses below) is a bit more complicated, due to the possible noncompactness of $\hat{\mathcal{B}}$ and the appearance of reduced homology groups and might be skipped. (Let $S^d = \mathbb{R}^d \cup \{\infty\}$ and $\hat{\mathcal{B}}^c = \hat{\mathcal{B}} \cup \{\infty\}$ denote 1-point compactifications; then the Alexander duality yields the following isomorphism of reduced (co)homology groups $\tilde{H}_0(S^d \setminus \hat{\mathcal{B}}^c) \cong \tilde{H}^{d-1}(\hat{\mathcal{B}}^c)$. (See [11, Section 8.15, Chapter VIII]). The set $\hat{\mathcal{B}}$ is a closed real algebraic set; hence $\tilde{H}_0(S^d \setminus \hat{\mathcal{B}}^c) \cong \tilde{H}_0(\mathbb{R}^d \setminus \hat{\mathcal{B}})$ and $b_i(\hat{\mathcal{B}}^c) = b_i(\hat{\mathcal{B}})$, for all $i > 0$. Finally, note that the rank of the zeroth homology group is one plus that of the reduced one.)

We claim that the degree of $\hat{\mathcal{B}}$ is of order $n^2\delta^{2m+1}$. The set $\hat{\mathcal{B}}$ is the union of $\binom{n}{2}$ (real algebraic) sets $\hat{\mathcal{S}}_{i,j}$, n sets $\hat{\mathcal{S}}_i$, and n sets $\hat{\mathcal{E}}_i$. The orders of the degrees of the $\hat{\mathcal{S}}_i$ and the $\hat{\mathcal{E}}_i$ are lower than those of the $\hat{\mathcal{S}}_{i,j}$; hence it suffices to estimate the degree of $\hat{\mathcal{S}}_{i,j}$. So let $\tilde{\mathcal{S}}_{i,j} \subset \mathbb{R}^d \times \mathbb{R}^{m_i+m_j}$ be the real algebraic set defined by the defining equations of $\mathcal{S}_{i,j}$ (omitting the existential quantifier). The restriction of the projection $\pi : \mathbb{R}^d \times \mathbb{R}^{m_i+m_j} \to \mathbb{R}^d$ to $\tilde{\mathcal{S}}_{i,j}$ yields the semialgebraic set $\mathcal{S}_{i,j}$. Complexifying the defining equations of $\tilde{\mathcal{S}}_{i,j}$ and taking the real part of the projection π onto $\mathbb{C}^d$ of the resulting zero-set yields a closed real algebraic set $\hat{\mathcal{S}}_{i,j} \subset \mathbb{R}^d$ which contains the semialgebraic set $\mathcal{S}_{i,j}$. Suppose that $\operatorname{codim}\hat{\mathcal{S}}_{i,j} = 1$ (otherwise the complement of $\hat{\mathcal{S}}_{i,j}$ is connected, and we are finished), and let $\mathcal{L} \subset \mathbb{R}^d$ be any line. Now there are two cases: (1) the set $A := \pi^{-1}(\mathcal{L}) \cap \tilde{\mathcal{S}}_{i,j}$ consists of isolated points (the "generic case") and (2) $\dim A = e > 0$. Let $\bar{\pi} : \mathbb{R}^d \times \mathbb{R}^{m_i+m_j} \to \mathbb{R}^{m_i+m_j}$ denote the projection onto the second factor, let $\bar{\mathcal{L}} \subset \mathbb{R}^{m_i+m_j}$ be any linear subspace of codimension e such that $\bar{\pi}^{-1}(\bar{\mathcal{L}})$ is not contained in A and set $\bar{A} := A \cap \bar{\pi}^{-1}(\bar{\mathcal{L}})$. The sets A in case 1 and $\bar{A}$ in case 2 are discrete point sets, and the restriction of π to these point sets onto the set of intersection points $\hat{\mathcal{S}}_{i,j} \cap \mathcal{L}$ is surjective. The degree of $\hat{\mathcal{S}}_{i,j}$ is therefore bounded by the number of points of A (or $\bar{A}$ in case 2). Inspecting the defining equations of these sets, we get from Bezout's theorem that

$$\deg \hat{\mathcal{S}}_{i,j} \le (2\delta_i - 1)^{m_i} \cdot (2\delta_j - 1)^{m_j} \cdot 2\max(\delta_i, \delta_j)$$

(counting both real and complex roots with their multiplicities).

For the proof of the second part of the proposition, consider the following real algebraic set:

$$\Sigma_{F_i} := \{(p, x) : df_i(p, x) = 0\} \subset \mathbb{R}^d \times \mathbb{R}^{m_i}.$$

The set Σ_{F_i} is the critical set of the family F_i of all distance-squared functions f_i on X_i. The fibers $\pi^{-1}(p) \cap \Sigma_{F_i}$ of the projection $\pi : \mathbb{R}^d \times \mathbb{R}^{m_i} \to \mathbb{R}^d$ correspond to the critical points of f_i from p. The restriction of π to Σ_{F_i} is a covering map whose branch-locus is the (preimage of the) evolute $\mathcal{E}_i$ and which is finite-to-one off the branch-locus. The number of points in each fiber $\pi^{-1}(p) \cap \Sigma_{F_i}$ is therefore finite and constant for all points p in a connected region of $\mathbb{R}^d \setminus \mathcal{E}_i$. The same is true for the total number c of critical points of a collection $\{f_i\}_{1 \le i \le n}$ of distance-squared functions on $X := \cup X_i$ for all p in a single connected region of $\mathbb{R}^d \setminus \cup\mathcal{E}_i$. Furthermore, the indices $\nu_1, \dots, \nu_c$ are invariant within a connected region of $\mathbb{R}^d \setminus (\cup\mathcal{E}_i) \cup (\cup\mathcal{S}_{i,j})$, because c is constant and permutations of indices can only occur along the intersurface level bifurcation sets $\mathcal{S}_{i,j}$. Finally, let U be any connected region of $\mathbb{R}^d \setminus \cup\mathcal{E}_i$ and consider the union of the n bundles $\cup_{i=1}^n \pi^{-1}(U) \cap \Sigma_{F_i}$. This is a semialgebraic set consisting of c disjoint components of dimension d, and these components are the graphs of continuous maps $h_j : U \to \mathbb{R}^{m_i}$, $1 \le j \le c$ (these facts are established by arguments that are quite similar to the proof of the first main structure theorem in [4, Chapter 2.2]; in fact, most stratification schemes of semialgebraic sets seem to be based on some version of this theorem). The composition of the h_j with the projection $\bar{\pi} : \mathbb{R}^d \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ is a continuous map, which implies that the c critical points of the collection of distance-squared functions $x \mapsto f_i(p, x)$, $1 \le i \le n$, vary continuously with $p \in U$. The continuity of the maps $p \mapsto \xi_{l,\nu_l}(p)$ for all p within a single connected region of $\mathbb{R}^d \setminus \mathcal{B}$ then follows from the results above and the fact that the permutation of the critical points of a single function $x \mapsto f_i(p, x)$ can only occur on $\mathcal{S}_i$. $\square$

3. Contact of X with spheres and the definition of $\mathcal{B}$ for zero-sets X. The fibers of the distance-squared function from a point $p \in \mathbb{R}^d$ are $(d-1)$-spheres of varying radius r, given by $\{x \in \mathbb{R}^d : \|x - p\|^2 - r^2 = 0\}$. The conditions for an A_k-singularity of the distance-squared function, which appear in the definition of the bifurcation set $\mathcal{B}$, can be reformulated in more geometric terms involving the contact between a family of such spheres and a collection $X = \cup X_i$ of algebraic sets. Using these more geometric conditions, we can easily define, and compute, the bifurcation set $\mathcal{B}$ in the case of algebraic sets X_i given as zero-sets of polynomials $h_i^j \in \mathbb{Q}[x] = \mathbb{Q}[x^1, \dots, x^d]$, $1 \le j \le d - m_i$.

We first consider the special case of algebraic hypersurfaces (codimension 1) where the local and level bifurcation sets of the distance-squared function are the well-known evolutes and symmetry sets of classical differential geometry (section 3.1). In section 3.2 we consider the more general case of arrangements of algebraic sets X_i of codimension $1 \le d - m_i \le d - 1$ which are complete intersections (i.e., are defined by $d - m_i$ polynomials). Note that the case of points X_i (of codimension d) can be handled as in section 2.

3.1. Arrangements of hypersurfaces, evolutes, and symmetry sets. First, recall that a hypersurface X_i in d-space has $d-1$ (not necessarily distinct) principal curvatures κ_j and directions d_j which are the eigenvalues and eigendirections of the Weingarten map. (The Weingarten map $W_p : T_pX_i \to T_pX_i$, $v \mapsto -\nabla_v N$ measures the rate of change of the normal direction N along a direction v in the tangent space of X_i at p.) A $(d-1)$-sphere is a curvature sphere at $x \in X_i$ if its center lies on the normal line through x and its radius r is the inverse of one of the principal curvatures of X_i at x. The unique great circle in this curvature sphere whose tangent line at x is oriented along the principal direction associated to $1/r$ is an osculating circle. The evolute (or focal surface) $\mathcal{E}_i$ of X_i is the locus of centers of such osculating circles and of the curvature spheres containing them (for each surface patch of X_i there are generically $d-1$ sheets of the evolute, one for each principal curvature).

The distance-squared function from $p \in \mathbb{R}^d$ to X_i has an A_k-singularity ($k \geq 1$) at $x \in \mathbb{R}^d$ if and only if there exists a circle with center p having $(k+1)$-point contact with X_i at x. The order of contact is ≥ 2 if p lies on the normal line to X_i at x and ≥ 3 if, in addition, the circle is an osculating circle. The local bifurcation set $\mathcal{E}_i$ consists of points p for which the distance-squared function to X_i has an $A_{\geq 2}$-singularity; such points are centers of osculating circles (and of curvature spheres). The local bifurcation set $\mathcal{E}_i$ is therefore the evolute of X_i. The relation between singularities of the distance-squared function, normal singularities of submanifolds (i.e., singularities of the exponential map of the normal bundle), and the possible types of contact between these submanifolds and spheres was first studied by Porteous; see [16] and [17].

The intra- and intersurface level bifurcation sets $\mathcal{S}_i$ and $\mathcal{S}_{i,j}$ are loci of centers of bitangent spheres touching $X = \cup X_i$ in two distinct points (the spheres can shrink to a point as their centers tend to the self-intersection locus of X). If both points of tangency lie on a single surface X_i then the center belongs to $\mathcal{S}_i$, otherwise it belongs to $\mathcal{S}_{i,j}$. Clearly, the distance-squared function from a center of a bitangent sphere has the two points of tangency as its critical points, and the corresponding critical values are given by the square of the radius of the bitangent sphere. The locus of centers of bitangent spheres of a hypersurface is known as symmetry set in the differential geometry literature, and the singularities of such symmetry sets of plane curves and of surfaces in 3-space have been classified by Bruce, Giblin, and Gibson [6]. (In the pattern recognition literature, the symmetry set of a plane curve is also known under the names skeleton, medial axis, and symmetric axis transform.)

Using these geometrical descriptions of the local bifurcation sets $\mathcal{E}_i$ and of the level bifurcation sets $\mathcal{S}_i$ and $\mathcal{S}_{i,j}$, we can now define the bifurcation set of the distance-squared functions for arrangements of algebraic hypersurfaces given as zero-sets $X_i = h_i^{-1}(0)$. Below, $V \| W$ denotes the condition that the pair of vectors V, W in $\mathbb{R}^d$ is parallel (obviously, this condition involves the vanishing of $d-1$ functions involving the components of the vectors), and $S(p,x,r) := \|x-p\|^2 - r^2$ defines a $(d-1)$-sphere with center p and radius r. The fact that (at least) one of the principal curvatures of X_i at x is equal to $1/r$ is equivalent to the vanishing of the following two equations:

$$Q_i(x,u) := \det \begin{pmatrix} (d^2 h_i(x) - u \cdot I) & dh_i(x) \\ (dh_i(x))^t & 0 \end{pmatrix}$$

(where I denotes the $d \times d$ identity matrix) and

$$R_i(x,u,r) := u^2 r^2 - \|dh_i(x)\|^2.$$

(The condition $Q_i = R_i = 0$ can be deduced easily from the standard formula for the principal curvatures of a hypersurface defined as zero-set; see, e.g., [23, p. 204]. Note that the derivation of this formula [23, pp. 202–204] is for hypersurfaces in 3-space, but the d-dimensional case ($d \geq 2$) is analogous.)

Using this notation, the local bifurcation sets (evolutes) are defined as follows:

$$\begin{aligned}\mathcal{E}_i := \{p \in \mathbb{R}^d : \exists x,u,r : & h_i(x) = S(p,x,r) = Q_i(x,u) = R_i(x,u,r) = 0, \\ & dh_i(x) \| dS(p,x,r)\}.\end{aligned}$$

The level bifurcation sets (symmetry sets) are given by

$$\begin{aligned}\mathcal{S}_i := \mathrm{cl}\{p \in \mathbb{R}^d : \exists x_1 \neq x_2 : & h_i(x_k) = 0, dh_i(x_k) \| (x_k - p), k = 1,2; \\ & \|x_1 - p\|^2 = \|x_2 - p\|^2\}\end{aligned}$$

and

$$\mathcal{S}_{i,j} := \{p \in \mathbb{R}^d : \exists x_i, x_j : h_k(x_k) = 0, dh_k(x_k) \| (x_k - p), k = i, j; \\ \|x_i - p\|^2 = \|x_j - p\|^2\}.$$

The estimates in Propositions 2.1, 2.2, and 2.3 for arrangements of parametrized surfaces, in terms of n and δ, have the following analogues, (i)–(iii) of 3.1, in the case of $(d-1)$-dimensional zero-sets.

PROPOSITION 3.1. *Let $\mathcal{B}$ denote the bifurcation set of the family of distance-squared functions on a collection $X = \cup_{i=1}^n X_i$ of algebraic hypersurfaces of maximal degree Δ. Then the following holds:* (i) *the degree of $\cup \partial \mathcal{S}_i'$ is at most of order $n \cdot \Delta^{2d}$;* (ii) *the number of critical points of the distance-squared function from any point $p \in \mathbb{R}^d \setminus \cup \mathcal{E}_i$ to X is at most of order $n \cdot \Delta^d$; and* (iii) *the number of connected regions of $\mathbb{R}^d \setminus \mathcal{B}$ and the size of $\mathcal{A}(\mathcal{B})$ are at most of order $n^{2d} \cdot \Delta^{2d^2}$.*

Proof. For statement (i), we modify the defining equations of $\mathcal{S}_i$, as in the case of parametrized sets X_i, by blowing up the diagonal $\{x_1 = x_2\} \subset \mathbb{R}^{2d}$ by setting $x_2 := x_1 + \lambda \cdot \omega$, where $\omega \in \mathbb{P}^{d-1}$, and by dividing certain generators of the resulting ideal by suitable powers of λ. Let $\tilde{\mathcal{S}_i}$ be the zero-set of these modified defining equations and let $\mathcal{S}_i'$ denote the semialgebraic set that coincides with $\tilde{\mathcal{S}_i}$ off the diagonal $\{\lambda = 0\}$; then again $\partial \mathcal{S}_i' = \tilde{\mathcal{S}_i} \cap \{\lambda = 0\}$ (see the proof of Proposition 2.1). For statements (ii) and (iii) we simply follow the proofs of Propositions 2.2 and 2.3 using the new definitions of the components of $\mathcal{B}$. $\square$

3.2. Arrangements of algebraic sets of higher codimension. Let $X_i = h_i^{-1}(0)$ be the m_i-dimensional zero-set of a polynomial map $h_i := (h_i^1, \ldots, h_i^{d-m_i}) : \mathbb{R}^d \to \mathbb{R}^{d-m_i}$. The distance-squared function from p to X_i has an A_k-singularity at x if and only if there exists a $(d-m_i)$-sphere with center p having $(k+1)$-point contact with X_i at x. Algebraically, the order of contact (or intersection multiplicity) between X_i and a $(d-m_i)$-sphere, with defining equations $s^1(\xi) = \cdots = s^{m_i}(\xi) = 0$, at x is equal to the dimension of the vector space

$$\mathbb{R}[\xi] / \langle h_i^1(\xi - x), \ldots, h_i^{d-m_i}(\xi - x), s^1(\xi - x), \ldots, s^{m_i}(\xi - x) \rangle.$$

It is easy to see that such a sphere has at least 2-point contact with X_i at x if its center p lies in the normal space $N_x X_i = x + \operatorname{span}\{dh_i^1(x), \ldots, dh_i^{d-m_i}(x)\}$ of X_i at x (this assumes that x is a regular point of X_i, but the algebraic definition of the intersection multiplicity above is also valid for the singular locus of X_i).

We can now define the local bifurcation set $\mathcal{E}_i$ for complete intersections X_i and give an estimate for its degree. The point p lies in the normal space of X_i at x if $x - p \in \operatorname{span}\{dh_i^1(x), \ldots, dh_i^{d-m_i}(x)\}$, which means that all $(d-m_i+1) \times (d-m_i+1)$ minors of $\binom{dh_i(x)}{x-p}$ have to vanish. Note that only m_i of these minors are independent and that each of them has degree $O(\Delta_i)$. Let $\mathcal{M}_i := (\mathcal{M}_i^1, \ldots, \mathcal{M}_i^{m_i}) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{m_i}$ be a d-parameter family of polynomial maps, depending on the variables x and parameters p, whose component functions are such independent minors. Then $\varphi_i := (h_i, \mathcal{M}_i) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(p, x) \mapsto \varphi_i(p, x)$ is a d-parameter family of polynomial maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. Using the algebraic definition of the intersection multiplicity, one checks that the simple roots in x of φ_i correspond to points of X_i having 2-point contact with $(d-m_i)$-spheres with center p through x. Roots of higher multiplicity correspond to points x in which the order of contact is at least 3-point; hence we define

$$\mathcal{E}_i := \{p \in \mathbb{R}^d : \exists x : \varphi_i(p,x) = \det d\varphi_i(p,x) = 0\}.$$

The product of the degrees of these defining equations of $\mathcal{E}_i$ is at most $O(\Delta_i^{2(m_i+1)})$.

The level bifurcation set of the family of all distance-squared functions to a pair of complete intersections $X_i = h_i^{-1}(0)$ and $X_j = h_j^{-1}(0)$ of dimension m_i and m_j is given by

$$\mathcal{S}_{i,j} := \{p \in \mathbb{R}^d : \exists x, \bar{x} : \varphi_i(p,x) = \varphi_j(p,\bar{x}) = 0, \|x-p\|^2 = \|\bar{x}-p\|^2\}$$

and has degree at most $O(\Delta_i^{m_i+1}\Delta_j^{m_j+1})$. The level bifurcation set of a single set X_i is given by

$$\mathcal{S}_i := \mathrm{cl}\{p \in \mathbb{R}^d : \exists x \neq \bar{x} : \varphi_i(p,x) = \varphi_i(p,\bar{x}) = 0, \|x-p\|^2 = \|\bar{x}-p\|^2\}$$

and has degree at most $O(\Delta_i^{2(m_i+1)})$. Recall that $\mathcal{S}_i$ is the projection of the algebraic set $\tilde{\mathcal{S}}_i := \mathcal{S}'_i \cup \partial\mathcal{S}'_i$. The set $\tilde{\mathcal{S}}_i$ is the closure of the difference of two algebraic sets $U \setminus V$, where U is the zero-set of the defining equations of $\mathcal{S}_i$, omitting the inequations $x \neq \bar{x}$, and where V is defined by the equations of $\mathcal{S}_i$ and by $x = \bar{x}$. Hence, $\tilde{\mathcal{S}}_i = Z(I(U) : I(V))$ is an algebraic set of degree at most $\deg U \sim O(\Delta_i^{2(m_i+1)})$, and its projection $\mathcal{S}_i$ is a semi-algebraic subset of an algebraic set of degree $O(\Delta_i^{2(m_i+1)})$.

Next recall that the boundary $\mathcal{S}'_i$ of $\tilde{\mathcal{S}}_i$ is contained in the diagonal $E := \{x = \bar{x}\} \subset \mathbb{R}^{2d}$. The subspace E is linear which implies that $\deg \tilde{\mathcal{S}}_i \cap E = \deg \tilde{\mathcal{S}}_i$ and that $\mathcal{S}'_i \subset \tilde{\mathcal{S}}_i \cap E$ has degree at most $O(\Delta_i^{2(m_i+1)})$.

Finally, note that the number of critical points of the distance-squared function from some fixed point $p \in \mathbb{R}^d \setminus \mathcal{E}_i$ is finite and bounded above by the degree of the map φ_i, which is $O(\Delta_i^{m_i+1})$. Summing up, we have the following proposition.

PROPOSITION 3.2. *Let $\mathcal{B}$ denote the bifurcation set of the family of distance-squared functions on a collection $X = \cup_{i=1}^n X_i$ of algebraic sets of maximal degree Δ and maximal dimension m. Then the following holds:* (i) *the degree of $\cup\partial\mathcal{S}'_i$ is at most of order $O(\Delta^{2(m+1)})$;* (ii) *the number of critical points of the distance-squared function from any point $p \in \mathbb{R}^d \setminus \cup\mathcal{E}_i$ to X is at most of order $n \cdot \Delta^{m+1}$; and* (iii) *the number of connected regions of $\mathbb{R}^d \setminus \mathcal{B}$ and the size of $\mathcal{A}(\mathcal{B})$ are at most of order $n^{2d} \cdot \Delta^{2(m+1)d}$*

Remarks. 1. Note that the estimates i, ii, and iii yield in the case of hypersurfaces $(m = d-1)$ the same estimates as in Proposition 3.1 (i), (ii), and (iii).

2. The estimate (i) implies for arrangements of plane curves (where $m = 1$) that there are at most $O(n \cdot \Delta^4)$ endpoints of the level bifurcation set. But, again using the fact that these endpoints correspond to curvature extrema of the curves X_i, one checks that actually there are at most $O(n \cdot \Delta)$ such endpoints.

3. It is also interesting to compare these estimates for arrangements of zero-sets with the corresponding bounds in the special case of parametrized m_i-surfaces given in section 2. Not surprisingly, the combinatorial complexities (fixing the degrees Δ or δ) are the same. However, in terms of algebraic complexity, the estimates in Propositions 2.1, 2.2, and 2.3 for arrangements of parametrized surfaces are sharper than the corresponding ones in Proposition 3.2 This can be seen using the following fact.

LEMMA 3.3. *The degree Δ_i of a parametrized m_i-surface X_i given by*

$$x \mapsto X_i(x) := (X_i^1(x), \dots, X_i^d(x)), \quad \delta_i := \sup_j \deg X_i^j$$

is of order $\delta_i^{m_i}$ (which implies, for arrangements of such surfaces, that $\Delta \sim O(\delta^m)$).

Proof. Let $\mathcal{L}$ be a $(d-m_i)$-dimensional linear subspace of $\mathbb{R}^d$ not contained in X_i, and let $\mathcal{L}$ be given as zero-set of some linear map $L = (L_1, \dots, L_{m_i}) : \mathbb{R}^d \to \mathbb{R}^{m_i}$. By Bézout's theorem, $L \circ X_i : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ has at most $\delta_i^{m_i}$ roots (counting multiplicities, complex roots, and roots at infinity), hence $|\mathcal{L} \cap X_i| \sim O(\delta_i^{m_i})$. □

4. Determining the connected regions of $\mathbb{R}^d \setminus \mathcal{B}$, and applications to proximity queries. This section consists of two parts: in subsection 4.1 we sketch the exact symbolic computation of connected regions of $\mathbb{R}^d \setminus \mathcal{B}$ of constant description size for arrangements of algebraic sets defined by polynomials with rational coefficients. And in subsection 4.2 we discuss how this partition of $\mathbb{R}^d$ can be used to efficiently answer proximity queries for points with algebraic number coordinates. In terms of combinatorial complexity (where the degrees and coefficient sizes of the defining polynomials of the algebraic sets are bounded by some constant), computing the partition takes $O(n^{4d-6+\epsilon})$ (for $d \geq 3$) or $O(n^{4+\epsilon})$ (for $d = 2$) expected time (here ϵ is some small positive constant), and answering a proximity query takes $O(\log n)$ time. In dimensions 2 and 3, the time for computing the partition almost matches the number of regions of $\mathbb{R}^d \setminus \mathcal{B}$ (in the worst case), but in higher dimensions the computation time is much larger than the number of regions. In section 6 we shall study certain partitions, cut out by subsets of $\mathcal{B}$, which have a lower combinatorial complexity but can still be used for the same proximity problems.

4.1. Determining the partition. The bifurcation set $\mathcal{B}$ of the family of distance-squared functions between points $p \in \mathbb{R}^d$ and a collection of algebraic sets is a semi-algebraic set which is the projection of a real algebraic set $\tilde{\mathcal{B}} \subset \mathbb{R}^d \times \mathbb{R}^a$, where $a \leq 2m$ (for parametrized m_i-surfaces, $m := \sup m_i$) or $a = 2d$ (for zero-sets). The defining equations of the components $\tilde{\mathcal{E}}_i$, $\tilde{\mathcal{S}}_i$, $\tilde{\mathcal{S}}_{ij}$ of $\tilde{\mathcal{B}}$ are described in sections 2 and 3 and are polynomials with rational coefficients (we assume that the X_i are defined by polynomials over $\mathbb{Q}$). If π denotes the restriction to $\tilde{\mathcal{B}}$ of the obvious projection from $\mathbb{R}^d \times \mathbb{R}^a$ to $\mathbb{R}^d$ and if $\hat{\mathcal{B}} \subset \mathbb{R}^d$ is a closed real algebraic set containing $\mathcal{B}$, then we have the following set-up for the algorithm below (which consists of three steps):

$$\begin{array}{ll} \tilde{\mathcal{B}} & \subset \mathbb{R}^d \times \mathbb{R}^a \\ \big\downarrow{\scriptstyle \pi} & \\ \mathcal{B} & \subset \hat{\mathcal{B}} \subset \mathbb{R}^d \end{array}$$

1. Eliminate $x^1, \dots, x^a$ between the defining equations of $\tilde{\mathcal{B}}$. Result: the defining equations of the real algebraic set $\hat{\mathcal{B}} \subset \mathbb{R}^d$.
2. Decompose $\mathbb{R}^d$ into connected regions (of constant combinatorial complexity) such that each such region lies in a single component of $\mathbb{R}^d \setminus \hat{\mathcal{B}}$ (and hence of $\mathbb{R}^d \setminus \mathcal{B}$).
3. Optional step: determine the connected regions of $\mathbb{R}^d \setminus \mathcal{B}$ by deleting the "branches" of $\hat{\mathcal{B}} \setminus \mathcal{B}$ from $\hat{\mathcal{B}}$.

In these steps we use known techniques; here we discuss their complexity and give some references.

Step 1. The set $\tilde{\mathcal{B}}$ has $2n + \binom{n}{2}$ components; the combinatorial complexity of the elimination is therefore $O(n^2)$. Next, we consider the algebraic complexity. Recall from section 3.2 that, for zero-sets X_i of codimension ≥ 2, the generators of the ideals $I(\tilde{\mathcal{S}}_i)$ have to be precomputed from certain ideal quotients. In all other cases the defining equations of the components of $\tilde{\mathcal{B}}$ are already known. For parametrized surfaces one has to eliminate m_i (for $\tilde{\mathcal{B}}_l = \tilde{\mathcal{E}}_i$), $2m_i$ (for $\tilde{\mathcal{B}}_l = \tilde{\mathcal{S}}_i$), or $m_i + m_j$ variables (for $\tilde{\mathcal{B}}_l = \tilde{\mathcal{S}}_{i,j}$). During the elimination, which can use either multipolynomial resultants (see, for example, [7]) or Gröbner bases, one can remove repeated factors, because the later steps of the algorithm only require information about the radicals of the elimination ideals $I(\hat{\mathcal{B}}_l) := I(\tilde{\mathcal{B}}_l) \cap \mathbb{Q}[p]$. (Note that the worst-case computation time is $D^{O(v)}$ for the multipolynomial resultant and $D^{2^{O(v)}}$ for Gröbner bases, where $D \leq \delta$ or Δ is the maximal degree of the input polynomials and $v = d + a$ the number of variables.)

Step 2. We mention two algorithms in (i) and (ii) below which can be used to compute a partition of $\mathbb{R}^d$ into connected regions, of constant combinatorial complexity, in the complement of $\hat{\mathcal{B}}$. The first is the classical one and has been used to compute the examples shown in section 5; the second is more efficient. Recall that $\hat{\mathcal{B}}$ is the union of the zero-sets of $N \sim O(n^2)$ polynomials of maximal degree $D \sim O(\delta^{2m+1})$ (for parametrized X_i) or $O(\Delta^{2(m+1)})$ (for zero-sets X_i).

(i) The cylindrical algebraic decomposition of Collins [9] yields at most $(ND)^{2^d}$ d-cells in the complement of $\mathbb{R}^d \setminus \hat{\mathcal{B}}$. The cells are diffeomorphic to open d-cubes (so that the number of lower dimensional cells in their closure is independent of N) and can be determined in $L^3(ND)^{2^d}$ time (where L denotes the maximal coefficient size of the input polynomials).

(ii) Chazelle et al. [8] (see also [22, Theorem 8.23]) describe a stratification which yields d-cells in the complement of $\hat{\mathcal{B}}$ whose closure contains a number of lower dimensional cells which does not depend on N. Assuming that the maximal degree D and the bit lengths of all polynomials arising during the computation are bounded by some constant, this stratification consists of at most $O(N^{2d-3+\epsilon})$ (for $d \geq 3$) or $O(N^{2+\epsilon})$ (for $d = 2$) cells which can be determined deterministically in $O(N^{2d+1})$ time or by a randomized algorithm in $O(N^{2d-3+\epsilon})$ (for $d \geq 3$) or $O(N^{2+\epsilon})$ (for $d = 2$) expected time (here ϵ denotes an arbitrarily small positive constant). Furthermore, given some point $p \in \mathbb{R}^d$, the cell containing p can be determined in $O(\log N)$ time. The drawback of this stratification procedure is that, considering the degree D as a variable, the number of cells and the running time become doubly exponential in d with base D.

Remark. The algorithm of Grigor'ev and Vorobjov [14] can be used to find the representative points of a partition of $\mathbb{R}^d \setminus \hat{\mathcal{B}}$ into connected regions, and this algorithm produces at most $(ND)^{d^2}$ such points in $L^{O(1)}(ND)^{O(d^2)}$ time. But it is not clear whether the number of lower-dimensional cells in the closure of each of these regions is independent of N (also, some extra work would be required to compute the region boundaries).

Step 3. The algorithms in Rieger [19, 20] can be adapted to determine the connected regions in the complement of $\mathcal{B}$. The adapted algorithm is based on a very coarse "stratification" (in which the strata can have singularities) of $\hat{\mathcal{B}}$ consisting of $O(n^2) \cdot P$ "branches" (where P is a polynomial in the degrees of the X_i) such that each "branch" either lies entirely in $\mathcal{B}$ or in $\hat{\mathcal{B}} \setminus \mathcal{B}$. Picking a "good" sample point q in each "branch," one can count the number k of real roots of the specialization $I(\tilde{\mathcal{B}})_{p=q}$

(using results from real algebra): if $k > 0$, then the "branch" belongs to $\mathcal{B}$, otherwise we delete it. The running time of this procedure is quadratic in n and polynomial in the remaining parameters. However, a region in the complement of $\hat{\mathcal{B}}$ could have up to $O(N) = O(n^2)$ "branches" of $\hat{\mathcal{B}}$ in its closure, and in the proximity queries discussed next it is important that the regions have a constant number of cells in their closures.

4.2. Answering proximity queries. We now discuss how the decompositions above can be used to answer proximity queries exactly (i.e., without numerical errors). Let $p \in \mathbb{R}^d$ be a point whose coordinates $(\alpha^1, \ldots, \alpha^d)$ are algebraic numbers, represented by minimal polynomials $m^j(t) = 0$ and isolating intervals with rational endpoints. Given p and a set of defining polynomials of X with rational coefficients we would like to do the following:

1. find the k nearest sets X_i;
2. find the nearest point in X;
3. and, provided that X is compact, find the farthest point in X (and hence the smallest sphere with center p enclosing X).

For all three problems we first decompose $\mathbb{R}^d$ into regions which lie in a single connected component in the complement of $\cup\mathcal{S}_{i,j}$ (or $\mathcal{B}$ or $\hat{\mathcal{B}}$—the latter two possibilities yield finer decompositions but with the same "leading term" with respect to the asymptotic complexity in the number of regions). For problem 1 we store for each region the k nearest X_i (for any sample point in the region), for problem 2 the nearest X_i, and for problem 3 the farthest. This completes the preprocessing.

Next, given p, we use the algorithm in [8] to find the region containing p. This takes $O(\log n)$ time, assuming that the degrees and coefficient sizes of the defining polynomials of X and of the minimal polynomials m^j are bounded by some constant. This solves problem 1 already. In the case of problem 2 (respectively, 3) we now know the set X_i in the arrangement that contains the nearest (respectively, farthest) point $q \in X$ from p.

The only remaining task, then, is to determine the coordinates $(\beta^1, \ldots, \beta^d)$ of the point $q \in X_i$. The coordinates are algebraic numbers, and we want to compute their minimal polynomials and isolating intervals. We know from sections 2 and 3 that the critical points of the distance-squared function from p to X_i are either the real roots of $df_i^1(p,x) = \cdots = df_i^{m_i}(p,x) = 0$, where $df_i^j \in \mathbb{Q}(\alpha^1, \ldots, \alpha^d)[x]$, (for parametrized m_i-surfaces X_i) or of $\varphi_i^1(p,x) = \cdots = \varphi_i^d(p,x) = 0$, where $\varphi_i^j \in \mathbb{Q}(\alpha^1, \ldots, \alpha^d)[x]$, (for zero-sets X_i). The real roots $\xi_1, \ldots, \xi_s$ of these systems are isolated and have algebraic coordinates whose minimal polynomials and isolating intervals can be determined by computing a diagonal basis of the systems and by isolating the roots of univariate polynomials with algebraic coefficients (using primitive element methods and the modified Uspenski algorithm). For parametrized surfaces X_i we have to determine the root ξ_j, $1 \le j \le s$, for which the algebraic number $f_i(p, \xi_j)$ is minimal (problem 2) or maximal (problem 3); and the result is $q = X_i(\xi_j)$, whose coordinates are algebraic numbers. For zero-sets X_i we have to determine the ξ_j for which $\|\xi_j - p\|^2$ is minimal or maximal; and the result is $q = \xi_j$.

Remarks. 1. The decomposition of d-space into connected regions in the complement of $Y := \cup\mathcal{S}_{i,j}$ is finer than it needs to be for answering the above queries. In section 6 we study certain subsets R_k and $\tilde{R}_k$ of Y which cut out far fewer regions (see

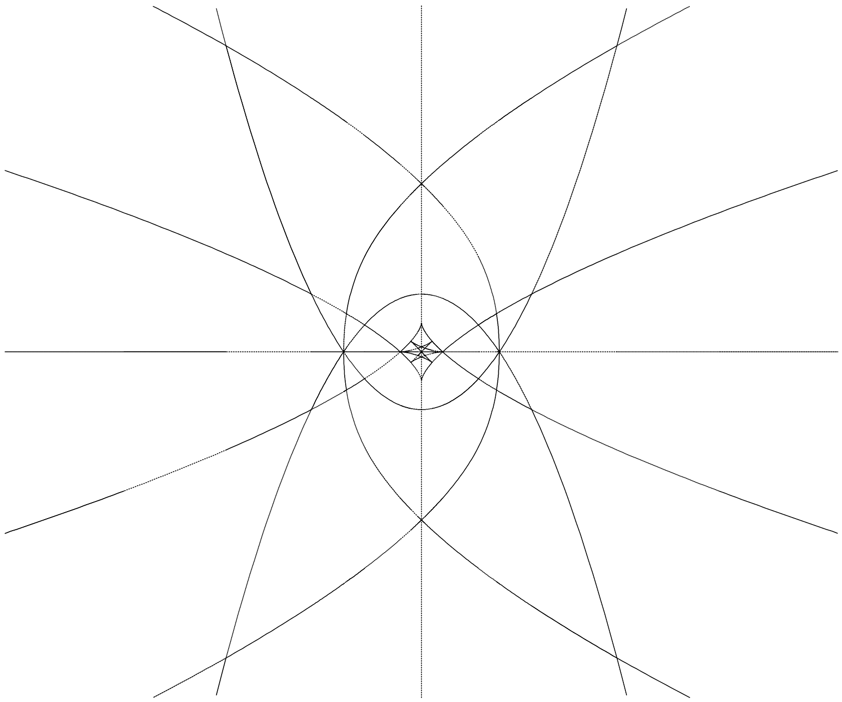

FIG. 2. *The set $\hat{\mathcal{B}}$ for a pair of parabolas.*

Proposition 6.1 for precise statements). Using the above exact symbolic methods, one can replace Y by the following sets: in problem 1 by R_k, in problem 2 by R_1, and in problem 3 by $\tilde{R}_1$.

2. However, in problems 2 and 3, if one replaces the exact symbolic computation of the nearest and farthest point $q \in X$ from p by the numerical curve-tracing procedure sketched below, then the decomposition into regions of $\mathbb{R}^d \setminus Y$ is too coarse, and one must replace Y by the full bifurcation set $\mathcal{B}$! The numerical procedure consists of the following: determine one sample point p' for each region in the complement of $\mathcal{B}$ (or $\hat{\mathcal{B}}$) and the corresponding nearest or farthest point $q' \in X$. After determining the region containing p (as before), one knows that any path in this region joining its sample point p' with p corresponds to a unique path in X joining the corresponding nearest/farthest points q' and q. This follows from the fact that the critical points of the distance-squared function are isolated in the complement of $\mathcal{B}$ and the continuity of the map which assigns to p its nearest/farthest point in X (Proposition 2.3). This would not be the case if we replace $\mathcal{B}$ by Y.

5. Some examples for arrangements in the plane. The first example, in Figure 2, shows a pair of parabolas $X_1(x) = (x, x^2 - 1)$, $X_2(x) = (x, 1 - x^2)$ together with the set $\hat{\mathcal{B}}$ which contains the bifurcation set $\mathcal{B}$. It should be noted that most curves in Figure 2, except for the intercurve level bifurcation set, are already known from Figure 1: the first parabola X_1 and the sets $\hat{\mathcal{E}}_1 = \mathcal{E}_1$ (a cusp-shaped curve) and $\hat{\mathcal{S}}_1$ (a vertical line) are exactly as shown in Figure 1, and turning Figure 1 upside down yields X_2, $\mathcal{E}_2$, and $\hat{\mathcal{S}}_2$ ($\hat{\mathcal{S}}_2$ coincides with $\hat{\mathcal{S}}_1$). The set $\hat{\mathcal{S}}_{1,2}$, which contains the intercurve level bifurcation set $\mathcal{S}_{1,2}$ (but we do not know whether it is equal to it), consists of a horizontal line through the origin and the zero-set Z of an irreducible

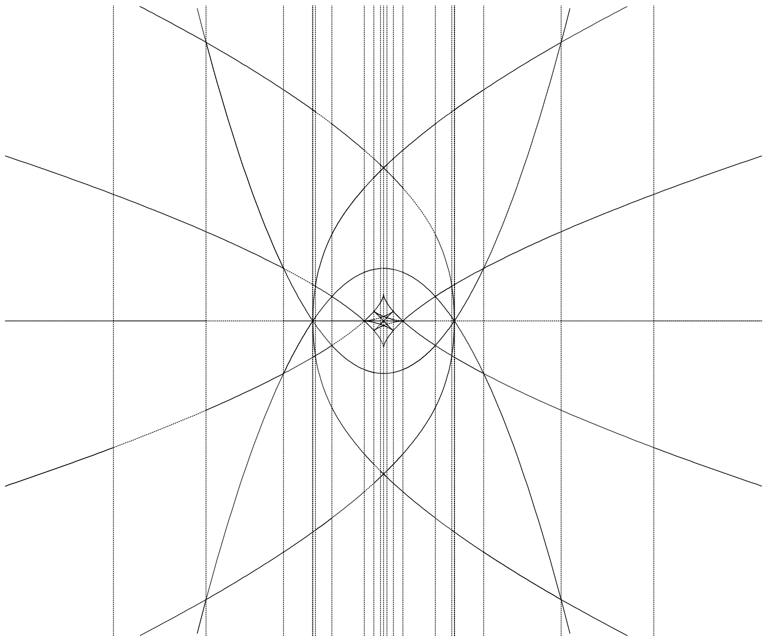

FIG. 3. *Cylindrical algebraic decomposition of $\hat{\mathcal{B}}$ and a pair of parabolas.*

(over $\mathbb{Q}$) degree 12 polynomial. The set Z has three real components: a compact curve with 6 cusps and a pair of nonsingular curves passing through the intersection points of the parabolas. Figure 3 shows a cylindrical algebraic decomposition of the plane into regions in the complement of $\mathbb{R}^2 \setminus \hat{\mathcal{B}} \cup X_1 \cup X_2$, which are arranged in "vertical cylinders." The cylinders are bounded by vertical tangent lines or by vertical lines passing through singular points. The regions within a cylinder $I \times \mathbb{R}$, where I is an interval on the x-axis, are separated by nonintersecting function graphs over I.

The second example, in Figure 4, shows the set $\hat{\mathcal{S}}_{1,2}$ for a parabola $X_1(x) = (x, x^2)$ and a point $X_2 = (1, 2)$. Note that the Voronoi diagram of X_1, X_2 consists of just two regions: the region cut out by $\hat{\mathcal{S}}_{1,2}$ containing the point X_2 and the complement of the closure of this region. The curve $\hat{\mathcal{S}}_{1,2}$ has 2 cusps, which correspond to centers of osculating circles of X_1 which pass through the point X_2. Figure 5 illustrates this fact: the cusps of $\hat{\mathcal{S}}_{1,2}$ lie on the evolute $\mathcal{E}_1 = \hat{\mathcal{E}}_1$ of the parabola X_1 (and hence are centres of osculating circles having A_2-contact with X_1).

6. Regions of $\mathbb{R}^d \setminus \mathcal{B}$ and Voronoi regions. The estimates in sections 2 and 3 for the size of $\mathcal{A}(\mathcal{B})$ (in terms of n, d, and the degrees and dimensions of the X_i) yield upper bounds for the complexity of the kth-order Voronoi diagram. They also bound the complexity of Voronoi diagrams of collections $X^s := \cup_i X_i^s$ of closed semialgebraic sets $X_i^s \subset \mathbb{R}^d$, each given as unions and intersections of $O(1)$ "elementary" sets of the form $\{x \in \mathbb{R}^d : h(x) \ ? \ 0\}$, where $? \in \{=, <, >, \leq, \geq\}$ (actually, any such X_i^s can be defined by choosing $? \in \{=, <\}$). For each X_i^s we can define a real algebraic set $X_i \supset X_i^s$, using the $O(1)$ polynomials h, such that $\dim X_i = \dim X_i^s$. If $\mathcal{B}^s$ and $\mathcal{B}$ are the bifurcation sets of X^s and $X := \cup_i X_i$, then $\mathcal{B}^s \subset \mathcal{B}$; in fact $\mathcal{B} \setminus \mathcal{B}^s$ consists of the closure of certain $(d-1)$-cells of $\mathcal{B}$. The size of the arrangement $\mathcal{A}(\mathcal{B}^s)$ (and hence of the kth-order Voronoi diagram of X^s) is therefore bounded above by $|\mathcal{A}(\mathcal{B})|$.

A comparison of the bounds for the size of $\mathcal{A}(\mathcal{B})$ and of Voronoi diagram $\mathcal{A}(V_k)$

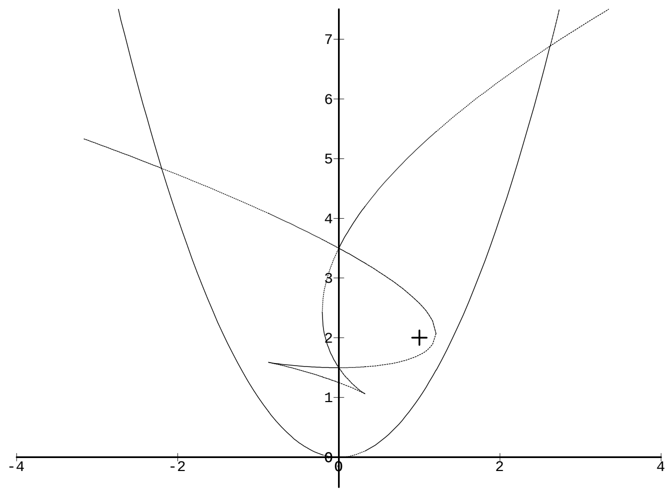

FIG. 4. *The set* $\hat{\mathcal{S}}_{1,2}$ *for a parabola* X_1 *and the point* $X_2 = (1, 2)$ *(marked by a cross).*

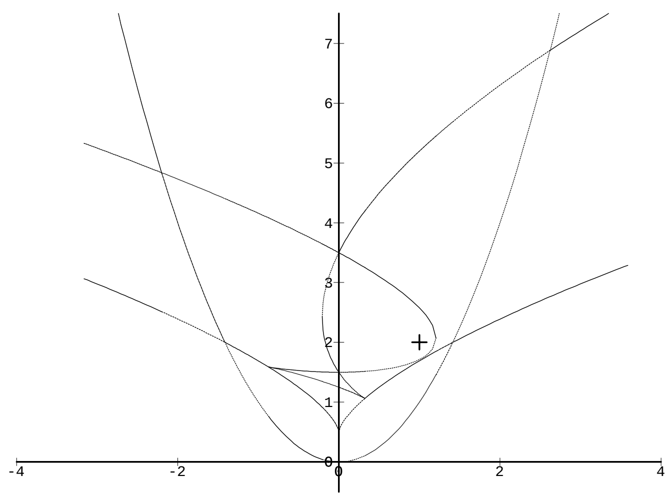

FIG. 5. *The sets* $\hat{\mathcal{S}}_{1,2}$ *and* $\hat{\mathcal{E}}_1$ *for a parabola* X_1 *and the point* $X_2 = (1, 2)$.

reveals, however, a considerable gap—at least in the special cases where something about the complexity of the Voronoi diagram is known. The works on Voronoi diagrams of arrangements of (semi-)algebraic sets study the combinatorial complexity (i.e., assume that the degrees of the algebraic sets in an arrangement are bounded above by some constant). In this case, $|\mathcal{A}(\mathcal{B})| \sim O(n^{2d})$. On the other hand, Sharir and Agarwal show in [22, Appendix 7.1] that the size of the first-order Voronoi diagram of n disjoint convex semialgebraic sets of "constant description size" (i.e., defined by $O(1)$ polynomial (in)equations of bounded degree and coefficient size) in d-space is $O(n^{d+\epsilon})$ (for any $\epsilon > 0$). And Alt and Schwarzkopf [1] show that the first-order Voronoi diagram of n points and disjoint parametrized algebraic curve segments in the plane, which also do not have self-intersections, has $O(n)$ size and can be constructed by a randomized algorithm in $O(n \log n)$ (expected) time. Their algorithm concentrates on the combinatorial aspect of the problem and assumes that the semialgebraic level bifurcation sets $\mathcal{S}_i$ and $\mathcal{S}_{i,j}$ (in our notation) can be determined by some numerical polynomial equation solver. We have seen in previous sections that the bifurcation set can also be determined by exact symbolic methods.

On the other hand, the first-order Voronoi diagram of the $dn \sim \Theta(n)$ intersecting hyperplanes $X_{ij} := \{x^i = j\}$, where $1 \leq i \leq d$ and $1 \leq j \leq n$, has $\Theta(n^d)$ connected d-dimensional regions (recall that $(x^1, \ldots, x^d)$ are coordinates in $\mathbb{R}^d$). Hence we have the lower bound $|\mathcal{A}(V_1)| \sim \Omega(n^d)$.

The goal of the present section, then, is to study the gap in the combinatorial complexities of $\mathcal{A}(\mathcal{B})$ and of $\mathcal{A}(V_k)$. To this end, we shall derive a bound for the combinatorial complexity of certain intermediate sets $V_k \subset R_k \subset \mathcal{B}$, which we are going to define next.

Roughly speaking, the sets R_k are constructed by deleting from the intersurface level bifurcation set $Y := \cup_{1 \leq i < j \leq n} \mathcal{S}_{i,j} \subset \mathcal{B}$ certain "branches" that cannot belong to V_k. In order to describe this construction, we need the following notation. Recall the notation for an $A^r_{\geq 1}$-singularity of the distance squared-function (see section 1.1). For a family of distance-squared functions $f(p, x)$ parametrized by the coordinates of the points $p \in \mathbb{R}^d$, we denote the set of points p for which f has a singularity of type $A^r_{\geq 1}$ by $\mathcal{L}(A^r_{\geq 1})$. Geometrically, it is the locus of centers of r-tangent spheres (i.e., spheres touching the algebraic set X in r distinct points). Now set

$$Y_r := Y \cap \mathcal{L}(A^r_{\geq 1}).$$

Furthermore, let $Y_{r_1,\ldots,r_s}$ denote the locus of common intersection points of s such sets Y_{r_i}. In order to avoid redundancies, let us agree that the indices are nonincreasing, i.e., $r_i \geq r_{i+1}$. It is also convenient to define the "closure" of $Y_{r_1,\ldots,r_s}$ as

$$\bar{Y}_{r_1,\ldots,r_s} := \bigcup \{Y_{a_1,\ldots,a_t} : t \geq s, a_i \geq r_i, 1 \leq i \leq s\}. \qquad (*)$$

Note that the points p of the s-fold self-intersection locus of $Y_{r_1,\ldots,r_s}$ of $Y = \bar{Y}_2$ are centers of s concentric r_i-tangent spheres (where $r_i \geq 2$). Also note that $p \in \bar{Y}_{r_1,\ldots,r_s} \setminus Y_{r_1,\ldots,r_s}$ if and only if there are more than s such spheres or some sphere has more than r_i points of tangency. To save breath, we shall often refer to the "k smallest r_i-tangent spheres" with common center $p \in Y_{r_1,\ldots,r_s}$, $k \leq s$, rather than to the "subset of the set of s simultaneous $A^r_{\geq 1}$-singularities with the k smallest critical values." Finally, let $S_{c_1,\ldots,c_t}$ be the locus of common intersections of t sets X_i of codimension $c_i := d - m_i$, which corresponds to the intersection locus of $\binom{t}{2}$ branches of Y, and define its "closure" $\bar{S}_{c_1,\ldots,c_t}$ as in $(*)$. Each branch consists of centers of spheres tangent to a pair of intersecting sets X_i, X_j whose radius tends to zero as the center approaches $X_i \cap X_j$, such spheres will be called *vanishing spheres.* We are now ready to construct the sets R_k.

First, we decompose the intersurface level bifurcation set Y into certain branches which, for generic arrangements X, will be $(d-1)$-dimensional. Let $B(Y)$ denote the set of connected components (branches) of $Y \setminus \bar{Y}_3 \cup \bar{S}_{1,1}$. Note that all points of such a branch lie either in $V_k \subset Y$ or in $Y \setminus V_k$, because for all these points we have a pair of critical points of the distance-squared function whose critical value is distinct from all other critical values.

Next, we decompose the self-intersection locus of Y into connected components of i-fold intersections, $i = 2, 3, \ldots, s$ and compare the radii of ≥ 2-tangent spheres associated to the i branches of $B(Y)$ passing through an i-fold intersection point. Let us call the set of $X_j \in \{X_1, \ldots, X_n\}$ containing the r points of tangency of an r-tangent sphere the *support set* of this sphere and the smallest sphere among a set of concentric r-tangent spheres with the same support set the *minimal sphere*. We also consider any vanishing sphere to be minimal. If, at any point p of the self-intersection locus, the ≥ 2-tangent sphere associated with some branch of $B(Y)$ does not belong to the k smallest minimal spheres with center p (including the vanishing sphere if $p \in S_{c_1,\ldots,c_t}$) and distinct support sets, then this branch cannot belong to V_k. Deleting all such branches from Y yields the set R_k. To be a bit more precise, let L be the set of "strata" of the "stratification" (the reason for the quotes will be explained in the remark below) of the self-intersection locus of Y into connected components of $Y_{r_1,\ldots,r_s}$ $(s, r_i \geq 2)$, $S_{c_1,\ldots,c_t}$ $(t \geq 2)$, and $S_{c_1,\ldots,c_t} \cap Y_{r_1,\ldots,r_s}$ $(t, r_i \geq 2,\ s \geq 1)$. For any $l \in L$, let l_k denote the set of branches $b \in B(Y)$ passing through l which correspond to the k smallest minimal ≥ 2-tangent spheres with center in l, which by definition have distinct support sets (if there are fewer than k minimal spheres with distinct radius, then l_k contains all branches through l that correspond to some minimal sphere). We can now define

$$R_k := \{b \in B(Y) : b \in l_k, \text{for all } l \in L : l \subset \mathrm{cl} b\} \cup \bar{Y}_3 \cup \bar{S}_{1,1}.$$

The principal result of the present section is based on an enumeration of the connected components of the self-intersection locus of Y, on the one hand, and of those components that also belong to R_k, on the other hand. We give an outline of our enumeration technique. Given an arrangement $X = \cup_{i=1}^n X_i$, there are $\Pi_{i=1}^s \binom{n}{r_i}$ sets $\bar{Y}_{r_1,\ldots,r_s}$, and each of them has a constant number of connected components (recall that the maximal degree of the defining equations of X and the ambient dimension are assumed to be fixed). The number of connected components of $Y_{r_1,\ldots,r_s}$ depends on the number of connected components of all the (lower dimensional) sets

$$Y_{a_1,\ldots,a_t} \subset \bar{Y}_{r_1,\ldots,r_s} \setminus Y_{r_1,\ldots,r_s}$$

in its boundary. For "large enough" s and $r_1, \ldots, r_s$ (see more precise statements below), the boundary of $Y_{r_1,\ldots,r_s}$ will be empty, so that $Y_{r_1,\ldots,r_s}$ has as many connected components as $\bar{Y}_{r_1,\ldots,r_s}$. We call a connected component of such a nonempty set $Y_{r_1,\ldots,r_s}$, whose boundary is empty, a *maximal component*, and $Y_{r_1,\ldots,r_s}$ a *maximal set.* (Note that its index set $r_1, \ldots, r_s$ is maximal, with respect to the natural partial order of $\mathbb{N}^s$, among the nonempty sets $Y_{a_1,\ldots,a_s}$. Among these nonempty sets, however, it will be the one with minimal dimension.) Likewise, the combinatorial complexity of the closures of the sets $S_{c_1,\ldots,c_t}$ and $S_{c_1,\ldots,c_t} \cap Y_{r_1,\ldots,r_s}$ is $O(n^t)$ and $O(n^{t+\Sigma r_i})$, respectively, and the complexity of the interiors of these sets will depend on the number of components in their boundary (for Σr_i and Σc_j sufficiently large we get, again, maximal sets with empty boundary). By inductively deleting the lower dimensional boundary components from $Y = \bar{Y}_2$, beginning with the maximal components, whose boundary is empty, we obtain a "stratification" of Y whose "strata" are the connected components of the sets $Y_{r_1,\dots,r_s}$, $S_{c_1,\dots,c_t}$ and their intersections. The number of strata obtained in this way is of the order of the number of maximal sets (recall that a given maximal set has $O(1)$ connected components, in terms of combinatorial complexity).

Remarks (refining stratification to get genuine stratification). 1. First, the rough idea. The strata of this stratification of Y can have singularities along the intersection of Y with the local bifurcation set $\mathcal{E} := \cup\mathcal{E}_i$. For example, the intersurface level bifurcation curve Y in Figure 2 has a component C with six cusps contained in $Y \cap \mathcal{E}$ (C is the small compact curve in the center of the figure). So what is going on here? The strata of C are connected components of $Y_{r_1,\dots,r_s}$ and correspond to points $p \in \mathbb{R}^d$ for which the distance-squared function to X has s simultaneous singularities of type $A^{r_i}_{\geq 1}$. But any r-tuple of singular points with the same critical values belongs to $A^r_{\geq 1}$: for example, the regular branches of C in Figure 2, which are of type $A_1^2 = \{A_1, A_1\}$, but also the cusps, which are of type $\{A_2, A_1\}$. Roughly speaking, one can further subdivide the components (i.e., the strata) of Y into submanifolds (i.e., genuine strata) by requiring that its singular points $\Sigma_1, \dots, \Sigma_{r_i}$, $1 \leq i \leq s$, are of the same local type (for example, we distinguish A_1 from A_2 points). For each original stratum we then obtain $O(1)$ genuine "equisingular" strata (the number of local types on a given stratum depends on the degree of the X_i and the ambient dimension, but not on n). With this understood, we shall no longer make the distinction between a stratification and its genuine refinement.

2. The meaning of equisingular (for readers familiar with singularity theory). We are considering, in general, nonversal d-parameter families of functions. As d increases, the standard groups of equivalences for functions, such as $\mathcal{R}$ and $\mathcal{K}$, will quickly yield an infinite number of equisingular strata (due to the appearance of moduli). Recall that a $\mathcal{K}$-orbit consists, in general, of several $\mathcal{R}$-orbits; we can only expect for the class of quasi-homogeneous functions that the $\mathcal{K}$- and $\mathcal{R}$-orbits coincide (by a result of Saito [21]). The appropriate definition of equisingular stratum, which yields a finite number of smooth strata, therefore involves the union of $\mathcal{K}$-modular strata (minus certain exceptional strata of higher codimension).

We can now state the main result of section 6.

PROPOSITION 6.1. *For any arrangement of parametrized or implicitly defined algebraic sets (whose degrees are bounded by some constant) in d-space consisting of n elements of any positive codimension the following hold:*

(i) $V_k \subset R_k \subset Y \subset \mathcal{B}$, $1 \leq k \leq n-1$.
(ii) *The size of* $\mathcal{A}(R_k)$ *and hence the number of connected regions of* $\mathbb{R}^d \setminus R_k$ *are bounded above by* $O(\min(n^{d+k}, n^{2d}))$.
(iii) *The combinatorial complexity of* $Y_{2,\dots,2}$ *(d twos) is* $\Pi_{j=1}^d \binom{n}{2} \sim O(n^{2d})$ *and represents the "leading term" in the combinatorial complexity of* $\mathcal{A}(\mathcal{B})$.

Proof. Statement (i) simply follows from the definitions of these sets. For the proof of statements (ii) and (iii) it is convenient to distinguish generic and nongeneric arrangements X, which are defined as follows. Let $\mathcal{X}$ be the space of arrangements $X \subset \mathbb{R}^d$ of n zero-sets X_i of codimension c_i of maximal degree Δ (or of n m_i-surfaces X_i parametrized by polynomials ofdegree $\leq \delta$). $\mathcal{X}$ can be identified with some semialgebraic subset of the finite dimensional space of coefficients of $\sum_{i=1}^{n} c_i$ polynomials in d variables of degree $\leq \Delta$ (or of nd polynomials in $\sum_{i=1}^{n} m_i$ variables of degree $\leq \delta$). (Note that not all choices of coefficients yield m_i-dimensional real algebraic sets X_i.) Now define W to be the union of the following sets (corresponding to degenerate X for which Y has "excess intersection"):

$$\left\{ X \in \mathcal{X} : \exists s \geq 1, \exists r_i \geq 2 : \dim Y_{r_1,\ldots,r_s} > d + s - \sum_{i=1}^{s} r_i \right\},$$

$$\left\{ X \in \mathcal{X} : \exists t \geq 2, \exists c_i \geq 1 : \dim S_{c_1,\ldots,c_t} > d - \sum_{i=1}^{t} c_i \right\},$$

and

$$\left\{ X \in \mathcal{X} : \exists s, c_i \geq 1, \exists t, r_j \geq 2 : \dim(S_{c_1,\ldots,c_t} \cap Y_{r_1,\ldots,r_s}) > d + s - \sum_{i=1}^{t} c_i - \sum_{j=1}^{s} r_j \right\}.$$

(Note that X denotes both a subset of $\mathbb{R}^d$ as well as a point of $\mathcal{X}$, but the meaning of X should be clear from the context.) One shows, using the defining equations of these sets, that W is a Zariski closed subset of $\mathcal{X}$. We shall therefore say that an element X in $\mathcal{X} \setminus W$ is generic and one in W nongeneric.

First, assume that X is generic and consider the following two stratifications of $\mathbb{R}^d$. In the first, take as strata of dimension 0 to $d-1$ the connected components of the sets $Y_{r_1,\ldots,r_s}$, $S_{c_1,\ldots,c_t}$, $S_{c_1,\ldots,c_t} \cap Y_{r_1,\ldots,r_s}$ and as d-dimensional strata the connected components in the complement of $Y = Y_2$. In the second stratification, we discard the connected components of $Y_{r_1,\ldots,r_s}$ that do not belong to R_k and take the connected components of $\mathbb{R}^d \setminus R_k$ as d-dimensional strata.

For the 0-dimensional maximal sets $Y_{r_1,\ldots,r_s}$ we have, by the genericity of X, the relation $\Sigma_{i=1}^{s} r_i = d + s$. For the 0-dimensional maximal sets $S_{c_1,\ldots,c_t}$ and $S_{c_1,\ldots,c_t} \cap Y_{r_1,\ldots,r_s}$ we have in the worst case of hypersurface arrangements (where all $c_i = 1$) the relations $t = d$ and $t + \Sigma r_i = d + s$. Hence, there are at most $\Pi_{i=1}^{s} \binom{n}{r_i} \sim O(n^{d+s})$ such maximal sets, and each of them has $O(1)$ connected components. For the first stratification (whose union of strata of dimension less than d is Y), the relation $\Sigma r_i = d + s$, where all $r_i \geq 2$, implies that $s \leq d$. For the second stratification (whose union of strata of dimension $< d$ is R_k) we have, by the definition of R_k, $s \leq \min(k, d)$. Let e_i denote the number of i-dimensional strata. Then, for all $0 \leq i \leq d-1$, $e_i \sim O(n^{2d})$ (for the stratification of Y) and $e_i \sim O(n^{\min(d+k,2d)})$ (for the stratification of R_k).

For statement (ii) of the proposition we now consider the second stratification. We claim that the number e_d of connected regions of $\mathbb{R}^d \setminus R_k$ is also $O(n^{\min(d+k,2d)})$. Taking a 1-point compactification $S^d = \mathbb{R}^d \cup \{\infty\}$ and adding at most $O(n^{\min(d+k,2d)})$ cells to the induced stratification of R_k in the d-sphere, we get a cell complex K whose Euler characteristic is equal to $\chi(S^d) = 1 + (-1)^d$. Note that e_d is bounded above by the number of d-cells of K; this implies (ii).

For statement (iii), note that $Y_{2,\ldots,2}$ (with d twos) is the maximal set of the highest combinatorial complexity in the stratification of Y satisfying the relation $\Sigma r_i = d + s$, and its number of connected components is of order n^{2d}.

For nongeneric arrangements $X \in W$, we consider a "linear deformation" X_t, $t \in (-\epsilon, +\epsilon)$, of $X = X_0$ such that X_0 is the only nongeneric element—linear in the sense that $t \mapsto X_t$ defines a line in the space of coefficients which can be identified with $\mathcal{X}$. (Such a deformation can be obtained, for example, by constructing a stratification of the semialgebraic set W and by restricting a line in the normal space of the stratum containing X to some sufficiently small open neighborhood.) Consider the union U of any of the semialgebraic sets $U_t = Y_t$ or $(R_k)_t$ associated to X_t; U is a semialgebraic subset of $\mathbb{R}^d \times (-\epsilon, +\epsilon)$. We claim that the combinatorial complexity of the degenerate arrangement $\mathcal{A}(U_0)$ is of the same order as that of its generic deformation $\mathcal{A}(U_t)$, for small $t \neq 0$, which implies the desired bounds in the degenerate case.

The claim follows from the following argument. First, we want to check that all strata of U_0 lie in the closure of some stratum of $U \setminus U_0$. Given a pair of closed, connected subsets $A, B \subset \mathbb{R}^d$ and any point $q \in A$ there exists a sphere tangent to A at q and to B in some point q'. Let A_t and B_t be subsets of $X_t \subset \mathbb{R}^d \times (-\epsilon, +\epsilon)$ such that A_0, B_0 are connected subsets of $X = X_0$. Our assumptions about the algebraic set X imply that the dimensions of A_t and B_t are constant for t in some open neighborhood I of 0 (in particular the sets remain nonempty over the reals). By the geometric fact above, there exist points $q_t \in A_t$ and $q'_t \in B_t$ supporting bitangent spheres with centers p_t, such that the sets $\{q_t : t \in I\}$, $\{q'_t : t \in I\}$, and $\{p_t : t \in I\}$ are connected and p_0 is any point of U_0. Next, let $\epsilon > 0$ be small enough such that U is transverse to all hyperplanes $t = c$, for any constant $|c| < \epsilon$, except $t = 0$. (U is, in general, a singular semialgebraic set, and transverse means that the hyperplane in question is transverse to all the strata of a suitable stratification of U, e.g., a stratification satisfying the Whitney condition (b). See the book by Goresky and MacPherson [13, Part I, Chapters 1.2–1.8] for a good introduction to stratification theory.) So the numbers of strata of dimension $0 \leq i \leq d$ in $\mathcal{A}(U_t)$ are locally constant for $t \in (-\epsilon, 0)$ and $t \in (0, \epsilon)$; denote these numbers by r_- and r_+, respectively. Hence there are $r_- + r_+$ strata in $\mathcal{A}(U \setminus U_0)$ and therefore at most that many strata of $\mathcal{A}(U_0)$. □

Remarks. 1. For compact arrangements $X = \cup X_i$ we can define regions analogous to the usual kth-order Voronoi regions, such that for all points within each region the farthest k sets in the arrangement do not change. If $\tilde{V}_k$ is the union of the boundaries of these regions and $\tilde{R}_k$ the set analogous to R_k, except that the k smallest minimal spheres are replaced by the k largest maximal spheres, then Proposition 6.1 above holds for $\tilde{R}_k$ and $\tilde{V}_k$ in place of R_k and V_k.

2. One can get a sharper bound for the (expected) size of $\mathcal{A}(V_k)$, where $k \sim O(1)$, in the average case. Set $\mu_p(X_i) := \inf_{q \in X_i} \|q - p\|^2$. The sets $Y_{r_1,\dots,r_s} \cap R_k$ and $(Y_{r_1,\dots,r_s} \cap S_{c_1,\dots,c_t}) \cap R_k$, $s \leq k$, can only belong to V_k if the critical values of the s $A^{r_i}_{\geq 1}$-singularities of the distance-squared function from $p \in Y_{r_1,\dots,r_s}$ are smaller than all but $k - s \sim O(1)$ of the minima $\mu_p(X_j)$ of the $O(n)$ sets X_j that do not belong to the support sets of one of these $A^{r_i}_{\geq 1}$-singularities (for $p \in \cap_{i=1}^{t} X_i$, we include the intersecting X_i in the support set). Now suppose for the moment that there exists a "good" probability measure on the space $\mathcal{X}$ of arrangements (definition follows below) such that if we pick some $X = \cup X_i \in \mathcal{X}$ "at random," then $\Pr[\mu_p(X_i) < \mu_p(X_j)] = 1/2$ (actually it is enough if this probability is different from 0 and 1). One then checks that the probability that the critical values of all s $A^{r_i}_{\geq 1}$-singularities are smaller than $O(n)$ minima $\mu_p(X_j)$ is $O(n^{-s})$. Looking at the proof of Proposition 6.1 we now see the following: if M sets X are picked independently from some "good" distribution on $\mathcal{X}$ then, as $M \to \infty$, the expected size of $\mathcal{A}(V_k)$ is $O(n^d)$.

The "good" probability measures on $\mathcal{X}$ are defined as follows. Let $\mathcal{X}^{m_i}$ denote the space of real algebraic sets of dimension m_i of some bounded degree (recall that $\mathcal{X}^{m_i}$ can be identified with a semialgebraic subset of the space of coefficients of the defining polynomials of such sets); then $\mathcal{X} := \times_{i=1}^{n} \mathcal{X}^{m_i}$. Let $\mathcal{M}_i$ be a probability measure on $\mathcal{X}^{m_i}$ (for example, the uniform distribution on some compact subset B of $\mathcal{X}^{m_i}$ of bounded coefficients). We say that the collection of probability measures $\mathcal{M}_i$, $1 \leq i \leq n$, is "good" if the following hold. 1. For each pair $(\mathcal{M}_i, \mathcal{M}_j)$, the Lebesgue measure on $\mathcal{X}^{m_i} \times \mathcal{X}^{m_j}$ is *absolutely continuous* with respect to the product measure $\mathcal{M}_i \times \mathcal{M}_j$. 2. For any given $p \in \mathbb{R}^d$ and all ordered pairs i, j the sets $\{(X_i, X_j) : \mu_p(X_i) < \mu_p(X_j)\}$ have non-zero Lebesgue measure in $\mathcal{X}^{m_i} \times \mathcal{X}^{m_j}$. Conditions 1 and 2 imply that $\Pr[\mu_p(X_i) < \mu_p(X_j)] \neq 0, 1$, but seem quite strong. Note, however, that the conditions are trivially satisfied in an important special case: if all sets X_i in an arrangement have the same dimension and are chosen from a single, but arbitrary, distribution, then, by symmetry, $\Pr[\mu_p(X_i) < \mu_p(X_j)] = 1/2$.

Acknowledgments. I am very grateful to the referees for their detailed comments that contained several corrections and many other useful suggestions. The author acknowledges financial support from the Max–Planck Gesellschaft and the research foundation FAPESP, São Paulo, Brazil.

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